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Soliton Asymptotics of Rear Part of Non-Localized Solutions of the Kadomtsev-Petviashvili Equation

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Abstract

We construct non-localized, real global solutions of the Kadomtsev-Petviashvili-I equation which vanish for $x \to -\infty$ and study their large time asymptotic behavior. We prove that such solutions eject (for $t \to \infty$) a train of curved asymptotic solitons which move behind the basic wave packet.

1 Introduction

The study of long-time asymptotic behavior of solutions of nonlinear evolution equations has attracted growing attention in last years and a number of papers have been devoted to this problem (see review paper [8] and references therein). Interest in this problem was especially stimulated by the discovery of the inverse scattering transform method [11, 19, 1, 10]. In particular, one remarkable result obtained by this method was the proof that any localized (i.e. rapidly decreasing as $|x| \to \infty$) solution $u(x,t)$ of the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0$$

splits into a finite number of solitons when time tends to infinity [17] (see also [18, 16]). Other nonlinear evolution equations with one space variable integrable by the inverse scattering transform also exhibit similar splitting. This phenomenon is an argument in favor of the physical interpretation of solitons as stable "long-living" particles.

The simplest non-localized solution is of step-like form, i.e. the solution with following asymptotic behavior:

$$u(x,t) = \begin{cases} 0, & x \to +\infty \\ -c^2, & x \to -\infty. \end{cases}$$

It was proved in [12] that the step-like solution of the KdV equation splits into $\left[\frac{N+1}{2}\right]$ soliton-like objects in the neighbourhood

$$G_N^+(t) = \{x \in \mathbb{R} \mid x > 4c^2 t - N \ln t\}$$

of solution front ($N \in \mathbb{Z}^+$ and $[\,\cdot\,]$ denotes integer part). The form of these objects is similar to ordinary soliton but their velocities depend on t. In contrast with ordinary solitons they are not exact solutions of the KdV equation, however they satisfy it with increasing accuracy when $t \to \infty$. For this reason such objects are called "asymptotic solitons". The number of these asymptotic solitons increases to infinity when $t \to \infty$ if the observation domain in the neighbourhood of the the solution front is extended correspondently.

The same phenomenon of generation of asymptotic solitons trains on the solution front takes place (under certain conditions) for non-localized solutions of more general form as well as for other nonlinear evolution equations integrable by the inverse scattering transform [13]. Roughly speaking this phenomenon can be considered as a manifestation of the fact that any non-localized initial data consists of an infinite number of solitons which are gradually ejected at the front (first is the most rapid of them). An important condition for the ejection is the existence of wide "living area" for solitons where they can propagate without collisions. In the case where the non-localized initial data vanish for $x \to \infty$ this area is a positive beam (at, ∞) for some suitable $a > 0$.

Another type of non-localized solutions of the KdV equation which vanish as $x \to -\infty$ was considered in [7]. It has been proved that under some conditions trains of asymptotic solitons are formed on a tail of the solution as $t \to \infty$. These solitons move to the right following behind the basic wave packet. Physically it can be treated as an ejection of the slower solitons from a non-localized initial perturbation (initial data $u(x,0)$). This phenomenon takes place under certain conditions on the spectrum of the Schrödinger operator whose potential is the initial data $u(x,0)$. This condition is the existence of a gap between the continuous spectrum of multiplicity one, which is provided by a non-trivial asymptotic behavior of $u(x,0)$ as $x \to \infty$, and the continuous spectrum of multiplicity two (positive half-axis).

In [5, 2, 6, 15] similar problems were studied for the Kadomtsev-Petviashvili equations (KP-I and KP-II):

$$\frac{\partial}{\partial x}\Big(u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\Big) + \frac{3}{4}\alpha^2 u_{yy} = 0, \tag{1.1}$$

where $\alpha = i$ for KP-I and $\alpha = 1$ for KP-II. These are equations with two spatial variables x, y integrable by inverse scattering transform. See also [3, 4] for the Johnson equation and for the modified Kadomtsev-Petviashvili-I equation (mKP-I).

Following the Zakharov-Shabat scheme of the dressing method we look for solutions in the form:

$$u(x,y,t) = 2\frac{d}{dx}K^{\pm}(x,x,y,t), \tag{1.2}$$

where the function $K^{\pm}(z,x,y,t)$ is a solution of the Marchenko integral equation:

$$K^{\pm}(z,x,y,t) + F(z,x,y,t) \pm \int_x^{\pm\infty} K^{\pm}(s,x,y,t)F(z,s,y,t)\,ds = 0 \tag{1.3}$$

viewed as an equation in $z > x$ ($z < x$) with parameters x, y, t. The kernel $F(z,x,y,t)$ in (1.3) satisfies the system of linear differential equations:

$$\begin{cases} F_t + F_{xxx} + F_{zzz} = 0, \\ \alpha F_y + F_{xx} - F_{zz} = 0. \end{cases} \tag{1.4}$$

Choosing solutions of this system in an appropriate way and solving the integral equation (1.3) (with the sign "+" or "−") one can construct solutions of the KP equations which vanish as $x \to +\infty$ or $x \to -\infty$. For the KP-I equation global real solutions vanishing as $x \to +\infty$ were constructed in [2] (see also [6]). It was proved that in the neighbourhoods of their fronts:

$$x > C(Y)t - \frac{1}{a(Y)}\log t^N,$$

these solutions are asymptotically represented as follows:

$$u(x,y,t) = \sum_{n=1}^{N} 2a^2(Y)\left(\cosh\Big[a(Y)(x - C(Y)t + \frac{1}{a(Y)}\log t^{n-1/2}) - \log\varphi_n(Y)\Big]\right)^{-1} + \mathrm{O}(t^{-1/2+\varepsilon}) \tag{1.5}$$

where N is any natural number, $Y = y/t$, $a(Y)$, $C(Y)$ and $\varphi_n(Y)$ are some positive functions. This means that the trains of curved solitons are formed in the neighbourhood of the solution front because the ridges of these solitons at time t are located along curves $x = C(y/t)t - a^{-1}(y/t)\log t^{n-1/2} + \log\varphi_n(y/t)$ ($n = 1, 2, \dots$). The phenomenon of formation of curved solitons trains nearly solution fronts is observed also for the KP-II equation, however in this case even solitons have a singularity [5].

This paper is devoted to the study of non-localized solutions of the KP-I equation which vanish as $x \to -\infty$. We construct a global real solution of this type and prove that a back part of the solution splits into curved asymptotic solitons. The dressing method of Zakharov-Shabat described above is not suitable for the investigation of long-time asymptotics of the tail of solutions. Therefore we use another method based on an integral equation in the plane of spectral parameters. This approach is also suitable for the study of long-time asymptotics of non-localized solutions in the neighbourhood of the solution front, which was first shown in [14] for the KdV equation.

The paper is organized as follows. In Section 2 we prove the existence of a global real solution of the KP-I equation, vanishing as $x \to -\infty$. In Section 3 we reduce the problem to a degenerated integral equation and obtain a determinant formula for the solution. In Section 4 we study the asymptotic behavior of the determinant formula as $t \to \infty$ and prove an ejection of curved solitons which move behind the basic wave packet.

2 Construction of global solutions vanishing as $x \to -\infty$

Let Ω be a domain in the complex plane $\mathbb{C}$ ($k = p + iq$) with smooth boundary $\Gamma = \partial\Omega$ located in the right half plane at positive distance from the imaginary axis. Let us define the function $E(p,q) := E(p,q,x,y,t)$ on Ω as follows:

$$E(p,q) = e^{p(x - f(p,q,Y)t)}, \tag{2.1}$$

with

$$f(p,q,Y) = p^2 - 3q^2 - 2qY, \tag{2.2}$$

where $x, y, t \in \mathbb{R}$, and $Y = y/t$ are parameters. Let us now consider the integral equation (with respect to $\psi(p,q) := \psi(p,q,x,y,t)$):

$$\psi(p,q) + \int_\Omega \frac{1}{\lambda + \bar{k}}\, E(p,q)E(\mu,\nu)\psi(\mu,\nu)g(\mu,\nu)\, d\mu\, d\nu = E(p,q), \tag{2.3}$$

where $\lambda = \mu + i\nu \in \Omega$, $k = p + iq \in \Omega$, $\bar{k} = p - iq$ and $g(\mu,\nu)$ is a smooth positive function on $\overline{\Omega}$. Furthermore, if Ω is unbounded, $g(\mu,\nu)$ satisfies

$$\int_\Omega e^{c\mu(\mu^2+\nu^2)} g(\mu,\nu)\, d\mu\, d\nu < \infty \tag{2.4}$$

for any $c > 0$.

