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On Weak Convergence of Locally Periodic Functions

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Abstract

We prove a generalization of the fact that periodic functions converge weakly to the mean value as the oscillation increases. Some convergence questions connected to locally periodic nonlinear boandary value problems are also considered.

1 Introduction

In the the proof of the reiterated homogenization results obtained in [14] (see also [15]) the following two facts were used (the exact defintions and properties are given later in Section 2 and 3):

- If (v_h) is a sequence of Y-periodic functions in $L^p_{\mathrm{loc}}(R^n)$, $p > 1$, such that $v_h \to v$ weakly in $L^p(Y)$ and if w_h is defined as $w_h(x) = v_h(hx)$, then $w_h \to (1/|Y|) \int_Y v(x)\,dx$.
- If

  $$-\mathrm{div}(a(x, hx, \xi + Du^\xi_h)) = 0$$

  on $Y = [0, y]^n$, $u^\xi_h \in W^{1,p}_{\mathrm{per}}(Y)$, then $u^\xi_h \to u^\xi_0$ weakly in $W^{1,p}_{\mathrm{per}}(Y)$, where u^ξ_0 is the solution of the corresponding limit-equation. Here a is monotone, continuous and satisfies suitable coerciveness and growth conditions in the third variable and periodic in the second.

We have not found proofs of these facts in the literature. The aim of this paper is to present such proofs. Moreover, we show that the first statement also holds for the case $p = 1$.

The two facts described above are used in the proof of the reiterated homogenization result for monotone operators, see [14] and [15]. The solution u^ξ_h is used to define a sequence of functions similar to the ones in Tartar's celebrated method of oscillated test functions (see e.g. the book [8]). The first fact described above in combination with compensated compactness is used to analyze the asymptotic behavior of this sequence of functions.

For information concerning reiterated homogenization we recommend the papers [14] and [15] and the references given there. Concerning explicit engineering applications see e.g. [4].

2 A weak convergence result

Let us first recall the following lemma (for the proof see e.g. [12]).

Lemma 1. *Let $\{u_h\}$ be a sequence in $L^1(\Omega)$. The following statements are equivalent:*

1. *every subsequence of $\{u_h\}$ contains a subsequence which converges weakly in $L^1(\Omega)$.*
2. *for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $h \in \{1, 2, ...\}$ it holds that*

$$\int_{\{u_h \geq t_\varepsilon\}} |u_h| \, dx \leq \varepsilon,$$

where $\{u_h \geq t_\varepsilon\}$ denotes the set $\{x \in \Omega : u_h(x) \geq t_\varepsilon\}$.

The following Proposition is a generalization of the well-known fact that a periodic function converges weakly to its mean value as the oscillation increases.

Theorem 1. *Let $1 \leq p \leq \infty$ and let $u_h \in L^p_{loc}(R^n)$ be Y periodic for $h \in N$. Moreover, suppose that $u_h \to u$ weakly in $L^p(Y)$ (weak-$*$ if $p = \infty$) as $h \to \infty$. Let w_h be defined by $w_h(x) = u_h(hx)$. Then as $h \to \infty$ it holds that*

$$w_h \to \frac{1}{|Y|} \int_Y u(x) dx$$

weakly in $L^p(\Omega)$ (weak-$*$ if $p = \infty$).

Proof. We first consider the case $1 < p \leq \infty$. For simplicity we put $Y = (0,1)^n$, i.e. the unit cube in R^n, since the general case is principally the same. Let $Y_h^k = (1/h)\,(k + Y)$, where $k \in Z^n$, i.e. the translated image of $1/hY$ by the vector k/h. We note that $\left|Y_h^k\right| = (1/h)^n$. Let $\phi \in C_0(\Omega)$ and ϕ^h the function which takes a constant value equal to the value $\phi(k/h)$ in each cell Y_h^k. Due to the uniform continuity of ϕ on the compact set $\overline{\Omega}$, we obtain that $\phi^h \to \phi$ uniformly on Ω. Thus,

$$\int_\Omega \left|\phi^h - \phi\right|^q dx \leq \int_\Omega \left(\max_{x \in \Omega} \left|\phi^h(x) - \phi(x)\right|\right)^q dx \to 0, \tag{2.1}$$

as $h \to \infty$, i.e. $\phi^h \to \phi$ in $L^q(\Omega)$. Since ϕ has compact support in Ω we have that each cell Y_h^k, for which $\phi(k/h) \neq 0$, is contained Ω for sufficiently large values of h. This and the Y-periodicity of u_h implies that

$$\begin{aligned}
\int_\Omega u_h(hx)\,\phi^h(x)dx &= \int_{\mathbf{R}^n} u_h(hx)\,\phi^h(x)dx \\
&= \sum_{k\in\mathbf{Z}^n} \int_{Y_h^k} u_h(hx)\,\phi(k/h)dx \\
&= \sum_{k\in\mathbf{Z}^n} \left(\frac{1}{|Y_h^k|}\int_{Y_h^k} u_h(hx)\,dx\right)\left(\int_{Y_h^k}\phi(k/h)dx\right) \\
&= \frac{1}{|Y|}\int_Y u_h(x)\,dx \sum_{k\in\mathbf{Z}^n}\left(\int_{Y_h^k}\phi(k/h)dx\right) \qquad (2.2) \\
&= \frac{1}{|Y|}\int_Y u_h(x)\,dx \int_\Omega \phi^h(x)dx,
\end{aligned}$$

for sufficiently large values of h. Moreover, we have that $u_h(h\cdot)$ is bounded in $L^p(\Omega)$. This fact is shown as follows: Define the index set I_h as

$$I_h = \left\{k \in Z^n : Y_h^k \cap \Omega \neq \emptyset\right\}.$$

Since Ω is bounded there exists a constant K such that the number of elements in I_h is less than Kh^n. We obtain that

$$\begin{aligned}
\int_\Omega |u_h(hx)|^p\,dx &\leq \sum_{k\in I_h}\int_{Y_h^k}|u_h(hx)|^p\,dx \\
&= \sum_{k\in I_h}\left(\frac{1}{h}\right)^n\int_{k+Y}|u_h(x)|^p\,dx \qquad (2.3) \\
&\leq K\int_Y |u_h(x)|^p\,dx.
\end{aligned}$$

