

# On Integrability of Differential Constraints Arising from the Singularity Analysis

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## Abstract

Integrability of differential constraints arising from the singularity analysis of two (1+1)-dimensional second-order evolution equations is studied. Two nonlinear ordinary differential equations are obtained in this way, which are integrable by quadratures in spite of very complicated branching of their solutions.

The Weiss-Kruskal algorithm of singularity analysis [1], [2] is an effective tool for testing the integrability of PDEs. Certainly, most of PDEs do not pass the Painlevé test. If the test fails at its first step, when the leading exponents are determined, it is often possible to improve the dominant behavior of solutions by a transformation of dependent variables and then continue running the test. If the test fails at its second step, when the positions of resonances are determined, one still may hope to improve those positions by a transformation of hodograph type [3] and continue the analysis again. But if the test fails at its third step, when the consistency of recursion relations is checked, one only have to introduce some logarithmic terms into the expansions of solutions, and this is generally believed to be a clear symptom of nonintegrability of the tested PDE [4].

If the recursion relations are not consistent at a resonance, there arise some differential constraints on those arbitrary functions which have entered the singular expansions at lower resonances. A conjecture exists, formulated by Weiss [5], that the differential constraints, arising in the singularity analysis of nonintegrable equations, are always integrable themselves. In the present paper, we study this interesting conjecture on the basis of the following two (1+1)-dimensional second-order evolution equations:

$$u_t = u_{xx} + uu_x - u^3, \quad (1)$$

$$u_t = u_{xx} - u^2. \quad (2)$$

Let us study the equation (1) first. Starting the Weiss-Kruskal algorithm, we determine that a hypersurface  $\phi(x, t) = 0$  is non-characteristic for (1) if  $\phi_x \neq 0$  (we set  $\phi_x = 1$ ), and that the general solution of (1) depends on two arbitrary functions of one variable. Then we substitute the expansion

$$u = u_0(t)\phi^\alpha + \dots + u_n(t)\phi^{n+\alpha} + \dots \quad (3)$$

into (1) and find admissible branches and positions of resonances in them. Besides the Taylor expansions with  $\alpha = 0, 1$ , governed by the Cauchy-Kovalevskaya theorem, the equation (1) admits the following two singular branches:

$$\alpha = -1, \quad u_0 = 1, \quad (n+1)(n-3) = 0; \quad (4)$$

$$\alpha = -1, \quad u_0 = -2, \quad (n+1)(n-6) = 0; \quad (5)$$

both being generic.

In the case (4), the coefficients of (3) are determined by the recursion relations

$$(n-1)(n-2)u_n + \frac{1}{2}(n-2) \sum_{i=0}^n u_i u_{n-i} - \sum_{i=0}^n \sum_{j=0}^{n-i} u_i u_j u_{n-i-j} - (n-2)\phi' u_{n-1} - u'_{n-2} = 0, \quad n = 0, 1, 2, \dots, \quad (6)$$

where  $u_{-2} = u_{-1} = 0$  formally, and the prime denotes the derivative with respect to  $t$ . We find from (6) that  $u_1 = \frac{1}{4}\phi'$  at  $n = 1$ ,  $u_2 = -\frac{1}{16}\phi'^2$  at  $n = 2$ , whereas  $u_3(t)$  is not determined and the differential constraint (or ‘‘compatibility condition’’) on  $\phi$ ,

$$\phi'' = \frac{1}{2}\phi'^3, \quad (7)$$

arises at  $n = 3$ . Consequently, the equation (1) does not pass the Painlevé test for integrability. We see, however, that the differential constraint (7) is integrable by quadratures, and this supports the conjecture of Weiss.

For the branch (5), we have the same recursion relations (6) and obtain  $u_1, \dots, u_5$  from them. Then, at  $n = 6$ ,  $u_6(t)$  remains undetermined, but the following differential constraint on  $\phi$  arises:

$$\phi' \phi''' + \frac{4}{9}\phi''^2 - \frac{173}{45}\phi'^3 \phi'' + \frac{142}{225}\phi'^6 = 0. \quad (8)$$

By introducing the new variable  $v(t) = \phi'$ , we rewrite (8) as the second-order ODE

$$vv'' + \frac{4}{9}v'^2 - \frac{173}{45}v^3 v' + \frac{142}{225}v^6 = 0. \quad (9)$$

Is this ODE integrable? Let us try to answer this question, using the Ablowitz-Ramani-Segur algorithm of the singularity analysis of ODEs [6], the predecessor of the Weiss-Kruskal algorithm. Substituting into (9) the expansion

$$v = v_0 \psi^\beta + \dots + v_n \psi^{n+\beta} + \dots \quad (10)$$

with  $\psi' = 1$  and constant coefficients  $v_i$ ,  $i = 0, \dots, n, \dots$ , we find the following singular branches and positions of resonances in them:

$$\beta = \frac{9}{13}, \quad \forall v_0 : v_0 \neq 0, \quad (n+1)n = 0; \quad (11)$$

$$\beta = -\frac{1}{2}, \quad v_0^2 = -\frac{5}{2}, \quad \left(n + \frac{37}{6}\right)(n+1) = 0; \quad (12)$$

$$\beta = -\frac{1}{2}, \quad v_0^2 = -\frac{155}{284}, \quad (n+1)\left(n - \frac{1147}{852}\right) = 0. \quad (13)$$

We see that the ODE (9) does not possess the Painlevé property. In principle, the rational values of  $\beta$  and  $n$  (11)–(13) allow the equation (9) to possess the weak Painlevé property, but we will not study this possibility because the weak Painlevé property not always implies integrability [4].