Lemma 2.1. *There exists a unique solution $\psi(p,q) = \psi(p,q,x,y,t)$ of* (2.3) *which is C^∞ with respect to $x, y, t \in \mathbb{R}$.*

Proof. $L^2_g(\Omega)$ denotes the Hilbert space of complex valued functions on Ω with norm

$$||\varphi|| := \left\{ \int_\Omega |\varphi(\mu,\nu)|^2 g(\mu,\nu) d\mu\, d\nu \right\}^{\frac{1}{2}}.$$

Let $\mathbf{A}$ be an operator on $L^2_g(\Omega)$, depending on parameters $x, y, t \in \mathbb{R}$, as follows:

$$[\mathbf{A}\varphi](p,q) = \int_\Omega \frac{E(p,q)E(\mu,\nu)}{\lambda + \bar{k}} \varphi(\mu,\nu) g(\mu,\nu)\, d\mu\, d\nu. \tag{2.5}$$

According to (2.1), (2.2) and (2.4), $\mathbf{A}$ is Hilbert-Schmidt for any values of the parameters x, y, t. For $\varphi \in L^2_g(\Omega)$ we can write

$$(\mathbf{A}\varphi, \varphi) = \int_\Omega E(p,q) \int_\Omega \frac{E(\mu,\nu)}{\lambda + \bar{k}} \varphi(\mu,\nu) g(\mu,\nu)\, d\mu\, d\nu\, \bar{\varphi}(p,q) g(p,q)\, dp\, dq,$$

where $(\cdot\,,\,\cdot)$ denotes the inner product in $L^2_g(\Omega)$ and $\lambda = \mu + i\nu$, $\bar{k} = p - iq$. Using the equality

$$\frac{1}{\lambda + \bar{k}} = \int_{-\infty}^0 e^{(\lambda + \bar{k})s}\, ds,$$

which is true because $\operatorname{Re}\lambda > 0$, $\operatorname{Re} k > 0$, we obtain

$$(\mathbf{A}\varphi,\varphi) = \int_{-\infty}^{0} ds \left| \int_{\Omega} E(\mu,\nu)\varphi(\mu,\nu)e^{(\mu+i\nu)s} g(\mu,\nu)\, d\mu\, d\nu \right|^2 \geq 0. \tag{2.6}$$

It follows from (2.6) that the operator $\mathbf{A}$ is positive. Therefore, the homogeneous equation $\varphi+\mathbf{A}\varphi = 0$ has only the trivial solution $\varphi \equiv 0$. Since $\mathbf{A}$ is Hilbert-Schmidt, hence compact in $L^2_g(\Omega)$, the inhomogeneous equation

$$\psi + \mathbf{A}\psi = \zeta \tag{2.7}$$

has a unique solution in $L^2_g(\Omega)$ for any $\zeta \in L^2_g(\Omega)$. Let us note now that the integral equation (2.3) has the same form as (2.7) with $\zeta = E(p,q) = E(p,q,x,y,t)$ belonging to the space $L^2_g(\Omega)$ due to (2.1). Therefore, (2.3) has a unique solution $\psi(p,q)$, depending on the parameters x, y, t. The first derivatives $D\psi$ of this solution with respect to x, y, t also satisfy (2.7) with right-hand side

$$\zeta = DE(p,q) - \int_{\Omega} D\, \frac{E(p,q)E(\mu,\nu)}{\lambda+\bar{k}} \psi(\mu,\nu)g(\mu,\nu)\, d\mu\, d\nu$$

belonging to the space $L^2_g(\Omega)$. This proves their existence. Existence of high order derivatives is proved by induction. ■

Corollary 2.1. *The inverse operator* $(\mathbf{I}+\mathbf{A})^{-1}$ *exists and its norm in* $\mathcal{L}(L^2_g(\Omega))$ *is uniformly bounded with respect to* $x, y, t \in \mathbb{R}$*:*

$$||(\mathbf{I}+\mathbf{A})^{-1}|| \leq 1. \tag{2.8}$$

Let us now define the function

$$u(x,y,t) = -2\frac{\partial}{\partial x} \int_{\Omega} E(p,q,x,y,t)\psi(p,q,x,y,t)g(p,q)\, dp\, dq, \tag{2.9}$$

where $\psi(p,q,x,y,t)$ is the solution of the integral equation (2.3), and $E(p,q,x,y,t)$ is determined by (2.1) and (2.2).

Lemma 2.2. *The function* (2.9) *is a solution of the KP-I equation* (1.1). *It is a real solution, vanishing as* $x \to -\infty$.

Proof. Let us multiply (2.3) by $E(p,q,z,y,t)e^{iq(z-x)}g(p,q)$ and integrate with respect to $p, q \in \Omega$. Then, using

$$\frac{1}{\lambda+\bar{k}} = \int_{-\infty}^{x} e^{(\lambda+\bar{k})(s-x)}\, ds,$$

together with (2.1), (2.2) we obtain

$$K(z,x,y,t) + \int_{-\infty}^{x} F(z,s,y,t)K(s,x,y,t)ds + F(z,x,y,t) = 0, \tag{2.10}$$

where

$$K(z,x,y,t) = -\int_{\Omega} E(p,q,z,y,t)e^{iq(z-x)}\psi(p,q,x,y,t)g(p,q)\,dp\,dq,$$

$$F(z,x,y,t) = \int_{\Omega} E(p,q,z,y,t)E(p,q,x,y,t)e^{iq(z-x)}g(p,q)\,dp\,dq.$$

Taking into account (2.1), (2.2), and (2.4) it is easy to show that the function $F(z,x,y,t)$ satisfies equations (1.4). It is also obvious that formula (2.9) and equation (2.10) (with respect to $K(z,x,y,t)$, $z < x$) correspond to the equations (1.2), (1.3) (with the "$-$" sign). Thus according to the Zakharov-Shabat dressing method, u defined by (2.9) is a solution of the KP-I equation. Taking into account that the domain Ω is contained in the right half plane at positive distance from the imaginary axis and that the function $g(p,q)$ satisfies inequality (2.4) it is easy to prove that solution (2.9) vanishes for $x \to -\infty$.

This solution is real. Indeed according to (2.9) it is sufficient to prove $\operatorname{Im}(E,\bar{\psi}) = 0$, where ψ is solution of the integral equation (2.3) and $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in $L^2_g(\Omega)$. Let us remind that equation (2.3) in $L^2_g(\Omega)$ takes the form (2.7) with $\mathbf{A}$ a positive operator and real right hand side $\zeta = E$. Then we obtain

$$(E,\bar{\psi}) = \overline{(\zeta,\psi)} = \overline{(\psi + \mathbf{A}\psi,\psi)} = (\psi,\psi) + (\mathbf{A}\psi,\psi) \geq 0.$$

Hence $\operatorname{Im}(E,\bar{\psi}) = 0$ and the lemma is proved. ■

Thus for a given function $g(p,q) \geq 0$, formulas (2.9) and (2.1)-(2.3) represent a global real solution $u(x,y,t)$ of the KP-I equation. This solution decays as $x \to -\infty$, but its behavior as $x \to +\infty$ is unknown.