Now it follows that $u_h(h\cdot)$ is bounded in $L^p(\Omega)$ by taking into account that any weakly convergent sequence is bounded. By Hölder's inequality we have that

$$\begin{aligned}
&\left|\int_\Omega\left(u_h(hx) - \frac{1}{|Y|}\int_Y u(x)\,dx\right)\phi(x)dx\right| \\
\leq\; &\left(\int_\Omega\left|u_h(hx) - \frac{1}{|Y|}\int_Y u(x)\,dx\right|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega\left|\phi(x)-\phi^h(x)\right|^q dx\right)^{\frac{1}{q}} \\
&+\left|\int_\Omega\left(u_h(hx) - \frac{1}{|Y|}\int_Y u_h(x)dx\right)\phi^h(x)dx\right| \\
&+\left|\frac{1}{|Y|}\int_Y (u_h(x)-u(x))\,\phi^h(x)\,dx\right||\Omega|.
\end{aligned}$$

This together with (2.1), (2.2) and (2.3) implies that

$$\int_\Omega\left(u_h(hx) - \frac{1}{|Y|}\int_Y u(x)\,dx\right)\phi(x)dx \to 0,$$

as $h\to\infty$ for every $\phi \in C_0(\Omega)$. By using a density argument it also holds for every $\phi \in L^q(\Omega)$ and we are done.

Let us now turn to the case $p = 1$. Let u_h^i be defined as follows

$$u_h^i = \begin{cases} u_h & \text{if } u_h(x) < t_{1/i} \\ \\ 0 & \text{if } u_h(x) \geq t_{1/i} \end{cases} .$$

According to Lemma 1 there exists a constant $t_{1/i} > 0$ for each positive integer i such that

$$\int_\Omega \left|u_h^i - u_h\right| dx = \int_{\{u_h \geq t_{1/i}\}} |u_h| \, dx \leq \frac{1}{i}, \tag{2.4}$$

for all $h, i \in N$. By a diagonalization argument each subsequence of (h) has a subsequence, denoted by $(h^{'})$, such that $u_{h'}^i$ converges weak* in $L^\infty(Y)$ to some function u^i for every i. It is easy to see that the proof for the case $(1 < p \leq \infty)$ also holds if (h) is replaced with (h'), which implies that

$$u_{h'}^i(h' \cdot) \to \frac{1}{|Y|} \int_Y u^i(x) dx \tag{2.5}$$

weak* in $L^\infty(\Omega)$ for every i. Let $v \in L^\infty(\Omega)$. Then

$$\begin{aligned}
& \limsup_{h' \to \infty} \left| \int_\Omega v(x) \left(u_{h'}\left(h'x\right) dx - \left(\frac{1}{|Y|} \int_Y u(x) dx \right) \right) dx \right| \qquad (2.6) \\
= \; & \limsup_{i \to \infty} \limsup_{h' \to \infty} \left| \int_\Omega v(x) \left(u_{h'}\left(h'x\right) dx - \left(\frac{1}{|Y|} \int_Y u(x) dx \right) \right) dx \right| \\
\leq \; & \limsup_{i \to \infty} \limsup_{h' \to \infty} \left| \int_\Omega v(x) \left(u_{h'}\left(h'x\right) - u_{h'}^i\left(h'x\right) \right) dx \right| \\
& + \limsup_{i \to \infty} \limsup_{h' \to \infty} \left| \int_\Omega v(x) \left(u_{h'}^i\left(h'x\right) dx - \left(\frac{1}{|Y|} \int_Y u(x) dx \right) \right) dx \right| .
\end{aligned}$$

Both of the last terms are zero. For the first term this is seen by replacing u_h with $u_{h'}^i - u_{h'}$ in (2.3) and using (2.5) and (2.4) to obtain that

$$\begin{aligned}
\left| \int_\Omega v(x) \left(u_{h'}\left(h'x\right) - u_{h'}^i\left(h'x\right) \right) dx \right| & \leq \|v(x)\|_\infty \int_\Omega \left| u_{h'}^i\left(h'x\right) - u_{h'}\left(h'x\right) \right| dx \\
& \leq \|v(x)\|_\infty K \int_Y \left| u_{h'}^i(x) - u_{h'}(x) \right| dx \\
& \leq \|v(x)\|_\infty \frac{K}{i} .
\end{aligned}$$

From this it is clear that

$$\limsup_{i \to \infty} \limsup_{h' \to \infty} \left| \int_\Omega v(x) \left(u_{h'}\left(h'x\right) - u_{h'}^i\left(h'x\right) \right) dx \right| = 0.$$

For the second term we use (2.5), the weak lower semicontinuity of the $L^1(\Omega)$ norm and

(2.4) in order to obtain that

$$\begin{aligned}
& \limsup_{h'\to\infty}\left|\int_\Omega v(x)\left(u^i_{h'}\left(h'x\right)dx-\left(\frac{1}{|Y|}\int_Y u(x)dx\right)\right)dx\right| \\
\leq\ & \frac{1}{|Y|}\int_Y \left|u^i(x)-u(x)\right|dx\left|\int_\Omega v(x)dx\right| \\
\leq\ & \frac{1}{|Y|}\liminf_{h'\to\infty}\int_Y \left|u^i_{h'}(x)-u_{h'}(x)\right|dx\left|\int_\Omega v(x)dx\right| \\
\leq\ & \frac{1}{|Y|\,i}\left|\int_\Omega v(x)dx\right|\to 0
\end{aligned}$$

as $i\to\infty$. Summing up from (2.6) we have that any subsequence of $(u_h(h\cdot))$ contains a subsequence $\left(u_{h'}(h'\cdot)\right)$ which converges weakly to $|Y|^{-1}\int_Y u(x)dx$ in $L^1(\Omega)$. Thus this is also true for the whole sequence $\{u_h(h\cdot)\}$.