Before proceeding with the integrability of (9), let us obtain one more ODE, with even worse analytic properties, from the PDE (2). Using the expansion (3), we find that the equation (2) admits the singular generic branch

$$\alpha = -2, \quad u_0 = 6, \quad (n+1)(n-6) = 0. \quad (14)$$

Then, constructing the recursion relations and checking their consistency at the resonance  $n = 6$ , we obtain the differential constraint

$$\phi' \phi''' + \frac{3}{8} \phi''^2 - \frac{27}{20} \phi'^3 \phi'' + \frac{9}{200} \phi'^6 = 0. \quad (15)$$

Consequently, the PDE (2) does not pass the Painlevé test for integrability. Again, we rewrite (15) by  $v(t) = \phi'$  in the form

$$vv'' + \frac{3}{8} v'^2 - \frac{27}{20} v^3 v' + \frac{9}{200} v^6 = 0, \quad (16)$$

use the expansion (10), and find for the ODE (16) the following singular branches and positions of resonances in them:

$$\beta = \frac{8}{11}, \quad \forall v_0 : v_0 \neq 0, \quad (n+1)n = 0; \quad (17)$$

$$\beta = -\frac{1}{2}, \quad v_0^2 = \frac{5}{2}(-3 - \sqrt{6}), \quad \left(n + \frac{27}{8}(2 + \sqrt{6})\right)(n+1) = 0; \quad (18)$$

$$\beta = -\frac{1}{2}, \quad v_0^2 = \frac{5}{2}(-3 + \sqrt{6}), \quad (n+1)\left(n + \frac{27}{8}(2 - \sqrt{6})\right) = 0. \quad (19)$$

Like the equation (9), the ODE (16) does not possess the Painlevé property. But, unlike the ODE (9), which can possess the weak Painlevé property, the equation (16) is characterized by irrational positions of resonances, i.e. its solutions exhibit some infinite branching.

Thus, due to the results of the Painlevé test, we should not expect the equations (9) and (16) to be integrable. Nevertheless, these two ODEs turn out to be integrable by quadratures.

Let us consider the ODE

$$vv'' + av'^2 + bv^3v' + cv^6 = 0 \quad (20)$$

with constant  $a, b, c$ . After the change of variables

$$\{t, v(t)\} \rightarrow \{v, w(v)\} : v' = w(v), \quad (21)$$

which can be inverted as  $t = \int \frac{dv}{w(v)}$ , the ODE (20) becomes

$$vww_v + aw^2 + bv^3w + cv^6 = 0. \quad (22)$$

Changing the variables again,

$$\{v, w(v)\} \rightarrow \{y, z(y)\} : v = f(y), w = g(y)z(y), \quad (23)$$

we rewrite (22) in the form

$$zz_y + \left(\frac{gy}{g} + \frac{af_y}{f}\right)z^2 + \frac{bf^2f_y}{g}z + \frac{cf^5f_y}{g^2} = 0. \quad (24)$$

Then, if  $a \neq -3$ , we choose

$$f = \exp\left(-\frac{y}{a+3}\right), \quad g = \epsilon f^3, \quad \epsilon = \text{constant} \neq 0 \quad (25)$$

in (24) and obtain the ODE

$$zz_y - z^2 - \frac{b}{\epsilon(a+3)}z - \frac{c}{\epsilon^2(a+3)} = 0, \quad (26)$$

which is integrable by quadrature. In the case of (9), setting

$$f = \exp\left(-\frac{9}{31}y\right), \quad g = -\frac{1}{155}\exp\left(-\frac{27}{31}y\right), \quad (27)$$

we have

$$y = \int \frac{z dz}{(z+31)(z+173)}. \quad (28)$$

In the case of (16), setting

$$f = \exp\left(-\frac{8}{27}y\right), \quad g = \frac{1}{15}\exp\left(-\frac{24}{27}y\right), \quad (29)$$

we have

$$y = \int \frac{z dz}{z^2 - 6z + 3}. \quad (30)$$

The integrals (28) and (30) reveal the origin of the observed complicated branching of solutions of (9) and (16).

Consequently, the differential constraints (7), (8) and (15) are integrable by quadratures, and this supports the conjecture of Weiss. This interesting conjecture deserves further study based on more examples of nonintegrable PDEs.

The ODEs (9) and (16) may illustrate the fact that the analytic properties of integrable equations can be very complicated. In this relation, the following two papers should also be mentioned. Lemmer and Leach [7] studied the class of ODEs  $Y'' + YY' + KY^3 = 0$  with constant  $K$  (this class is equivalent to (20) with  $a = 1$ ) and found that the Lie integrability in quadratures is wider than the Painlevé integrability. More recently, Ramani, Grammaticos and Tremblay [8] found that the Painlevé property is not necessary for integrability of a large class of linearizable systems.

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