

Dynamic Inventory and Pricing Policy with Ordering and Price Adjustment Cost

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Abstract—In this paper, we study a dynamic inventory control and pricing optimization problem in a periodic review inventory system with fixed ordering cost and price adjustment cost. At the same time, the ordering quantity is limited. We show that the optimal inventory control is partially characterized by an (s, s', p) policy in four regions, and the optimal pricing policy is dependent on the inventory level after the replenishment decision.

Keywords—optimization; ordering cost; price adjustment cost.

I. INTRODUCTION

In this article, we consider a dynamic inventory control and pricing optimization problem in a periodic review inventory system with fixed ordering cost and price adjustment cost. At the same time, the ordering quantity is limited.

This problem is related to the optimal control of a single product system with finite capacity and setup cost. Several studies have been conducted on this problem. For instance, Chen and Lambrecht (1996) point out the generally known result is that the optimal policy can only be partially characterized in the form of X - Y bands. When the inventory level is below the first band X , then produce/order the capacity, and when the inventory level is over the second band Y , produce/order nothing. If the inventory level is between the two bands, the ordering policy is complicated and depends on the instance. In Gallego and Scheller-Wolf (2000), the structure of the policy between the bands is further refined using two numbers s and s' in four possible regions. However, none of them have studied the pricing problem in the inventory control problem. Only Chao, Yang and Xu (2012) have studied a dynamic inventory and pricing optimization problem in a periodic review inventory system with setup cost and finite ordering capacity in each period. They show that the optimal inventory control is characterized by an (s, s', p) policy in four regions of the starting inventory level. But in their paper, the selling price can be adjusted without any cost.

In the reality, there must be some price adjustment cost when the price is changed. In the economics literature, there are two major types of price adjustment costs identified: managerial costs and physical costs. Rotemberg (1982), Levy et al. (1997), Slade and G.R.E.Q.A.M (1998), Aguirregabiria (1999), Bergen et al. (2003), Zbaracki et al. (2004) have shown both the two styles of costs are

significant in retailing and other industries. According to these empirical studies, Chen, Zhou and Chen (2011) consider a periodic-review inventory model with price adjustment costs that consist of both fixed and variable components. They develop the general model and characterize the optimal policies for two special scenarios, a model with inventory carryover and no fixed price-change costs and a model with fixed price-change costs and no inventory carryover. Although there is price adjustment cost, they do not consider the finite ordering capacity.

In this paper, we consider a random additive demand model and investigate the structure of the optimal inventory control and pricing policy in each period. We show that the optimal inventory policy is partially characterized by an (s, s', p) policy on four regions, in two of these regions the optimal policy is completely specified while in the other two, it is partially specified. More specifically, the optimal ordering quantity in the first region is the full capacity, while in the last region it is optimal to order nothing; in the two middle regions, the optimal decision is either to order to the maximum capacity, to order to at least a pre-specified level s' , or to order nothing. The optimal pricing policy $p(y)$ in each period is dependent on the inventory level after the replenishment decision, y , which is in general not a monotone function. The key concept utilized is strong CK -concavity, which is an extension of K -concavity, and was first introduced by Gallego and Scheller-Wolf (2000). The result is similar with the one in Chao, Yang and Xu (2012).

The rest of this paper is organized as follows. In the next section we present the model. The main result and the proofs are provided in Section 3, and concluding remarks are in Section 4.