3 Homogenization of some periodic boundary value problems

Before we state the result of this section we introduce some definitions and notations. Let Y and Z be a open bounded rectangles in R^n, $|E|$ denotes the Lebesgue measure of the set $E\subset R^n$ and $(\cdot,\cdot)$ is the Euclidean scalar product on R^n. Moreover, c will be a constant that may differ from one place to an other and $h\in N$. The function $\widetilde{\omega}:R\to R$ is an arbitrary function which is continuous, increasing and $\widetilde{\omega}(0)=0$. By $W^{1,p}_{per}(Y)$ we denote the set of all functions $u\in W^{1,p}(Y)$ with mean value zero which have the same trace on opposite faces of Y, $W^{1,p}_{per}(Z)$ is defined in the corresponding way. Every function $u\in W^{1,p}_{per}(Y)$ can be extended by periodicity to a function in $W^{1,p}_{loc}(R^n)$ (in this paper we will not make any distinction between the original function and its extension). Let us fix a function $a:Y\times R^n\times R^n\to R^n$ which fulfills the conditions:

1. $a(y,\cdot,\xi)$ is Z-periodic and Lebesgue measurable for every $\xi\in R^n$ and every $y\in R^n$,
2. There exists two constants $c_1,c_2>0$ and two constants α and β, with $0\leq\alpha\leq\min\{1,p-1\}$ and $\max\{p,2\}\leq\beta<\infty$ such that a satisfies the following boundedness, continuity and monotonicity assumptions:

$$a(y,z,0)=0 \text{ for a.e. } y,z\in R^n, \tag{3.1}$$

$$|a(y,z,\xi_1)-a(y,z,\xi_2)|\leq c_1(1+|\xi_1|+|\xi_2|)^{p-1-\alpha}|\xi_1-\xi_2|^\alpha, \tag{3.2}$$

$$(a(y,z,\xi_1)-a(y,z,\xi_2),\xi_1-\xi_2)\geq c_2(1+|\xi_1|+|\xi_2|)^{p-\beta}|\xi_1-\xi_2|^\beta \tag{3.3}$$

3. a is on the form $a(y,z,\xi)=\sum_{i=1}^N\chi_{\Omega_i}(y)a_i(y,z,\xi)$ and satisfies a continuity condition of the form

$$|a(y_1,z,\xi)-a(y_2,z,\xi)|^q\leq\omega(|y_1-y_2|)\left(1+|\xi|^p\right), \tag{3.4}$$

for $y_1,y_2\in\Omega_i$, $i=1,\dots,N$, a.e. $z\in R^n$ and every $\xi\in R^n$, and where $\omega:R\to R$ is continuous, increasing and $\omega(0)=0$.

By (3.1), (3.2), and (3.3) it follows that

$$|a(y,z,\xi)| \leq c\left(1+|\xi|^{p-1}\right), \tag{3.5}$$

$$|\xi|^p \leq c\left(1+(a(y,z,\xi),\xi)\right), \tag{3.6}$$

hold for $y \in R^n$, a.e. $z \in R^n$ and every $\xi \in R^n$.

We are now in the position to state the result in this section.

Theorem 2. *Let a satisfy (3.1), (3.2), (3.3) and (3.4). Moreover, let* (u_h^ξ) *be the solutions of*

$$\begin{cases} \int_Y (a(x,hx,\xi+Du_h^\xi), D\phi)\,dx = 0 \text{ for every } \phi \in W^{1,p}_{per}(Y), \\ u_h^\xi \in W^{1,p}_{per}(Y). \end{cases} \tag{3.7}$$

Then

$$\begin{aligned} u_h^\xi &\to u^\xi \text{ weakly in } W^{1,p}_{per}(Y), \\ a(x,hx,\xi+Du_h^\xi) &\to b(x,\xi+Du^\xi) \text{ weakly in } L^q(Y,R^n), \end{aligned}$$

as $h \to \infty$*, where* u^ξ *is the unique solution of*

$$\begin{cases} \int_Y (b(x,\xi+Du^\xi), D\phi)\,dx = 0 \text{ for every } \phi \in W^{1,p}_{per}(Y), \\ u^\xi \in W^{1,p}_{per}(Y). \end{cases}$$

The operator $b : Y \times R^n \to R^n$ *is defined as*

$$b(y,\tau) = \frac{1}{|Z|}\int_Z a(y,z,\tau+Dv^{\tau,y}(z))\,dz,$$

where $v^{\tau,y}$ *is the unique solution of the cell-problem*

$$\begin{cases} \int_Z (a(y,z,\tau+Dv^{\tau,y}(z)), D\phi)\,dz = 0 \text{ for every } \phi \in W^{1,p}_{per}(Z), \\ v^{\tau,y} \in W^{1,p}_{per}(Z). \end{cases} \tag{3.8}$$

Proof. We divide the proof into several steps.

Step 1. Let $\left\{\Omega_i^k \subset \Omega : i \in I_k\right\}$ denote a family of disjoint open sets with diameter less than $\frac{1}{k}$ such that $\left|\Omega \backslash \cup_{i\in I_k} \Omega_i^k\right| = 0$ and $\left|\partial\Omega_i^k\right| = 0$. We define the function a^k as

$$a^k(y,z,\xi) = \sum_{i\in I_k} \chi_{\Omega_i^k}(y)\, a(y_i^k,z,\xi),$$

where $y_i^k \in \Omega_i^k$. Consider the auxillary periodic boundary value problems (transmission problems)

$$\begin{cases} \int_Y (a^k(x,hx,\xi+Du_h^{k,\xi}), D\phi)\,dx = 0 \quad \text{for every } \phi \in W^{1,p}_{per}(Y), \\ u_h^{k,\xi} \in W^{1,p}_{per}(Y). \end{cases} \tag{3.9}$$