Our main goal is to study the asymptotic behavior of this solution as $t \to \infty$ in suitable neighbourhoods of the rear part of the solution, namely neighbourhoods of the form:

$$D_N(t) = \Big\{(x,y) \;\Big|\; -\infty < y < \infty,\ x < C(Y)t + \frac{1}{a(Y)}\log t^N\Big\}.$$

We show that in such domains and for large t ($t > T(N)$) the solution $u(x,y,t)$ has the asymptotic behavior described by (1.5), where $a(Y)$, $C(Y)$, $\varphi_n(Y)$ are expressed in terms of $g(p,q)$. This formula means that the solution splits into a sequence of curved solitons, which are formed in the neighbourhood of its trailing edge. To prove this asymptotic formula we first need to approximate the solution of the integral equation (2.3) by solutions of integral equations with appropriate degenerate kernels.

3 Integral equation with degenerate kernel

Let $k_0 = p_0 + iq_0 \in \Gamma$ be an arbitrary point of the boundary of Ω. We will use the following double power series expansion:

$$\frac{1}{\lambda + \bar{k}} = \sum_{i,j=0}^{\infty} C_{ij}(\lambda - k_0)^i(\bar{k} - \bar{k}_0)^j, \tag{3.1}$$

where

$$C_{ij} = (-1)^{i+j} \frac{(i+j)!}{i!j!(2p_0)^{i+j+1}}.$$

It is easy to check that (3.1) converges in the polydisk

$$\Pi(k_0) = \{(\lambda, k) \mid |\lambda - k_0| < p_0, \, |k - k_0| < p_0\}.$$

Below we will choose k_0 as a point of $\overline{\Omega}$ where the function $f(p, q, Y) = p^2 - 3q^2 - 2qY$ attains its minimal value. We suppose such a point exists and is unique. This point depends on parameter Y:

$$k_0 = k_0(Y) = p_0(Y) + iq_0(Y). \tag{3.2}$$

Let us denote by $C(Y)$ the value of the function $f(p, q, Y)$ at the point $k_0(Y)$, i.e.

$$C(Y) = f(p_0(Y), q_0(Y), Y) = \min_{(p,q)\in\Omega} f(p, q, Y) \tag{3.3}$$

and by $\chi_{N,Y}(p, q)$ the characteristic function of a subdomain $G_{N,Y} \subset \Omega$ such that

$$0 < \operatorname{dist}(k_0(Y), \Omega \setminus \overline{G}_{N,Y}) < \frac{p_0(Y)}{2}. \tag{3.4}$$

This subdomain depends on Y and on N. It will be precisely defined later.

Using the expansion (3.1) we represent the operator $\mathbf{A}$ (2.5) in the form:

$$\mathbf{A} = \mathbf{A}_N + \mathbf{B}_N + \mathbf{C}^1_N + \mathbf{C}^2_N, \tag{3.5}$$

where the operators $\mathbf{A}_N$, $\mathbf{B}_N$, $\mathbf{C}^1_N$, and $\mathbf{C}^2_N$ are defined by

$$\begin{aligned}
[\mathbf{A}_N\varphi](p, q) = \int_\Omega & E(p, q)E(\mu, \nu)\chi_{N,Y}(p, q)\chi_{N,Y}(\mu, \nu)g(\mu, \nu) \\
& \times \sum_{i,j=0}^{N} C_{ij}(\lambda - k_0)^i(\overline{k} - \overline{k}_0)^j\varphi(\mu, \nu)\, d\mu\, d\nu, \\
[\mathbf{B}_N\varphi](p, q) = \int_\Omega & E(p, q)E(\mu, \nu)\chi_{N,Y}(p, q)\chi_{N,Y}(\mu, \nu)g(\mu, \nu) \\
& \times \sum_{(i,j)\in\tilde{R}^{(N)}} C_{ij}(\lambda - k_0)^i(\overline{k} - \overline{k}_0)^j\varphi(\mu, \nu)\, d\mu\, d\nu, \\
[\mathbf{C}^1_N\varphi](p, q) = \int_\Omega & \frac{E(p, q)E(\mu, \nu)}{\lambda + \overline{k}}(1 - \chi_{N,Y}(\mu, \nu))g(\mu, \nu)\varphi(\mu, \nu)\, d\mu\, d\nu, \\
[\mathbf{C}^2_N\varphi](p, q) = \int_\Omega & \frac{E(p, q)E(\mu, \nu)}{\lambda + \overline{k}}(1 - \chi_{N,Y}(p, q))\chi_{N,Y}(\mu, \nu)g(\mu, \nu)\varphi(\mu, \nu)\, d\mu\, d\nu.
\end{aligned}$$

Here $\lambda = \mu + i\nu$, $k = p + iq$ and

$$\begin{aligned}
\tilde{R}^{(N)} &:= \{(i, j) \mid i, j \geq 0\} \setminus \{(i, j) \mid 0 \leq i, j \leq N\} \\
&= \{(i, j) \mid i \geq 0, \, j \geq N + 1\} \cup \{(i, j) \mid i \geq N + 1, \, j \geq 0\}.
\end{aligned}$$

Let us estimate the norm of the operators $\mathbf{B}_N$, $\mathbf{C}_N^1$ and $\mathbf{C}_N^2$ in the space $L_g^2(\Omega)$. To avoid unessential complications we assume that Ω is bounded. We will denote

$$a = \inf_{\lambda\in\Omega} \operatorname{Re}\lambda, \qquad b = \sup_{\lambda\in\Omega} \operatorname{Re}\lambda,$$
$$d(\xi) = (b+a)\xi + (b-a)|\xi|, \qquad \xi = x - C(Y)t,$$

where $C(Y)$ is defined by (3.3). The norms of the operators $\mathbf{C}_N^i$ $(i = 1, 2)$ on $L_g^2(\Omega)$ can be estimated as follows:

$$\begin{aligned}||\mathbf{C}_N^i||^2 &\le \iint_{\Omega\times\Omega} (|\lambda + \overline{k}|)^{-2} e^{2(p+\mu)\xi}\, e^{2p(C(Y)-f(p,q,Y))t}\, e^{2\mu(C(Y)-f(\mu,\nu,Y))t} \\ &\qquad \times (1 - \chi_{N,Y}(\mu,\nu))\, g(p,q)g(\mu,\nu)\, dp\, dq\, d\mu\, d\nu \\ &\le \frac{e^{2d(\xi)-2am_0t}}{(2a)^2}\, \hat{g}^2 (\operatorname{meas}\Omega)^2,\end{aligned}$$

where

$$\hat{g} = \max_{(p,q)\in\Omega} g(p,q) \text{ and } m_0 = \min_{(p,q)\in\Omega\setminus G_{N,Y}} [f(p,q,Y) - C(Y)].$$

According to (2.2), (3.3), (3.4), $m_0 > 0$. Hence,

$$||\mathbf{C}_N^i|| \le \hat{g} \times \operatorname{meas}\Omega \times e^{-am_0t/2} \tag{3.6}$$

for $\xi < \frac{am_0t}{4b}$. Taking into account that $\tilde{R}^{(N)} \subset \cup_{k\ge N+1}\{(i,j) \mid i+j = k,\ i,j \ge 0\}$ we can write (for $\lambda, k \in G_{N,Y}$):

$$\begin{aligned}\sum_{(i,j)\in\tilde{R}^{(N)}} |C_{ij}| \cdot |\lambda - k_0|^i |\overline{k} - \overline{k}_0|^j &\le \sum_{\substack{i,j\ge 0,\\ i+j\ge N+1}} |C_{ij}| \cdot |\lambda - k_0|^i |k - k_0|^j \\ &= \sum_{l=N+1}^{\infty} \sum_{i=0}^{l} \frac{l!}{i!(l-i)!(2p_0)^{l+1}} |\lambda - k_0|^i\, |k - k_0|^{l-i} \\ &= \frac{1}{2p_0} \sum_{l=N+1}^{\infty} \left(\frac{|\lambda - k_0| + |k - k_0|}{2p_0}\right)^l \\ &\le \frac{1}{p_0} \left(\frac{|\lambda - k_0| + |k - k_0|}{2p_0}\right)^{N+1}.\end{aligned}$$