II. MODEL

Consider a periodic review inventory system with price-dependent random demand in each period. There are N periods, with the first period being 1, and last period N . The demand in period n , $D_n(p_n)$, depends on the selling price in period n , p_n . Here we consider the additive demand (Petruzzi and Dada (1999), Chen and Simchi-Levi (2004)). That is, the demand in period n is

$$D_n(p_n) = d_n(p_n) + \varepsilon_n,$$

where the selling price in period n is p_n , $n=1, \dots, N$. ε_n is a random variable with mean zero and $d_n(p_n)$ is the average demand which is a decreasing function of p_n . The selling price p_n for period n is a decision variable, taken from interval $[\underline{p}, \bar{p}]$. Thus, as the selling price increases from \underline{p} to \bar{p} , the average demand decreases from $d(\underline{p})$ to $d(\bar{p})$. Let $\bar{d} = d(\underline{p})$ and $\underline{d} = d(\bar{p})$. This shows that determining the selling price p_n is equivalent to setting the average demand d_n , and we can simply optimize over the average demand d_n between \underline{d} and \bar{d} . Let $p_n = p_n(d_n)$ be the inverse function of $d_n(p_n)$, which is also decreasing function. Suppose $p_n(d_n)$ is linear on d_n . Then the expected revenue when setting the average demand to d_n is

$$r_n(d_n) = d_n \cdot p_n(d_n),$$

which is a concave function. The ordering leadtime is zero, that is, an order placed at the beginning of a period is received at the end of the period after demand is realized.

We assume that the government gives the product a guide price p_0 . The actual selling price p_n should be no larger than p_0 , which indicates $p_n \leq \bar{p} \leq p_0$. The cost of a price adjustment from the actual selling price to guide price in period n is denoted by $U_n(p_0 - p_n)$. It is commonly assumed that the variable cost is increases with the size of the price change, because the decision and internal communication costs are higher for larger price changes. Several forms of $U_n(\cdot)$ have been used in the economics literature, including piece wise linear functions and quadratic functions. Here we assume that $U_n(\cdot)$ is linear. Due to the linearity of decreasing function $p_n(d_n)$, it is easy to show $U_n(p_0 - p_n)$ is linear and increases on d_n . For convenience, let

$$U_n(p_0 - p_n(d_n)) = U_n(d_n).$$

The sequence of events during a period is as follows: 1) inventory level is reviewed and replenishment order is placed, 2) replenishment order arrives, 3) a selling price is set, 4) random demand is realized, and 5) all costs are computed.

We assume the fixed ordering cost is K , and the variable unit ordering cost is c in period n . There is a finite ordering capacity C for each period. That is, the ordering quantity for each period cannot exceed C . Unsatisfied demand in a period is fully backlogged. Let x_n be the inventory level at the beginning of period n before placing an order, which can be positive or negative because we consider backlog model, and let y_n be the inventory level after placing the order. Due to ordering capacity, we must have

$$x_n \leq y_n \leq x_n + C$$

A cost $h(x)$ is incurred at the end of period n if the inventory level after demand realization is x , which represents inventory holding cost if $x \geq 0$ and shortage cost if $x < 0$.

For convenience we let

$$G(y) = E[h(y - \varepsilon_n)].$$

Then, the period- n expected holding and shortage cost, given that the inventory level after replenishment is y_n and the expected demand for period n is d_n , is $G(y - d_n)$.

Denote by $v_n(x)$ the profit-to-go function at the beginning of time period n with inventory level x . Let $v_{N+1}(x) = 0$ for all x . Hence, for each $n=1, \dots, N$, we have

$$\begin{aligned} v_n(x) &= cx + \max_{x \leq y \leq x+C} \max_{\underline{d} \leq d \leq \bar{d}} \{ -K\mathbb{I}[y > x] + r_n(d) - cy \\ &\quad - U_n(d) - G(y - d) + \alpha E[v_{n+1}(y - d - \varepsilon_n)] \} \\ &= cx + \max_{x \leq y \leq x+C} \{ -K\mathbb{I}[y > x] - cy \\ &\quad + \max_{\underline{d} \leq d \leq \bar{d}} \{ r_n(d) - U_n(d) + W_n(y - d) \} \}, \end{aligned}$$

where

$$W_n(y) = -G(y) + \alpha E[v_{n+1}(y - \varepsilon_n)]$$

$\mathbb{I}[A]$ is the indicate function, taking value 1 if statement A is true and zero otherwise. α is the one-period discount factor, $\alpha \in [0, 1]$.