Then we have that

$$u_h^{k,\xi} \to u^{k,\xi} \text{ weakly in } W_{per}^{1,p}(Y),$$

$$a^k(x, hx, \xi + Du_h^{k,\xi}) \to b^k(x, \xi + Du^{k,\xi}) \text{ weakly in } L^q(Y, R^n),$$

where $u^{k,\xi}$ is the unique solution of the homogenized problem

$$\begin{cases} \int_Y (b^k(x, \xi + Du^{k,\xi}), D\phi)\, dx = 0 \text{ for every } \phi \in W_{per}^{1,p}(Y), \\ u^{k,\xi} \in W_{per}^{1,p}(Y). \end{cases} \tag{3.10}$$

The operator $b^k : Y \times R^n \to R^n$ is defined a.e. as

$$b^k(y, \tau) = \sum_{i=I_k} \chi_{\Omega_i^k}(y) \int_Z a(y_i^k, z, \tau + Dv^{\tau, y_i^k}(z))\, dz = \sum_{i=I_k} \chi_{\Omega_i^k}(y) b(y_i^k, \tau),$$

where v^{τ, y_i^k} is the unique solution of the cell problem

$$\begin{cases} \int_Z (a(y_i^k, z, \tau + Dv^{\tau, y_i^k}(z)), D\phi(z))\, dz = 0 \text{ for every } \phi \in W_{per}^{1,p}(Z), \\ v^{\tau, y_i^k} \in W_{per}^{1,p}(Z). \end{cases} \tag{3.11}$$

The proof of these convergence results follows by suitable modifications of well-known homogenization techniques. Indeed, let $\phi = u_h^{k,\xi}$ in (3.7) then it follows by (3.6), (3.1), (3.2) and Hölder's inequality that

$$\begin{aligned} \int_Y \left| \xi + Du_h^{k,\xi} \right|^p dx &\leq c \int_Y 1 + (a^k(x, hx, \xi + Du_h^{k,\xi}), \xi + Du_h^{k,\xi})\, dx \\ &= c \int_Y 1 + (a^k(x, hx, \xi + Du_h^{k,\xi}), \xi)\, dx \\ &\leq c \left(1 + \int_Y \left(1 + \left| \xi + Du_h^{k,\xi} \right| \right)^{p-1} dx \right) \\ &\leq c \left(1 + \left(\int_Y \left(\left| \xi + Du_h^{k,\xi} \right| \right)^p dx \right)^{\frac{1}{q}} \right). \end{aligned} \tag{3.12}$$

If $\left(\int_Y \left(\left| \xi + Du_h^{k,\xi} \right| \right)^p dx \right)^{1/q} \leq 1$ it is clear that the sequence of solutions $(u_h^{k,\xi})$ is bounded in $L^p(Y, R^n)$, so let us assume that $\left(\int_Y \left(\left| \xi + Du_h^{k,\xi} \right| \right)^p dx \right)^{1/q} \geq 1$, then (3.12) implies that

$$\int_Y \left| \xi + Du_h^{k,\xi} \right|^p dx \leq c$$

which means that $(u_h^{k,\xi})$ is bounded in . Since $\|D\cdot\|_{L^p(Y,R^n)}$ is an equivalent norm on $W_{per}^{1,p}(Y)$ it follows that there exists a constant $c > 0$ independent of h such that

$$\left\| u_h^{k,\xi} \right\|_{W_{per}^{1,p}(Y)} \leq c.$$

From the reflexivity of $W^{1,p}_{per}(Y)$ there exists a subsequence, still denoted by $(u_h^{k,\xi})$ such that

$$u_h^{k,\xi} \to u_*^{k,\xi} \text{ weakly in } W^{1,p}_{per}(Y).$$

Let us now define

$$\eta_h^{i,k,\xi} = a(x_i^k, hx, \xi + Du_h^{k,\xi}), \quad i \in I_k$$

By (3.2), (3.1), Hölder's inequality and (3.12) we have that $\eta_h^{i,k,\xi}$ is bounded in $L^q(\Omega_i^k, R^n)$. Indeed

$$\begin{aligned} \int_{\Omega_i^k} \left|\eta_h^{i,k,\xi}\right|^q dx &= \int_{\Omega_i^k} \left|a(x_i^k, hx, \xi + Du_h^{k,\xi})\right|^q dx \\ &\leq c\int_{\Omega_i^k} \left(1 + \left|\xi + Du_h^{k,\xi})\right|\right)^{q(p-1-\alpha)} \left|\xi + Du_h^{k,\xi})\right|^{q\alpha} dx \\ &\leq c\int_{\Omega_i^k} 1 + \left|\xi + Du_h^{k,\xi})\right|^p dx \leq c \end{aligned}$$

where c is a constant independent of h. This means that there exists a subsequence, still denoted by $(\eta_h^{i,k,\xi})$, and a $\eta_*^{i,k,\xi} \in L^q(\Omega_i^k, R^n)$ such that

$$\eta_h^{i,k,\xi} \to \eta_*^{i,k,\xi} \text{ weakly in } L^q(\Omega_i^k, R^n).$$

From our original problem (3.7) we have that

$$\begin{cases} \sum_{i\in I_k} \int_{\Omega_i^k} (a(x_i^k, hx, \xi + Du_h^{k,\xi}), D\phi)\, dx = 0 \text{ for every } \phi \in W^{1,p}_{per}(Y), \\ u_h^{k,\xi} \in W^{1,p}_{per}(Y). \end{cases}$$

In the limit we get

$$\sum_{i\in I_k} \int_{\Omega_i^k} (\eta_*^{i,k,\xi}, D\phi)\, dx = 0 \text{ for every } \phi \in W^{1,p}_{per}(Y).$$

Especially this means that

$$\int_{\Omega_i^k} (\eta_*^{i,k,\xi}, D\phi)\, dx = 0 \text{ for every } \phi \in C_0^\infty(\Omega_i^k), \quad i \in I_k.$$