Using this inequality we obtain

$$\begin{aligned}||\mathbf{B}_N||^2 &\le \iint_{\Omega\times\Omega} e^{\Phi}\, \frac{1}{p_0^2} \left(\frac{|\lambda - k_0| + |k - k_0|}{2p_0}\right)^{2(N+1)} \chi\, dp\, dq\, d\mu\, d\nu \\ &= \frac{1}{p_0^2} \sum_{\substack{i,j\ge 0,\\ i+j=2(N+1)}} |C_{ij}| J_i(x,Y,t) J_j(x,Y,t),\end{aligned} \tag{3.7}$$

where

$$\Phi = 2(p+\mu)[x - t(f(p,q,Y) - f(\mu,\nu,Y))]$$
$$\chi = \chi_{N,Y}(p,q)\chi_{N,Y}(\mu,\nu)g(p,q)g(\mu,\nu)$$
$$J_i(x,Y,t) = \int_{G_{N,Y}} e^{2p(x-f(p,q,Y)t)}|k-k_0|^i\, g(p,q)\,dp\,dq, \tag{3.8}$$

and the numbers C_{ij} are those introduced in (3.1).

We now define more precisely the subdomain $G_{N,Y} \subset \Omega$. We will suppose that Γ is defined by

$$\Gamma = \{(p,q) \mid \varphi(p,q) = 0\}$$

where $\varphi(k) = \varphi(p,q) \in C^2(\bar{\Omega})$ and that the curvature of Γ is everywhere positive. Since the hyperbola

$$H(Y) := \{(p,q) \mid f(p,q,Y) - C(Y) = 0\}$$

is tangent to Γ at the point $k_0 = k_0(Y) \in \partial\Omega$ we can introduce in some neighbourhood of k_0 new coordinates

$$r := 2p(f(p,q,Y) - C(Y)) =: F(p,q)$$
$$s := \Big(\frac{\partial\varphi}{\partial q}(k_0)(p-p_0) - \frac{\partial\varphi}{\partial p}(k_0)(q-q_0)\Big)||\nabla\varphi(k_0)||^{-1} =: \Phi(p,q).$$

It is evident that s is the projection of $(p-p_0, q-q_0)$ on the tangent to the boundary Γ at k_0. It is easy to check that the equation of Γ near $k_0 \in \Gamma$ is of the form

$$r = \alpha_0 s^2 + \mathrm{O}(s^3),$$

where

$$\alpha_0 = ||\nabla F(k_0)||\,\frac{\hat{\kappa}_0 \mp \kappa_0}{2}. \tag{3.9}$$

$\hat{\kappa}_0$ and κ_0 are the curvatures of $\partial\Omega$ and $H(Y)$ at the point k_0 respectively. The minus sign occurs when Γ and $H(Y)$ are on the same side of their common tangent. The plus sign occurs otherwise.

In any case $\alpha_0 > 0$. Hence, Γ is given by

$$s = \hat{s}_\pm(r), \qquad r \geq 0,$$
$$\hat{s}_\pm(r) = \pm\sqrt{r/\alpha_0} + \mathrm{O}(r).$$

Now, the subdomain $G_{N,Y} \subset \Omega$ is defined as follows (Fig. 1):

$$G_{N,Y} = \{(r,s) \mid 0 < r < \varepsilon_0(N),\ \hat{s}_-(r) < s < \hat{s}_+(r)\}$$

where $\varepsilon_0(N)$ is a small positive number, such that

$$\varepsilon_0(N) \leq \frac{\alpha_0 p_0^2}{16N^2\big(\big|\frac{\partial p}{\partial s}(k_0)\big|^2 + \big|\frac{\partial p}{\partial r}(k_0)\big|^2\big)}. \tag{3.10}$$

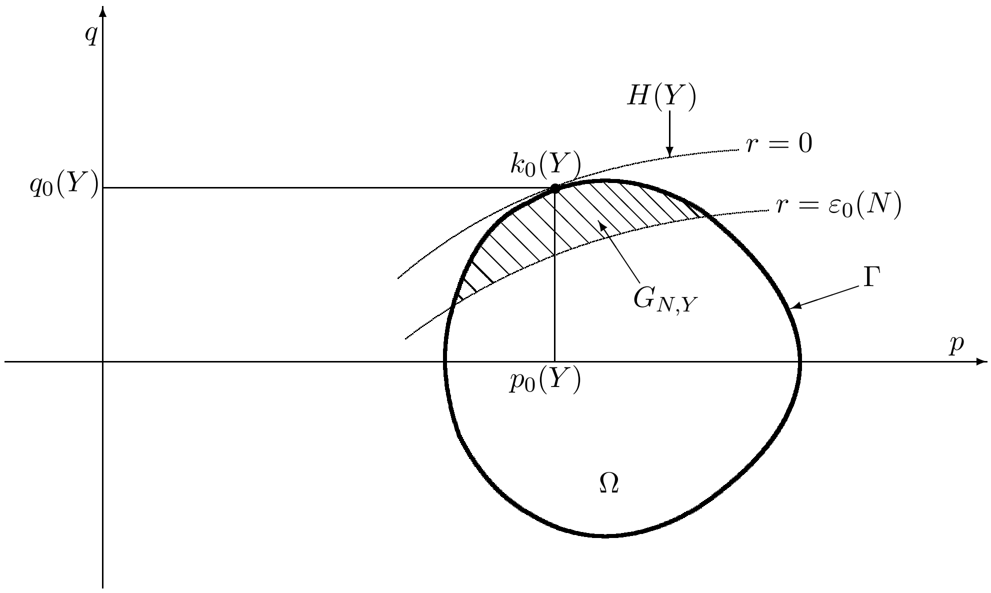

Figure 1. Ω, Γ and the subdomain $G_{N,Y}$

For $\varepsilon_0(N) > 0$ small enough both bounds in (3.4) are clearly fulfilled.

Let us pass to an estimation of the integrals $J_i(x,Y,t)$ $(i=1,2)$. Since for $k \in G_{N,Y}$

$$|k-k_0| = \sqrt{s^2 + \frac{r^2}{||\nabla F(k_0)||^2}}\,(1+\mathrm{O}(r+|s|))$$

we have

$$|k-k_0|^i \le 2^i\left(|s|^i + \frac{r^i}{||\nabla F(k_0)||^i}\right).$$

Due to (3.8) we obtain

$$J_i(x,Y,t) \le 2^i e^{2p_0\xi}\int_0^{\varepsilon_0(N)} e^{-rt}\,dr \times \int_{\hat{s}_-(r)}^{\hat{s}_+(r)} e^{2(p-p_0)\xi}\left(|s|^i + \frac{r^i}{||\nabla F(k_0)||^i}\right)\hat{g}(r,s)|w(r,s)|\,ds, \tag{3.11}$$

where $\xi = x - C(Y)t$, $\hat{g}(r,s) = g(p(r,s),q(r,s))$, and $w(r,s) = \dfrac{\partial(p,q)}{\partial(r,s)}$ is the Wronskian. Using Taylor's series one can write:

$$e^{2(p-p_0)\xi} = 1 + 2\left(\frac{\partial p}{\partial s}(k_0) + \frac{\partial p}{\partial r}(k_0)\right)E_0(r,s,\xi) \tag{3.12}$$

where $E_0(r,s,\xi)$ is bounded:

$$|E_0(r,s,\xi)| \le |\xi|e^{2\delta_0(N)|\xi|} \tag{3.13}$$

for $(r,s) \in G_{N,Y}$, small $\varepsilon_0(N)$ and

$$\delta_0(N) := 2||\nabla p(k_0)||\sqrt{\frac{\varepsilon_0(N)}{\alpha_0}}. \tag{3.14}$$