The objective is to characterize the optimal ordering and pricing strategy that maximizes the total expected discounted profit over the planning horizon.

III. ANALYSIS AND RESULTS

Before analysis, it is necessary for us to introduce an important definition.

Definition: Given non-negative constants C and K , we call the function G strong CK -concave if for all y , $0 \leq a < \infty$, $0 < b < \infty$, $\forall z \in [0, C]$,

$$-K + G(y + z) \leq G(y) + \frac{z}{b} \{ G(y - a) - G(y - a - b) \}$$

According to Gallego and Scheller-Wolf (2000), we can obtain some propositions about strong CK -concave:

Proposition: 1. If G is strong CK -concave, it is also strong DL -concave for any $0 \leq D \leq C$ and $L \geq K$.

2. If G is concave, it is strong CK -concave for any nonnegative C and K .

3. If G_1 is strong CK -concave, and G_2 is strong CL -concave, then for $\alpha, \beta \geq 0$, $\alpha G_1 + \beta G_2$ is strong $C(\alpha K + \beta L)$ -concave.

4. If G is strong CK -concave and X is a random variable such that $E[|G(y - X)|] < \infty$, then $E[G(y - X)]$ is strong CK -concave.

Given non-negative C and K and strong CK -concave functions $g_n(x, d_n(x))$ for any $n = 1, \dots, N$, let:

s_n be the point at which the function $g_n(x, d_n(x))$ reaches its maximum at period n , $n = 1, \dots, N$;

s_n be the minimum point at which no order is required by the optimal policy at period n , $n = 1, \dots, N$;

s'_n be the maximum point at which it is necessary to order some units for the optimal policy at period n and $s'_n \leq S_n$, $n = 1, \dots, N$

Clearly $-\infty \leq s_n \leq s'_n \leq S_n$.

Lemma 1. 1. $g_n(x, d_n(x))$ is non-decreasing on $(-\infty, s'_n]$ and strictly increasing on $(-\infty, s_n)$.

2. $g_n(x, d_n(x)) \geq \max_{x \leq y \leq x+C} g_n(y, d_n(y)) - K \quad \forall x > s'_n$

3. If $s'_n - C \leq s_n$,

$$v_n(x) = g_n(x + C, d_n(x + C)) + c_n x - K, \quad x < s'_n - C$$

$$v_n(x) = \max_{s'_n \leq y \leq x+C} g_n(y, d_n(y)) + c_n x - K, \quad s'_n - C \leq x < s_n$$

$$v_n(x) = \max\{g_n(x, d_n(x)) + c_n x, \max_{s'_n \leq y \leq x+C} g_n(y, d_n(y)) + c_n x - K\}, \quad s_n \leq x \leq s'_n$$

$$v_n(x) = g_n(x, d_n(x)) + c_n x, \quad x > s'_n$$

If $s'_n - C > s_n$,

$$v_n(x) = g_n(x + C, d_n(x + C)) + c_n x - K, \quad x < s_n$$

$$v_n(x) = \max\{g_n(x, d_n(x)) + c_n x, g_n(x + C, d_n(x + C)) + c_n x - K\}, \quad s_n \leq x \leq s'_n - C$$

$$v_n(x) = \max\{g_n(x, d_n(x)) + c_n x,$$

$$\max_{s'_n \leq y \leq x+C} g_n(y, d_n(y)) + c_n x - K\}, \quad s'_n - C \leq x \leq s'_n$$

$$v_n(x) = g_n(x, d_n(x)) + c_n x, \quad x > s'_n$$

The following result follows from the concavity of r and linearity of U , which can be proved using supermodularity.

Lemma 2. Suppose that $g_n(y, d)$ is jointly continuous in (y, d) . Then, there exists a $d_n(y)$ which maximizes $g_n(y, d)$ for any given y , such that $y - d_n(y)$ is a non-decreasing function of y .

According to Lemma 2 and Chao, Yang and Xu (2012), we can prove Lemma 3, which is essential in proving the main result of this paper.