If we now could show that

$$\eta_*^{i,k,\xi} = b(x_i^k, \xi + Du_*^{k,\xi}) \text{ for a.e } x \in \Omega_i^k, \tag{3.13}$$

then it follows by the uniqueness of the homogenized problem (3.10) that $u_*^{k,\xi} = u^{k,\xi}$. To this aim we define the function

$$w_h^{\tau,x_i^k}(x) = (\tau, x) + \frac{1}{h} v^{\tau,x_i^k}(hx),$$

where v^{τ,x_i^k} is defined as in (3.11). By periodicity we have that

$$\begin{aligned} w_h^{\tau,x_i^k} &\to (\tau,\cdot) \text{ weakly in } W^{1,p}(\Omega_i^k), \\ Dw_h^{\tau,x_i^k} &\to \tau \text{ weakly in } L^p(\Omega_i^k,R^n), \\ a(x_i^k,hx,Dw_h^{\tau,x_i^k}) &\to b(x_i^k,\tau) \text{ weakly in } L^q(\Omega_i,R^n). \end{aligned}$$

By the monotonicity of a_i we have for a fix τ that

$$\int_{\Omega_i}(a(x_i^k,hx,\xi+Du_h^{k,\xi})-a(x_i^k,hx,Dw_h^{\tau,x_i^k}),\xi+Du_h^{k,\xi}-Dw_h^{\tau,x_i^k})\phi\,dx\geq 0,$$

for every $\phi\in C_0^\infty(\Omega_i),\phi\geq 0$. By density we obtain that

$$(\eta_*^{i,k,\xi}(x)-b(x_i^k,\tau),\xi+Du_*^{k,\xi}(x)-\tau)\geq 0 \text{ for a.e. } x\in\Omega_i^k \text{ and for every } \tau\in R^n.$$

Since b^k is monotone and continuous, see Proposition 1, we have that b^k is maximal monotone and the crucial relation (3.13) follows. We have now proved step 1 up to a subsequence of $(u_h^{k,\xi})$. By the uniqueness of the solution of the homogenized equation (3.10) it follows that it is true for the whole sequence.

Step 2. Let us now prove that $u_h^\xi\to u^\xi$ weakly in $W_{per}^{1,p}(Y)$. Let $g\in\left(W_{per}^{1,p}(Y)\right)^*$, then

$$\begin{aligned} \lim_{h\to\infty}\left\langle g,u_h^\xi-u^\xi\right\rangle &= \lim_{k\to\infty}\lim_{h\to\infty}\left\langle g,u_h^\xi-u^\xi\right\rangle \\ &\leq \lim_{k\to\infty}\lim_{h\to\infty}\|g\|_{\left(W_{per}^{1,p}(Y)\right)^*}\left\|u_h^\xi-u_h^{k,\xi}\right\|_{W_{per}^{1,p}(Y)} \\ &\quad+\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g,u_h^{k,\xi}-u^{k,\xi}\right\rangle \\ &\quad+\lim_{k\to\infty}\|g\|_{\left(W_{per}^{1,p}(Y)\right)^*}\left\|u^{k,\xi}-u^\xi\right\|_{W_{per}^{1,p}(Y)}. \end{aligned}$$

It is enough to prove that all three terms on the right hand side are zero.

Term 1. Let us prove that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\|u_h^\xi-u_h^{k,\xi}\right\|_{W_{per}^{1,p}(Y)}=0. \tag{3.14}$$

By definition

$$\begin{aligned} \int_Y(a^k(x,hx,\xi+Du_h^{k,\xi}),D\phi)\,dx &= 0 \text{ for every } \phi\in W_{per}^{1,p}(Y), \\ \int_Y(a(x,hx,\xi+Du_h^{\xi}),D\phi)\,dx &= 0 \text{ for every } \phi\in W_{per}^{1,p}(Y). \end{aligned}$$

This implies that we for $\phi=u_h^{k,\xi}-u_h^\xi$ have

$$\begin{aligned} &\int_Z(a^k(x,hx,\xi+Du_h^{k,\xi})-a^k(x,hx,\xi+Du_h^\xi),Du_h^{k,\xi}-Du_h^\xi)\,dx \\ =\ &\int_Z(a(x,hx,\xi+Du_h^\xi)-a^k(x,hx,\xi+Du_h^\xi),Du_h^{k,\xi}-Du_h^\xi)\,dx. \end{aligned}$$

By using (3.3), and Hölder's reversed inequality on the left hand side and Hölder inequality (3.4) and the fact that (u_h^{ξ}) and $(u_h^{k,\xi})$ is bounded in $W_{per}^{1,p}(Y)$ on the right hand side we obtain that

$$\begin{aligned}
& c_2\left(\int_Y \left|Du_h^{k,\xi}-Du_h^{\xi}\right|^p dx\right)^{\frac{\beta}{p}} \\
& \times\left(\int_Y \left(1+\left|\xi+Du_h^{k,\xi}\right|+\left|\xi+Du_h^{\xi}\right|\right)^p dx\right)^{\frac{p}{p-\beta}} \\
\leq\; & c_2\int_Y \left(1+\left|\xi+Du_h^{k,\xi}\right|+\left|\xi+Du_h^{\xi}\right|\right)^{p-\beta}\left|Du_h^{k,\xi}-Du_h^{\xi}\right|^{\beta} dx \\
\leq\; & \left(\int_Y \left|(a(x,\frac{x}{\varepsilon_h},\xi+Du_h^{\xi})-a^k(x,\frac{x}{\varepsilon_h},\xi+Du_h^{\xi})\right|^q dx\right)^{\frac{1}{q}} \\
& \times\left(\int_Y \left|Du_h^{k,\xi}-Du_h^{\xi}\right|^p dx\right)^{\frac{1}{p}} \\
\leq\; & \widetilde{\omega}(\frac{1}{k})\left(\int_Y 1+\left|\xi+Du_h^{\xi})\right|^p dx\right)^{\frac{1}{q}}\left(\int_Y \left|Du_h^{k,\xi}-Du_h^{\xi}\right|^p dx\right)^{\frac{1}{p}} \\
\leq\; & \widetilde{\omega}(\frac{1}{k})\left(\int_Y \left|Du_h^{k,\xi}-Du_h^{\xi}\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\|D\cdot\|_{L^p(Y,R^n)}$ is an equivalent norm on $W_{per}^{1,p}(Y)$ this implies that

$$\left\|u_h^{k,\xi}-u_h^{\xi}\right\|_{W_{per}^{1,p}(Y)} \leq \widetilde{\omega}(\frac{1}{k}) \to 0 \tag{3.15}$$

as $k\to\infty$ uniformly in h. This means that we can change the order in the limit process in (3.14) and (3.14) follows by taking (3.15) into account.