Using this equality and the relations

$$g(k) = g(k_0) + \mathrm{O}(r+|s|) \qquad w(k) = w(k_0) + \mathrm{O}(r+|s|)$$

we obtain after integration of (3.11) over s:

$$\begin{aligned} J_i(x,Y,t) &\le \left(\frac{4}{\alpha_0}\right)^{\frac{i+1}{2}} \frac{g_0|w_0|}{i+1} e^{2p_0\xi} \int_0^{\varepsilon_0(N)} e^{-rt} r^{\frac{i+1}{2}} (1+\mathrm{O}(\sqrt{r}))\, dr \\ &\quad + |\xi| K_i(Y) e^{2p_0\xi + 2\delta_0(N)|\xi|} \int_0^{\varepsilon_0(N)} e^{-rt} r^{\frac{i+2}{2}} (1+\mathrm{O}(\sqrt{r}))\, dr \\ &\le K_{1i}(Y) \frac{e^{2p_0\xi}}{t^{\frac{i+3}{2}}} + K_{2i}(Y) \frac{e^{2p_0\xi+2\delta_0(N)|\xi|}(1+|\xi|)}{t^{\frac{i+4}{2}}} \end{aligned} \tag{3.15}$$

where K_{1i} and K_{2i} do not depend on ξ, t, and $g_0 = g(k_0)$, $w_0 = w(k_0)$. Here we have used the estimate

$$\int_0^{\varepsilon_0(N)} e^{-rt} r^d dr < \int_0^{\infty} e^{-rt} r^d dr = \frac{\Gamma(d+1)}{t^{d+1}}, \tag{3.16}$$

where $\Gamma(d)$ is the Euler Γ-function. From (3.7) and (3.15) we derive the following bound for the norm of $\mathbf{B}_N$:

$$||\mathbf{B}_N|| \le K_N(Y) \left[\frac{e^{2p_0\xi}}{t^{\frac{N+4}{2}}} + \frac{e^{2p_0\xi+\delta_0(N)|\xi|}(1+|\xi|)}{t^{\frac{N+4}{2}+\frac{1}{4}}} + \frac{e^{2p_0\xi+2\delta_0(N)|\xi|}(1+|\xi|)^2}{t^{\frac{N+5}{2}}} \right].$$

Therefore, in view of (3.10) and (3.14), we find

$$||\mathbf{B}_N|| \le \frac{K_N(Y)}{t^{1/2-\varepsilon}} \qquad (0 < \varepsilon < 1/4). \tag{3.17}$$

Now let us return to equation (2.3) (or (2.7)). Taking into account (2.5) and (3.5) we rewrite it in the form:

$$\psi = \mathbf{A}_N\psi + \mathbf{B}'_N\psi = e \tag{3.18}$$

where $\mathbf{B}'_N = \mathbf{B}_N + \mathbf{C}^1_N + \mathbf{C}^2_N$ and $e = E(p,q)$. According to (3.6) and (3.17) the norm of $\mathbf{B}'_N$ has the estimate

$$||\mathbf{B}'_N|| \le \frac{K'_N(Y)}{t^{1/2-\varepsilon}} \quad \text{if} \quad \xi = x - C(Y)t < \frac{1}{2p_0} \log t^{\frac{N+3}{2}+\varepsilon}. \tag{3.19}$$

Let us look for the solution of equation (3.18) in the form

$$\psi = \psi_N + \delta_N, \tag{3.20}$$

where ψ_N is the solution of the equation

$$\psi_N + \mathbf{A}_N \psi_N = e, \tag{3.21}$$

and therefore

$$\delta_N = -(\mathbf{I} + \mathbf{A})^{-1} \mathbf{B}'_N \psi_N. \tag{3.22}$$

According to (1.2) and (3.20) the solution of the KP-I equation is represented in the form:

$$u(x,t) = -2\frac{d}{dx}(\psi, e) = -2\frac{d}{dx}(\psi_N, e) - 2\frac{d}{dx}(\delta_N, e). \tag{3.23}$$

Taking into account (3.22), the fact that $\mathbf{A}$ is self-adjoint, and relations (3.18), (3.21) we can write

$$\begin{aligned}(\delta_N, e) &= \big((\mathbf{I} + \mathbf{A})^{-1} \mathbf{B}'_N \psi_N, e\big) = \big(\mathbf{B}'_N \psi_N, (\mathbf{I} + \mathbf{A})^{-1} e\big) \\ &= (\mathbf{B}'_N \psi_N, \psi) = (\mathbf{B}'_N \psi_N, \psi_N) + (\mathbf{B}'_N \psi_N, \delta_N) \\ &= (\mathbf{B}'_N \psi_N, \psi_N) - (\mathbf{B}'_N \psi_N, (\mathbf{I} + \mathbf{A})^{-1} \mathbf{B}'_N \psi_N).\end{aligned} \tag{3.24}$$

It follows from (3.21) that

$$||\psi_N||^2 + (\psi_N, \mathbf{A}\psi_N) - (\psi_N, \mathbf{B}'_N \psi_N) = (\psi_N, e).$$

Hence, due to the positivity of the operator $\mathbf{A}$,

$$||\psi_N||^2 - ||\mathbf{B}'_N|| \cdot ||\psi_N||^2 < |(\psi_N, e)|. \tag{3.25}$$

In the next section we will show that

$$|(\psi_N, e)| < KN, \tag{3.26}$$

where the constant K does not depend on x, y, t if $\xi = x - C(Y)t < \frac{1}{2p_0} \log t^{\frac{N+3}{2}+\varepsilon}$. According to (3.19) we have $||\mathbf{B}'_N|| \to 0$ as $t \to \infty$. Therefore taking into account (3.25) and (3.26) we get

$$||\psi_N||^2 < 2KN \tag{3.27}$$

for sufficiently large t. In turn, it follows from (3.24), (3.27) and (2.8) that

$$|(\delta_N, e)| \leq \frac{K_N(Y)}{t^{1/2-\varepsilon}}, \tag{3.28}$$

where K_N does not depend on x, y, t if $\xi = x - C(Y)t < \frac{1}{2p_0} \log t^{\frac{N+3}{2}+\varepsilon}$.

Remark 3.1. The same estimate is also valid for the derivative $\frac{\partial}{\partial x}(\psi_N, e)$. It can be proved using an analytic continuation of (ψ_N, e) in some strip $|\operatorname{Im} x| < \beta$.

Thus according to (3.23) we need to investigate the solution of (3.21). This equation is an integral equation with degenerate kernel:

$$A_N(p,q,\mu,\nu,x,Y,t) = \sum_{i,j=0}^{N} C_{ij}(\lambda - k_0)^i(\bar{k}-\bar{k}_0)^j \\ \times E(p,q)E(\mu,\nu)\chi_{N,Y}(p,q)\chi_{N,Y}(\mu,\nu)g(\mu,\nu) \tag{3.29}$$

and right-hand side

$$E(p,q) = E(p,q,x,Y,t) = e^{p(x-f(p,q,Y)t)}. \tag{3.30}$$

Due to the specific form of the kernel we look for a solution of (3.21) in the form:

$$\psi_N(p,q,x,Y,t) = \sum_{j=0}^{N} \psi_j^{(N)}(x,Y,t)(\bar{k}-\bar{k}_0)^j E(p,q)\chi_{N,Y}(p,q). \tag{3.31}$$

Substituting (3.31) into (3.21) and taking into account (3.29), (3.30) we obtain a system of linear algebraic equations for the functions $\psi_j^{(N)} = \psi_j^{(N)}(x,Y,t)$:

$$\psi_j^{(N)} + \sum_{l=0}^{N} A_{jl}^{(N)}\psi_l^{(N)} = \delta_{j0} \qquad j = 0,1,\ldots,N, \tag{3.32}$$

where $\delta_{00} = 1$ and $\delta_{j0} = 0$ for $j = 1,2,\ldots,N$,

$$A_{ij}^{(N)} = A_{ij}^{(N)}(x,Y,t) = \sum_{l=0}^{N} C_{il}J_{lj}(x,y,t), \tag{3.33}$$

and the integrals J_{lj} are defined by

$$J_{lj}(x,Y,t) = \int_{G_{N,Y}} E^2(p,q)(k-k_0)^l(\bar{k}-\bar{k}_0)^j g(p,q)\,dp\,dq. \tag{3.34}$$

The solution of the system (3.32) is given by

$$\psi_j^{(N)}(x,Y,t) = \frac{D_j^{(N)}(x,Y,t)}{D^{(N)}(x,Y,t)}, \tag{3.35}$$

where $D^{(N)}(x,Y,t) = \det[I^{(N)} + A^{(N)}(x,Y,t)]$ is the determinant of the matrix with entries $\delta_{ij} + A_{ij}^{(N)}(x,Y,t)$ $(i,j = 0,1,\ldots,N)$, and $D_j^{(N)}(x,Y,t)$ is the determinant of the matrix obtained by replacing the j-th column of $I^{(N)} + A_{ij}^{(N)}$ by the column $(1,0,\ldots,0)^\top$.