Lemma 3. If $W_n(y)$ is strongly CK -concave, then so is

$$g(y) = \max_{d \leq d \leq d} \{r_n(d) - U_n(d) + W_n(y - d)\}.$$

The following theorem characterizes the structure of the optimal inventory and pricing policy for each period.

Theorem 1. Suppose x is the starting inventory level at the beginning of period n . The optimal inventory and pricing strategy is characterized by two numbers s_n and s'_n and by the optimal pricing $p_n^*(y)$, such that $s_n \leq s'_n$ and $p_n^*(y)$ depends on the post-order inventory position y . If $s'_n - C \leq s_n$, then the optimal ordering policy is

i) order capacity C if $x < s'_n - C$;

ii) order at least up to s'_n if $s'_n - C \leq x < s_n$;

iii) either order nothing or order at least up to s'_n if $s_n \leq x \leq s'_n$; and

iv) order nothing if $x > s'_n$.

And if $s'_n - C > s_n$, then the optimal ordering policy is

i') order capacity C if $x < s_n$;

ii') either order nothing or order C if $s_n \leq x \leq s'_n - C$;

iii') either order nothing or order at least up to s'_n if $s'_n - C \leq x \leq s'_n$; and

iv') order nothing if $x > s'_n$.

Proof of Theorem 1. We prove by induction that $V_n(\cdot)$ is strongly CK -concave. This is clearly true for $n=N+1$ as $V_{N+1}(x) = 0$. Suppose it has been established for $n+1$ and we proceed to prove n .

It follows from the property of strong CK -concavity that $\alpha E[V_{n+1}(y - \varepsilon_n)]$ is strongly $C(\alpha K)$ -concave, and by the concavity of $-G_n(y)$ we obtain that $W_n(y)$ is strongly $C(\alpha K)$ -concave, hence it is also strongly CK -concave. By Lemma 3 we obtain that

$$H_n(y) = -cy + \max_{d \leq d \leq \bar{d}} \{r_n(d) - U_n(d) + W_n(y - d)\}.$$

is strongly CK -concave. Define s_n , s'_n and S_n by

$$s_n = \inf\{x \mid -K + \sup_{x \leq y \leq x+C} H_n(y) \leq H_n(x)\},$$

$$s'_n = \max\{x \mid -K + \sup_{x \leq y \leq x+C} H_n(y) \geq H_n(x)\},$$

$$S_n = \inf\{y \in \mathfrak{R} \mid H(y) = \sup_{y \in \mathfrak{R}} H_n(y)\}.$$

Then, clearly $s_n \leq s'_n$, and it can be shown similar to the argument in Gallego and Scheller-Wolf (2000) that, the optimal inventory strategy is determined by s_n and s'_n as described in the statement of Theorem 1 and that, by the strong CK -concavity of $H_n(y)$, $v_n(x)$ given by

$$v_n(x) = cx + \max_{x \leq y \leq x+C} \{-K\mathbb{I}[y > x] + H_n(y)\}$$

preserves strong CK -concavity.

The optimal pricing decision is determined by the maximizer in Lemma 2. Let

$$d_n^*(y) = \arg \max_{d \leq d \leq \bar{d}} \{r_n(d) - U_n(d) + W_n(y - d)\},$$

resulting in a state-dependent optimal average selling quantity. Recall that $p_n = p_n(d_n)$ is the inverse function of $d_n = d_n(p_n)$. The optimal pricing decision, when the replenished inventory level is y , is

$$p_n^*(y) = p_n(d_n^*(y)).$$

This concludes the proof of Theorem 1.

IV. CONCLUSION

In this paper, we consider a dynamic inventory control and pricing optimization problem in a periodic review inventory system with fixed ordering cost and price adjustment cost. At the same time, the ordering quantity is

limited. Here we assume that the guide price is larger than the actual selling price. And both the price function and price adjustment cost function are linear. In the future research, we will study the model with more complicate price adjustment cost.

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