Term 2. We observe that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g, u_h^{k,\xi}-u^{k,\xi}\right\rangle = 0,$$

as a direct consequence of Step 1.

Term 3. Let us prove that

$$\lim_{k\to\infty}\left\|u^{k,\xi}-u^{\xi}\right\|_{W_{per}^{1,p}(Y)} = 0. \tag{3.16}$$

By definition we have that

$$\begin{aligned}
\int_Y (b^k(x,\xi+Du^{k,\xi}),D\phi)\,dx &= 0 \text{ for every } \phi\in W_{per}^{1,p}(Y), \\
\int_Y (b(x,\xi+Du^{\xi}),D\phi)\,dx &= 0 \text{ for every } \phi\in W_{per}^{1,p}(Y).
\end{aligned}$$

Thus

$$\begin{aligned}
& \int_Y (b^k(x,\xi+Du^{k,\xi})-b^k(x,\xi+Du^{\xi}),D\phi)\,dx \\
=\; & \int_Y (b(x,\xi+Du^{\xi})-b^k(x,\xi+Du^{\xi}),D\phi)\,dx,
\end{aligned}$$

for every $\phi \in W^{1,p}_{per}(Y)$. Choose $\phi = u^{k,\xi} - u^{\xi}$ and take the strict monotonicity of b^k, see (3.22), into account on the left hand side and apply the Hölder inequality and (3.21) on the right hand side to obtain

$$\begin{aligned}
& c\left(\int_Y \left|Du^{k,\xi} - Du^{\xi}\right|^p dx\right)^{\frac{\beta}{p}} \times \\
& \left(\int_Y \left(1 + \left|\xi + Du^{k,\xi}\right| + \left|\xi + Du^{\xi}\right|\right)^p dx\right)^{\frac{p-\beta}{p}} \\
\leq\ & c\int_Y \left(1 + \left|\xi + Du^{k,\xi}\right| + \left|\xi + Du^{\xi}\right|\right)^{p-\beta} \left|Du^{k,\xi} - Du^{\xi}\right|^{\beta} dx \\
\leq\ & \left(\int_Y \left|b(x,\xi + Du^{\xi}) - b^k(x,\xi + Du^{\xi})\right|^q dx\right)^{\frac{1}{q}} \times \\
& \left(\int_Y \left|Du^{k,\xi} - Du^{\xi}\right|^p dx\right)^{\frac{1}{p}} \\
\leq\ & \widetilde{\omega}(\frac{1}{k})\left(\int_Y 1 + \left|\xi + Du^{\xi}\right|^p dx\right)^{\frac{1}{q}} \left(\int_Y \left|Du^{k,\xi} - Du^{\xi})\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

By using the fact that u^{ξ} and $u^{k,\xi}$ are bounded in $W^{1,p}_{per}(Y)$ it follows that

$$\left\|Du^{k,\xi} - Du^{\xi}\right\|_{L^p(Y,R^n)} \leq \widetilde{\omega}(\frac{1}{k}), \tag{3.17}$$

and the result follows by noting that $\|D\cdot\|_{L^p(Y,R^n)}$ is an equivalent norm on $W^{1,p}_{per}(Y)$.

Step 3. Next we prove that $a(x, hx, \xi + Du^{\xi}_h) \to b(x, \xi + Du^{\xi})$ weakly in $L^q(Y, R^n)$. In fact if $g \in (L^q(Y, R^n))^*$, then

$$\begin{aligned}
& \lim_{h\to\infty} \left\langle g, a(x,hx,\xi + Du^{\xi}_h) - b(x,\xi + Du^{\xi})\right\rangle \\
=\ & \lim_{k\to\infty}\lim_{h\to\infty} \left\langle g, a(x,hx,\xi + Du^{\xi}_h) - b(x,\xi + Du^{\xi})\right\rangle \\
\leq\ & \lim_{k\to\infty}\lim_{h\to\infty} \|g\| \left\|a(x,hx,\xi + Du^{\xi}_h) - a^k(x,hx,\xi + Du^{k,\xi}_h)\right\|_{L^q(Y,R^n)} \\
& + \lim_{k\to\infty}\lim_{h\to\infty} \left\langle g, a^k(x,hx,\xi + Du^{k,\xi}_h) - b^k(x,\xi + Du^{k,\xi}))\right\rangle \\
& + \lim_{k\to\infty} \|g\| \left\|b^k(x,\xi + Du^{k,\xi}) - b(x,\xi + Du^{\xi})\right\|_{L^q(Y,R^n)}.
\end{aligned}$$

It is sufficient to prove that all three terms on the right hand side are zero.