It follows from (3.31) and (3.35) that

$$(\psi_N, e) = \frac{F^{(N)}(x,Y,t)}{D^{(N)}(x,Y,t)}, \tag{3.36}$$

where $F^{(N)}(x,Y,t)$ is the determinant of the matrix obtained by replacing the first line of $I^{(N)} + A^{(N)}(x,Y,t)$ by the line $(J_0(x,Y,t), J_1(x,Y,t), \ldots, J_N(x,Y,t))$. Let us now note that according to (3.33), $A^{(N)}(x,Y,t)$ is the product of two $(N+1)\times(N+1)$ matrices:

$$A^{(N)}(x,Y,t) = C^{(N)} J^{(N)}(x,Y,t).$$

$C^{(N)}$ and $J^{(N)}(x,Y,t)$ are the $(N+1)\times(N+1)$ matrices with entries C_{ij} and $J_{lj}(x,Y,t)$, respectively. Taking this into account and setting C_{00} as a varying parameter in the matrix $C^{(N)}$ we obtain the determinant formula

$$(\psi_N, e) = \frac{\partial}{\partial C_{00}} \log\det[I^{(N)} + A^{(N)}(x,Y,t)]. \tag{3.37}$$

4 Asymptotic behavior of the solution for large time

First of all let us study the asymptotic behavior of the integrals (3.34).

Lemma 4.1. *The integrals $J_{ij}(x,Y,t)$ have the following asymptotic representation:*

$$J_{ij}(x,Y,t) = \frac{g_0|w_0|}{\alpha_0}\frac{h_0^i \bar{h}_0^j \Gamma\left(\frac{i+j+3}{2}\right)}{i+j+1}\left[1+(-1)^{i+j}\right]\frac{e^{2p_0\xi}}{t^{\frac{i+j+3}{2}}} + \frac{I_{ij}(\xi,Y,t)}{t^{\frac{i+j+4}{2}}}\ .$$

Here,

$$g_0 = g(k_0), \quad w_0 = w(k_0),$$
$$h_0 = \frac{1}{\alpha_0}\frac{\partial k}{\partial s}(k_0) = \frac{1}{\alpha_0}\left(\frac{\partial p}{\partial s} + i\frac{\partial q}{\partial s}\right)(k_0),$$

where α_0 is defined in (3.9)*, and the functions $I_{ij}(\xi,Y,t)$ satisfy*

$$|I_{ij}(\xi,Y,t)| \le K_{ij}(Y)(1+|\xi|)e^{2p_0\xi + 2\delta_0(N)|\xi|}$$

with $\delta_0(N)$ defined by (3.14).

Proof. Using (3.12) and taking into account that for $k \in G_{N,Y}$

$$k - k_0 = \frac{\partial k}{\partial s}(k_0)s + \frac{\partial k}{\partial r}(k_0)r + \mathrm{O}(r^2 + |s|^2)$$

we write down the integral (3.34) in the form:

$$\begin{aligned} J_{ij}(x,Y,t) &= g_0|w_0|k_1^i\bar{k}_1^j e^{2p_0\xi}\int_0^{\varepsilon_0(N)} e^{-rt}\int_{\hat{s}_-(r)}^{\hat{s}_+(r)} s^{i+j}[1+\mathrm{O}(r|s|^\beta)]\,ds\,dr \\ &+ 2g_0|w_0|k_1^i\bar{k}_1^j e^{2p_0\xi}\int_0^{\varepsilon_0(N)} e^{-rt}\int_{\hat{s}_-(r)}^{\hat{s}_+(r)} [p_1 s^{i+j+1} + p_2 r s^{i+j}][1+\mathrm{O}(r|s|^\beta)]\,ds\,dr \\ &= J_{ij}^1(x,Y,t) + J_{ij}^2(x,Y,t), \end{aligned} \tag{4.1}$$

where $k_1 = \dfrac{\partial k}{\partial s}(k_0)$, $p_1 = \dfrac{\partial p}{\partial s}(k_0)$, $p_2 = \dfrac{\partial p}{\partial r}(k_0)$ and $\beta = 0$ if $i + j = 0$ and $\beta = -1$ if $i + j \geq 1$. Integration over s of the first summand in (4.1) gives the following asymptotic equality:

$$J^1_{ij}(x, Y, t) = \frac{g_0|w_0|k_1^i \bar{k}_1^j e^{2p_0\xi}}{(i + j + 1)\alpha_0^{i+j+1}} \int_0^{\varepsilon_0(N)} e^{-rt} r^{\frac{i+j+1}{2}} [1 + (-1)^{i+j+1}][1 + \mathrm{O}(\sqrt{r})]\, dr.$$

Using the asymptotic relation (3.16) we find

$$J^1_{ij}(x, Y, t) = \frac{g_0|w_0|}{(i + j + 1)\alpha_0} \left(\frac{k_1}{\alpha_0}\right)^i \left(\frac{\bar{k}_1}{\alpha_0}\right)^j [1 + (-1)^{i+j+1}]\, \Gamma\left(\frac{i + j + 1}{2}\right) \frac{e^{2p_0\xi}}{t^{\frac{i+j+3}{2}}} + \mathrm{O}\left(\frac{e^{2p_0\xi}}{t^{\frac{i+j+4}{2}}}\right). \tag{4.2}$$

In the same way, taking into account inequality (3.13), we obtain the following estimate for the second summand in (4.1):

$$\begin{aligned} |J^2_{ij}(x, Y, t)| &\leq K'_{ij}(Y)|\xi|\, e^{2p_0\xi + 2\delta_0(N)|\xi|} \int_0^{\infty} e^{-rt} r^{\frac{i+j+2}{2}} [1 + \mathrm{O}(\sqrt{r})]\, dr \\ &\leq K_{ij}(Y) \frac{|\xi|\, e^{2p_0\xi + 2\delta_0(N)|\xi|}}{t^{\frac{i+j+4}{2}}}, \end{aligned} \tag{4.3}$$

where the functions $K_{ij}(Y)$ do not depend on ξ and t. The statement of the lemma follows from (4.1)-(4.3). ■

Now, using (3.37), let us study the large time asymptotic behavior of the function $D^{(N)}(x, Y, t) = \det[I^{(N)} + A^{(N)}(x, Y, t)]$.