Term 1. Let us show that

$$\lim_{k\to\infty}\lim_{h\to\infty} \left\|a(x,hx,\xi + Du^{\xi}_h) - a^k(x,hx,\xi + Du^{k,\xi}_h)\right\|_{L^q(Y,R^n)} = 0. \tag{3.18}$$

By using elementary estimates we find that

$$\begin{aligned}
&\int_Y \left|a^k(x,hx,\xi+Du_h^{k,\xi})-a(x,hx,\xi+Du_h^{\xi})\right|^q\,dx\\
\leq\ & c\int_Y \left|a^k(x,hx,\xi+Du_h^{k,\xi})-a^k(x,hx,\xi+Du_h^{\xi})\right|^q\,dx\\
&+c\int_Y \left|a^k(x,hx,\xi+Du_h^{\xi})-a(x,hx,\xi+Du_h^{\xi})\right|^q\,dx.
\end{aligned}$$

Hence, by applying the continuity conditions (3.2) and Hölder inequality to the first term and (3.4) to the second term we obtain that

$$\begin{aligned}
&\int_Y \left|a^k(x,hx,\xi+Du_h^{k,\xi})-a(x,hx,\xi+Du_h^{\xi})\right|^q\,dx\\
\leq\ & c\left(\int_Y \left(1+\left|\xi+Du_h^{k,\xi}\right|+\left|\xi+Du_h^{\xi}\right|\right)^p\,dx\right)^{\frac{p-1-\alpha}{p-1}}\\
&\times\left(\int_Y \left|Du_h^{k,\xi}-Du_h^{\xi}\right|^p\,dx\right)^{\frac{\alpha}{p-1}}+\widetilde{\omega}(\frac{1}{k})\int_Y 1+\left|\xi+Du_h^{\xi}\right|^p\,dx.
\end{aligned}$$

By using the fact that $u_h^{k,\xi}$ and u_h^{ξ} are bounded in $W_{per}^{1,p}(Y)$ and (3.15) it follows that

$$\left\|a(x,hx,\xi+Du_h^{\xi})-a^k(x,hx,\xi+Du_h^{k,\xi})\right\|_{L^q(Y,R^n)}\leq\widetilde{\omega}(\frac{1}{k})\to 0. \tag{3.19}$$

as $k\to\infty$ uniformly in h. This implies that we may change the order in the limit process in (3.18) and we obtain (3.18) by taking (3.19) into account.

Term 2. We observe that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g,a^k(x,hx,\xi+Du_h^{k,\xi})-b^k(x,\xi+Du^{k,\xi})\right\rangle=0,$$

as a direct consequence of Step 1.

Term 3. Let us show that

$$\lim_{k\to\infty}\left\|b^k(x,\xi+Du^{k,\xi})-b(x,\xi+Du^{\xi})\right\|_{L^q(Y,R^n)}=0. \tag{3.20}$$

We have that

$$\begin{aligned}
&\int_Y \left|b^k(x,\xi+Du_*^{k,\xi})-b_1(x,\xi+Du^{\xi})\right|^q\,dx\\
\leq\ & c\int_Y \left|b^k(x,\xi+Du_*^{k,\xi})-b^k(x,\xi+Du^{\xi})\right|^q\,dx\\
&+c\int_Y \left|b^k(x,\xi+Du^{\xi})-b_1(x,\xi+Du^{\xi})\right|^q\,dx.
\end{aligned}$$

By applying the continuity condition (3.23) and Hölders's inequality to the first term and the continuity condition (3.21) to the second term we see that

$$
\begin{aligned}
&\int_Y \left| b^k(x,\xi + Du^{k,\xi}) - b(x,\xi + Du^{\xi}) \right|^q \, dx \\
\leq \;& c \left(\int_Y \left(1 + \left| \xi + Du^{k,\xi} \right| + \left| \xi + Du^{\xi} \right| \right)^p \, dx \right)^{\frac{p-1-\gamma}{p-1}} \\
&\times \left(\int_Y \left| Du^{k,\xi} - Du^{\xi} \right|^p \, dx \right)^{\frac{\gamma}{p-1}} + \widetilde{\omega}(\frac{1}{k}) \int_Y \left| Du^{\xi} \right|^p \, dx.
\end{aligned}
$$

By using the fact that $u^{k,\xi}$ and u^{ξ} are bounded in $W^{1,p}_{per}(Y)$ and (3.17) it follows that

$$
\left\| b^k(x,\xi + Du^{k,\xi}) - b(x,\xi + Du^{\xi}) \right\|_{L^q(Y,R^n)} \leq \widetilde{\omega}(\frac{1}{k}) \to 0
$$

and we are done.

□

We remark that we have only considered the case when a satisfies (3.4) over the whole Y the piecewise case follows by using the technique used in step 1.

Proposition 1. *Let b be the homogenized operator defined in Theorem 2. Then*

(i) $b(\cdot,\xi)$ satisfies the continuity condition

$$
|b(y_1,\xi) - b(y_2,\xi)|^q \leq \widetilde{\omega}(|y_1 - y_2|) \left(1 + |\xi|^p \right). \tag{3.21}
$$

(ii) $b(x,\cdot)$ is strictly monotone, more precisely

$$
(b_1(y,\xi_1) - b_1(y,\xi_2), \xi_1 - \xi_2) \geq c \left(1 + |\xi_1| + |\xi_2| \right)^{p-\beta} | \, \xi_1 - \xi_2 |^{\beta}, \tag{3.22}
$$

$\xi_1, \xi_2 \in R^n$.

(iii) $b(x,\cdot)$ is Lipschitz continuous, more precisely

$$
|b(x,\xi_1) - b(x,\xi_2)| \leq c \left(1 + |\xi_1| + |\xi_2| \right)^{p-1-\gamma} | \, \xi_1 - \xi_2 |^{\gamma}, \tag{3.23}
$$

for every $\xi_1, \xi_2 \in R^n$, where $\gamma = \alpha/(\beta - \alpha)$.

(iv)

$$
b(x,0) = 0 \text{ for } x \in Z. \tag{3.24}
$$

Proof. (i): By the definition of b and Jensen's inequality we have that

$$
\begin{aligned}
&|b(y_1,\tau) - b(y_2,\tau)|^q \\
= \;& \left| \int_Z a(y_1,z,\tau + Dv^{\tau,y_1}(z)) \; - a(y_2,z,\tau + Dv^{\tau,y_2}(z)) \, dz \right|^q \\
\leq \;& c \int_Z |a(y_1,z,\tau + Dv^{\tau,y_1}(z)) \; - a(y_2,z,\tau + Dv^{\tau,y_1}(z))|^q \, dz \\
&+ c \int_Z |a(y_2,z,\tau + Dv^{\tau,y_1}(z)) - a(y_2,z,\tau + Dv^{\tau,y_2}(z))|^q \, dz.
\end{aligned}
$$

By applying (3.4) to the first term and (3.2) in combination with Hölder's inequality to the second term we obtain that

$$
\begin{aligned}
& |b(y_1,\tau)-b(y_2,\tau)|^q \le \widetilde{\omega}(|y_1-y_2|)\int_Z 1+|\tau+Dv^{\tau,y_1}|^p\ dz \\
+ \quad & c\left(\int_Z (1+|\tau+Dv^{\tau,y_1}|+|\tau+Dv^{\tau,y_2}|)^p\ dz\right)^{\frac{p-1-\alpha}{p-1}} \\
& \times\left(\int_Z |Dv^{\tau,y_1}-Dv^{\tau,y_2}|^p\ dz\right)^{\frac{\alpha}{p-1}}
\end{aligned}
\tag{3.25}
$$

Let us now study the two terms in (3.25) separately. The first term: (3.6), (3.8) and (3.5) yields

$$
\begin{aligned}
\int_Z |\tau+Dv^{\tau,y_1}|^p\ dz &\le c\int_Z 1+(a(y,z,\tau+Dv^{\tau,y_1}),\tau+Dv^{\tau,y_1})\,dz \\
&= c\int_Z 1+(a(y,z,\tau+Dv^{\tau,y_1}),\tau)\,dz \\
&\le c\int_Z 1+c\left(1+|\tau+Dv^{\tau,y_1}|^{p-1}\right)|\tau|\ dz.
\end{aligned}
$$

By using the Young inequality we obtain that

$$
\int_Z |\tau+Dv^{\tau,y_1}|^p\ dz \le c\,(1+|\tau|^p)\,. \tag{3.26}
$$

Let us now study the second term in (3.25): By definition we have that

$$
\begin{aligned}
\int_Z (a(y_1,z,\tau+Dv^{\tau,y_1}),D\phi)\,dz &= 0 \text{ for every } \phi\in W^{1,p}_{per}(Z), \\
\int_Z (a(y_2,z,\tau+Dv^{\tau,y_2}),D\phi)\,dz &= 0 \text{ for every } \phi\in W^{1,p}_{per}(Z).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& \int_Z (a(y_1,z,\tau+Dv^{\tau,y_1})-a(y_1,z,\tau+Dv^{\tau,y_2}),D\phi)\,dz \\
= \quad & \int_Z (a(y_2,z,\tau+Dv^{\tau,y_2})-a(y_1,z,\tau+Dv^{\tau,y_2}),D\phi)\,dz,
\end{aligned}
$$

for every $\phi\in W^{1,p}_{per}(Z)$. In particular, for $\phi=v^{\tau,y_1}-v^{\tau,y_2}$, we have that

$$
\begin{aligned}
& \int_Z (a(y_1,z,\tau+Dv^{\tau,y_1})-a(y_1,z,\tau+Dv^{\tau,y_2}),Dv^{\tau,y_1}-Dv^{\tau,y_2})\,dz \\
= \quad & \int_Z (a(y_2,z,\tau+Dv^{\tau,y_2})-a(y_1,z,\tau+Dv^{\tau,y_2}),Dv^{\tau,y_1}-Dv^{\tau,y_2})\,dz.
\end{aligned}
$$

By applying the reversed Hölder inequality and (3.3) on the left hand side and Schwarz's and Hölder's inequalities on the right hand side it follows that

$$
\begin{aligned}
& c\left(\int_Z |Dv^{\tau,y_1} - Dv^{\tau,y_2}|^p \, dz\right)^{\frac{\beta}{p}} \times \\
& \left(\int_Z (1+|\tau+Dv^{\tau,y_1}|+|\tau+Dv^{\tau,y_2}|)^p \, dz\right)^{\frac{p-\beta}{p}} \\
\leq \; & c\int_Z (1+|\tau+Dv^{\tau,y_1}|+|\tau+Dv^{\tau,y_2}|)^{p-\beta} |Dv^{\tau,y_1} - Dv^{\tau,y_2}|^\beta \, dz \\
\leq \; & \left(\int_Z |a(y_2,z,\tau+Dv^{\tau,y_2}) - a(y_1,z,\tau+Dv^{\tau,y_2})|^q \, dz\right)^{\frac{1}{q}} \\
& \times \left(\int_Z |Dv^{\tau,y_1} - Dv^{\tau,y_2}|^p \, dz\right)^{\frac{1}{p}},
\end{aligned}
$$

which means that

$$
\begin{aligned}
& \left(\int_Z |Dv^{\tau,y_1} - Dv^{\tau,y_2}|^p \, dz\right)^{\frac{\alpha}{p-1}} \\
\leq \; & c\left(\int_Z (1+|\tau+Dv^{\tau,y_1}|+|\tau+Dv^{\tau,y_2}|)^p \, dz\right)^{\frac{\alpha(\beta-p)}{(\beta-1)(p-1)}} \\
& \left(\int_Z |a(y_2,z,\tau+Dv^{\tau,y_2}) - a(y_1,z,\tau+Dv^{\tau,y_2})|^q \, dz\right)^{\frac{\alpha}{\beta-1}} \\
\leq \; & c\,(1+|\tau|^p)^{\frac{\alpha(\beta-p)}{(\beta-1)(p-1)}} \,\widetilde{\omega}(|y_1-y_2|)\,(1+|\tau|^p)^{\frac{\alpha}{\beta-1}} \\
\leq \; & \widetilde{\omega}(|y_1-y_2|)\,(1+|\tau|^p)^{\frac{\alpha}{p-1}} \,. \qquad (3.27)
\end{aligned}
$$

The result follows by taking (3.25), (3.26) and (3.27) into account.

(ii), (iii) and (iv): The proofs follows by similar arguments as in e.g. [5]. □

Remark 1. *By similar arguments it follows that (ii), (iii), and (iv) hold up to boundaries, for the homogenized operator b^k in Step 1.*

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