Lemma 4.2. *We have the following asymptotic relation*

$$\det[I^{(N)} + A^{(N)}(x, Y, t)] = 1 + \sum_{n=1}^{N} \det C^{(n)} \det \Gamma^{(n)}(Y) \frac{e^{2p_0\xi}}{t^{\frac{n(n+2)}{2}}} [1 + \delta_n(\xi, Y, t)]$$

where $C^{(n)}$ and $\Gamma^{(n)}(Y)$ are $n \times n$ matrices with entries ($i, j = 0, 1, \ldots, n - 1$)

$$\frac{(i + j)!}{i!j!(2p_0)^{i+j+1}} \quad \text{and} \quad \frac{g_0(Y)|w_0(Y)|}{\alpha_0(Y)} \frac{h_0^i(Y)\bar{h}_0^j(Y)\Gamma\left(\frac{i+j+3}{2}\right)}{i + j + 1},$$

respectively, and the functions $\delta_n(\xi, Y, t)$ satisfy

$$|\delta_n(\xi, Y, t)| \leq \frac{K_n(Y)}{t^{1/4}} \quad \text{if } \xi < \frac{1}{2p_0} \log t^{\frac{N+3}{2} + \varepsilon} \text{ with } 0 < \varepsilon < 1/4. \tag{4.4}$$

Proof. Let us denote by $\tilde{D}^{(N)}(x, Y, t; \lambda_0, \ldots, \lambda_N)$ the determinant of the matrix $\Lambda^{(N)} + A^{(N)}(x, Y, t)$, where $\Lambda^{(N)} = \mathrm{diag}(\lambda_0, \ldots, \lambda_N)$ is the diagonal matrix depending on $N + 1$ parameters $\lambda_0, \ldots, \lambda_N$. Clearly,

$$\tilde{D}^{(N)}(x, Y, t; 1, \ldots, 1) = D^{(N)}(x, Y, t) = \det\left[I^{(N)} + A^{(N)}(x, Y, t)\right].$$

This determinant is a polynomial with respect to the λ_k's:

$$\begin{aligned}\tilde{D}^{(N)}(x,Y,t;\lambda_0,\dots,\lambda_N) &= \lambda_0\dots\lambda_N + \hat{\lambda}_0\lambda_1\dots\lambda_N D_0^{(1)}(x,Y,t)\\ &+ \lambda_0\hat{\lambda}_1\lambda_2\dots\lambda_N D_1^{(1)}(x,Y,t) + \dots + \lambda_0\dots\lambda_{N-1}\hat{\lambda}_N D_N^{(1)}(x,Y,t)\\ &+ \hat{\lambda}_0\hat{\lambda}_1\lambda_2\dots\lambda_N D_{01}^{(2)}(x,Y,t) + \dots + \lambda_0\dots\hat{\lambda}_{i_1}\dots\hat{\lambda}_{i_n}\dots\lambda_N D_{i_1\dots i_n}^{(n)}(x,Y,t) + \dots\\ &+ D_{0\dots N}^{(N)}(x,Y,t),\end{aligned} \tag{4.5}$$

where $D_{i_1\dots i_n}^{(n)}(x,Y,t)$ is the determinant of the $n\times n$ matrix with entries $A_{i_r i_p}(x,Y,t)$ $(r,p=1,\dots,n)$; the hat means that the corresponding summand is absent. Taking into account (3.33) and using Lemma 4.1 we obtain

$$\begin{aligned}D_{i_1\dots i_n}^{(n)}(x,Y,t) &= \det C^{(n)}\det J^{(n)}(x,Y,t) + d^{(n)}(x,Y,t)\\ &= \det C^{(n)}\det\Gamma^{(n)}(Y)\frac{e^{2p_0 n\xi}}{t^{\frac{n(n+2)}{2}}} + d_1^{(n)}(x,Y,t),\end{aligned}$$

where $J^{(n)}(x,Y,t)$ is the $n\times n$ matrix with entries $J_{ij}^{(n)}(x,Y,t)$ defined by (3.34), $C^{(n)}$ and $\Gamma^{(n)}(Y)$ are defined in Lemma 4.2 ($\det C^{(n)}>0$, $\det\Gamma^{(n)}(Y)>0$ as Gram determinants), and the functions $d^{(n)}(x,Y,t)$, $d_1^{(n)}(x,Y,t)$ satisfy

$$|d^{(n)}(x,Y,t)|,\ |d_1^{(n)}(x,Y,t)| < K_d(Y)\frac{e^{2p_0 n\xi}}{t^{\frac{n(n+2)}{2}+1/2}}\left(1+\frac{e^{2\delta_0(N)|\xi|}}{t^{1/2}}\right). \tag{4.6}$$

The determinants $D_{i_1\dots i_n}^{(n)}$ with $i_1+i_2+\dots+i_n > \frac{n(n-1)}{2}$ also satisfy (4.6). Taking all this into account and setting $\lambda_i=1$ $(i=0,\dots,N)$ in (4.5), we obtain the assertion of Lemma 4.2. ■

Remark 4.1. A more precise analysis of the determinants $D_{i_1\dots i_n}^{(n)}$ shows that the derivatives of the functions with respect to ξ and to C_{00} have the same estimates as in (4.4).

Let us use the following equality, which is proved in [9]:

$$n\det C_0^{(n)} = \det C_1^{(n-1)},$$

where $C_0^{(n)}$ is the $n\times n$ matrix with entries $\dfrac{(i+j)!}{i!j!}$ $(i,j=0,\dots,n-1)$, and $C_1^{(n-1)}$ is the $(n-1)\times(n-1)$ matrix with entries $\dfrac{(i+j)!}{i!j!}$ $(i,j=1,\dots,n-1)$. This allows us to obtain the relation:

$$\frac{\partial}{\partial C_{00}}\det C^{(n)} = 2p_0 n\det C^{(n)}\big|_{C_{00}=(2p_0)^{-1}}, \tag{4.7}$$

where $C^{(n)}$ is as above.

Now taking into account (3.23), (3.28), (3.37), Lemma 4.2, Remarks 3.1 and 4.1 and (4.7) we obtain the following asymptotic formula for the solution:

$$u(x,y,t) = 2\frac{\partial^2}{\partial\xi^2}\log\left[1+\sum_{n=1}^{N}\det C^{(n)}\det\Gamma^{(n)}(Y)\frac{e^{2p_0 n\xi}}{t^{\frac{n(n+2)}{2}}}\right]_{\xi=x-C(Y)t} + \mathrm{O}\left(t^{-1/4}\right) \tag{4.8}$$

in the domain $\{(x,y)\in\mathbb{R}^2 \mid x<C(Y)t+\frac{1}{2p_0}\log t^{\frac{N+3}{2}+\varepsilon}\}$ as $t\to\infty$.

Remark 4.2. This asymptotic formula is uniform with respect to y because, in (3.28) and (4.4), $K_N(Y)$ is uniformly bounded with respect to y. That follows from the compactness of the contour Γ and the positivity of its curvature.

To push further the asymptotic analysis of the determinant formula (4.8) let us introduce notations:

$$\Delta_N(\xi,Y,t)=1+\sum_{n=1}^{N}R_n(Y)\frac{e^{2p_0n\xi}}{t^{n(n+2)/2}} \tag{4.9}$$

$$R_n(Y)=\det C^{(n)}\det\Gamma^{(n)}(Y)=\frac{\left(\frac{g_0|w_0|}{\alpha_0}\right)^n|h_0|^{n(n-1)}}{(2p_0)^{2n+2}\prod\limits_{i=0}^{n-1}(i!)^2}\Delta_1^{(n)}\Delta_2^{(n)}, \tag{4.10}$$

where $\Delta_1^{(n)}>0$, $\Delta_2^{(n)}>0$ are the determinants of the $n\times n$ matrices $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ with entries $\Gamma(i+j+1)$ and $\Gamma\big((i+j+3)/2\big)\frac{1+(-1)^{(i+j)}}{i+j+1}$ $(i,j=0,1,\dots,n-1)$, respectively. They are positive as Gram determinants.

From (4.8) and (4.9) it follows that

$$u(x,y,t)\sim u_N(x,Y,t)=2\frac{\partial^2}{\partial\xi^2}\log\Delta_N(\xi,Y,t)|_{\xi=x-C(Y)t}=\frac{\Delta_N''\Delta_N-(\Delta_N')^2}{\Delta_N^2} \tag{4.11}$$

and

$$\Delta_N''\Delta_N-(\Delta_N')^2=4p_0^2\sum_{n,l=0}^{N}\frac{(n-l)^2R_nR_le^{2(n+l)p_0\xi}}{t^{\frac{n(n+2)+l(l+2)}{2}}}\,. \tag{4.12}$$

Let us cover the domain $\xi<\frac{1}{2p_0}\log t^{\frac{N+3}{2}+\varepsilon}$ by the intervals

$$I_1(t)=\left\{-\infty<\xi<\frac{1}{2p_0}\log t^{2+\varepsilon}\right\},$$
$$I_2(t)=\left\{\frac{1}{2p_0}\log t^{2-\varepsilon}<\xi<\frac{1}{2p_0}\log t^{3+\varepsilon}\right\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$I_n(t)=\left\{\frac{1}{2p_0}\log t^{n-\varepsilon}<\xi<\frac{1}{2p_0}\log t^{(n+1)+\varepsilon}\right\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$I_{[\frac{N+1}{2}]}(t)=\left\{\frac{1}{2p_0}\log t^{[\frac{N+1}{2}]-\varepsilon}<\xi<\frac{1}{2p_0}\log t^{\frac{N+3}{2}+\varepsilon}\right\}.$$

Taking into account (4.11) and (4.12), we obtain

$$\Delta_N^2=\left[\frac{R_{n-1}(Y)e^{2(n-1)p_0\xi}}{t^{\frac{(n-1)(n+1)}{2}}}+\frac{R_n(Y)e^{2np_0\xi}}{t^{\frac{n(n+2)}{2}}}\right]\big(1+\mathrm{O}(t^{-1/2})\big)$$

and

$$\Delta''_N \Delta_N - (\Delta'_N)^2 = 4p_0^2 \frac{2R_n(Y)R_{n-1}(Y)e^{2(2n-1)p_0\xi}}{t^{n(n+2)+(n-1)(n+1)}} \left(1 + \mathrm{O}\left(t^{-1/2}\right)\right),$$

as $\xi \in I_n(t)$ and $t \to \infty$. Hence, by virtue of (4.9) and (4.11) we obtain

$$u(x, y, t) = \sum_{n=1}^{\left[\frac{N+1}{2}\right]} \frac{2p_0^2(Y)}{\cosh^2 \left[p_0(Y)\left(x - C(Y)t + \frac{1}{2p_0(Y)} \log t^{n+1/2} + x_n^0(Y)\right)\right]} + \mathrm{O}\left(t^{-1/4}\right), \quad (4.13)$$

where $x_n^0(Y) = \frac{1}{2p_0(Y)} \log \frac{R_n(Y)}{R_{n-1}(Y)}$ and $Y = y/t$.

Remark 4.3. The estimate (3.26) follows from (2.9), (4.8) and (4.13).

Thus we have proved the following result:

Theorem 4.1. *Let the contour* Γ *be compact, of class* C^2 *on* $\bar{\Omega}$, *and with everywhere positive curvature. Assume the function* $f(p, q, Y)$ *attains its minimal value at a unique point* $k_0(Y) = p_0(Y) + iq_0(Y) \in \Gamma$, *for any* $Y = y/t$.

Then the solution $u(x, y, t)$ *of the KP-I equation defined everywhere by* (2.3) *and* (2.9) *in the domain*

$$D_N = \left\{(x, y) \mid -\infty < y < \infty,\ x < C(Y)t + \frac{1}{2p_0(Y)} \log t^{[(N+1)/2]+1+\varepsilon}\right\} \quad (0 < \varepsilon < 1/4)$$

has the asymptotic behavior defined by (4.8) *and* (4.13) *for* $t \to \infty$.

References

[1] Ablowitz M J and Segur H, Solitons and the Inverse Scattering Transform, *Studies in Applied Mathematics* **4**, *SIAM*, Philadelphia, 1981.

[2] Anders I A, Curved Asymptotic Solitons of Kadomtsev–Petviashvili–I and Modified Kadomtsev–Petviashvili–I Equations, *Phys. D* **87**, 1-4 (1995), 160–167.

[3] Anders I and Boutet de Monvel A, Asymptotic Solitons of the Johnson Equation, *J. Nonlinear Math. Phys.* **7**, 3 (2000), 284–302.

[4] Anders I and Boutet de Monvel A, Soliton Asymptotics of Nondecaying Solutions of the Modified Kadomtsev–Petviashvili–I Equation, *J. Math. Phys.* **42**, 8 (2001), 3673–3690.

[5] Anders I A, Khruslov E Ya and Kotlyarov V P, Curved Asymptotic Solitons of the Kadomtsev–Petviashvili Equation, *Teoret. Mat. Fiz.* **99**, 1 (1994), 27–35 (English translation: *Theoret. and Math. Phys.* **99**, 1 (1994), 402–408).

[6] Anders I A, Khruslov E Ya and Kotlyarov V P, Soliton Asymptotics of Non-Localized Solutions of Non-Linear Evolutionary Equations, *Proceedings of the Second World Congress of Nonlinear Analysts, Part* **7** (Athens, 1996), *Nonlinear Anal.* **30**, 7 (1997), 3951–3961.

[7] Baranetskiĭ V B and Kotlyarov V P, Asymptotic Behavior in the Trailing Edge Domain of the Solution of the KdV Equation with an Initial Condition of the "Threshold Type", *Teoret. Mat. Fiz.* **126**, 2 (2001), 214–227 (English translation: *Theoret. and Math. Phys.* **126**, 2 (2001), 175–186).

[8] Deift P A, Its A R and Zhou X, Long-Time Asymptotics for Integrable Nonlinear Wave Equations, in Important Developments in Soliton Theory, Editors: Fokas A S and Zakharov V E, Springer Series Nonlinear Dynamics, Springer, Berlin, 1993, 181–204.

[9] Faddeev D K and Sominskiĭ I S, Sbornik zadach po vysshej algebre, Moskow, Nauka, 1977 (in Russian).

[10] Faddeev L D and Takhtajan L A, Hamiltonian Methods in the Theory of Solitons, Nauka, Moskow, 1986 (English translation: Springer, Berlin, 1987).

[11] Gardner C S, Green J M, Kruskal M D and Miura R M, Method for Solving the Korteweg-de Vries Equation, *Phys. Rev. Lett.* **19** (1967), 1095–1097.

[12] Khruslov E Ya, Asymptotics of the Solution of the Cauchy Problem for the Korteweg–de Vries Equation with Step-Like Initial Data, *Mat. Sbornik* **99(141)**, 2 (1976), 261–281 (in Russian).

[13] Khruslov E Ya and Kotlyarov V P, Soliton Asymptotics of Nondecreasing Solutions of Nonlinear Completely Integrable Equations, Advances in Soviet Mathematics: Spectral Operator Theory and Related Topics**19**, Amer. Math. Soc., 1994, 126–183.

[14] Khruslov E Ya and Stephan H, Splitting of Some Non-Localized Solutions of the Korteweg–de Vries Equation into Solitons, *Mat. Fiz. Anal. Geom.* **5**, 1/2 (1998), 49–67.

[15] Ostapenko D Yu, Pal-Val A P and Khruslov E Ya, Uniform Asymptotic Formulas for Curvilinear Solitons of the Kadomtsev–Petviashvili Equations, *Teoret. Mat. Fiz.* **108**, 2 (1996), 205–211 (English translation: *Theoret. and Math. Phys.* **108**, 2 (1996), 1013–1018).

[16] Schuur P C, Asymptotic Analysis of Soliton Problems. An Inverse Scattering Approach, Lecture Notes in Math. **1232**, Springer-Verlag, Berlin, 1986.

[17] Shabat A B, The Korteweg–de Vries Equation, *Dokl. Akad. Nauk SSSR* **211** (1973), 1310–1313 (English translation: *Sov. Math. Dokl.* **14** (1973), 1266–1269).

[18] Tanaka Sh, Korteweg–de Vries Equation; Asymptotic Behavior of Solutions, *Publ. Res. Inst. Math. Sci.* **10** (1974/75), 367–369.

[19] Zakharov V E, Manakov S V, Novikov S P and Pitaevskiĭ P, Theory of Solitons. The Inverse Scattering Method, Nauka, Moskow, 1980 (English translation: Plenum, New York, 1984).