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The Matrix Kadomtsev–Petviashvili Equation as a Source of Integrable Nonlinear Equations

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Abstract

A new integrable class of Davey–Stewartson type systems of nonlinear partial differential equations (NPDEs) in $2+1$ dimensions is derived from the matrix Kadomtsev–Petviashvili equation by means of an asymptotically exact nonlinear reduction method based on Fourier expansion and spatio-temporal rescaling. The integrability by the inverse scattering method is explicitly demonstrated, by applying the reduction technique also to the Lax pair of the starting matrix equation and thereby obtaining the Lax pair for the new class of systems of equations. The characteristics of the reduction method suggest that the new systems are likely to be of applicative relevance. A reduction to a system of two interacting complex fields is briefly described.

1 Introduction

New classes of evolution nonlinear partial differential equations (NPDEs) integrable by the inverse scattering method (S-integrable) have been found in the last years. These equations are known to be applicable to various branches of physics such as fluid dynamics, nonlinear optics, condensed matter physics and so on. The most famous examples are the Korteweg-de Vries and the nonlinear Schrodinger equations in $1+1$ dimensions and the Kadomtsev–Petviashvili and the Davey–Stewartson equations in $2+1$ dimensions [1].

A simple explanation of this coincidence (integrability and applicative relevance) is based on the observation that very large classes of evolution NPDEs in $1+1$ and $2+1$ dimensions, with a dispersive linear part, can be reduced, by a limiting procedure involving the wave modulation induced by weak nonlinear effects, to a very limited number of "universal" evolution NPDEs. Moreover, the same model equations obtained in this way appear in many applicative situations (for instance in plasma physics, nonlinear optics, hydrodynamics, etc.), where weakly nonlinear effects are important [2–5].

The reduction method preserves integrability and therefore the model equations are likely to be integrable. For example, it is sufficient that the very large class of equations from which they are obtainable contains just one S-integrable equation, provided the limiting procedure preserves integrability, so that the property of S-integrability is inherited through this limiting technique. Obviously, the last statement about the integrability is based on heuristic considerations and could not be characterized as a rigorous theorem.

No precise definition of integrability is available for evolution NPDEs, there being much difference between finding the general solution of a NPDE or solving an initial-value problem with given input data and boundary conditions. It would be possible to derive the spectral transform of the Davey–Stewartson equation from the spectral transform of the Kadomtsev–Petviashvili equation.

Thus this approach, besides explaining why certain model equations are integrable and applicable, provides a powerful tool to investigate the relation among different integrable equations, to test the integrability of nonlinear evolution PDEs and, most importantly, to identify integrable evolution equations that are likely to be of applicative relevance.

In previous papers, we applied this method to certain integrable equations in $2+1$ dimensions. The most interesting results are that the Davey-Stewarston equation [6–7] is the typical model equation in $2+1$ dimensions, while new integrable NPDEs can be obtained together with their Lax pair [8–11]. Moreover, we used the reduction method to derive two equations of applicative relevance in plasma physics [12–13].

The basic idea of the reduction method is to consider a nonlinear evolution PDE whose linear part is dispersive; as it is well known the linear evolution is most appropriately described in terms of Fourier modes and each Fourier mode evolves with constant amplitude and an associated group velocity, that represents the speed with which a wave packet peaked at that Fourier mode would move in configuration space. To evaluate the weak nonlinear effects it is convenient to consider a specific Fourier mode and follow it by going over to a frame of reference that moves with its group velocity. The weak nonlinear effects give rise to a modulation of the amplitude of that Fourier mode (that would remain constant in the absence of nonlinear effects). The modulation is best described in terms of rescaled "coarse-grained" and "slow" variables, that display the weak nonlinear effects on larger space and time scales; indeed, the first step of the reduction method is to use a moving frame of reference with the introduction of the slow variables:

$$\begin{aligned}
&\xi = \varepsilon^p (x - V_1 t), \qquad \eta = \varepsilon^p (y - V_2 t), \qquad \tau = \varepsilon^q t, \\
&p > 0, \qquad q > 0,
\end{aligned} \tag{1.1}$$

where $V_1 = V_1(K_1, K_2)$, $V_2 = V_2(K_1, K_2)$ are the components of the group velocity $\underline{V}\,(\underline{K}) \equiv (V_1(K_1, K_2), V_2(K_1, K_2))$ of the linearized equation, i.e. of the equation obtained by neglecting all the nonlinear terms, and ε is a "small" expansion parameter.

It is thereby seen that the function that represents the amplitude modulation satisfies, in terms of the rescaled, slow, variables, evolution equations having a universal character; since the coarse-grained nature of the new variables implies that only certain general features of the nonlinear interaction are important.

In this paper we expose an interesting extension of this approach and consider the matrix Kadomtsev–Petviashvili equation [14–15]

$$\begin{aligned}
&U_t + U_{xxx} - W_y + i\sqrt{3}\,[W, U] - 3\,\{U, U_x\} = 0, \\
&W_x = U_y,
\end{aligned} \tag{1.2}$$

where $[A, B] = AB - BA$, $U = U(x, y, t)$, $W = W(x, y, t)$ are $N \otimes N$ complex matrices and the subscripts denote partial differentiation.

By applying the reduction method, a new class of integrable matrix systems of evolution NPDEs depending on a real parameter λ is obtained

$$\begin{aligned}
&i\Psi_\tau + L\Psi - \lambda\left[\Psi,\Lambda\right] - \left[\Omega,\Psi\right] + \sqrt{3}\left\{\Lambda,\Psi\right\} - \left\{\Phi,\Psi^2\right\} = 0,\\
&i\Phi_\tau - L\Phi - \lambda\left[\Phi,\Lambda\right] - \left[\Omega,\Phi\right] - \sqrt{3}\left\{\Lambda,\Phi\right\} + \left\{\Psi,\Phi^2\right\} = 0,\\
&\left(3-\lambda^2\right)\Lambda_\xi + 2\lambda\Lambda_\eta - \Omega_\eta - \sqrt{3}\left\{\Psi,\Phi\right\}_\xi + \left[\Psi,\Phi\right]_\eta + \lambda\left[\Phi,\Psi\right]_\xi = 0,\\
&\Lambda_\eta = \Omega_\xi,
\end{aligned} \tag{1.3}$$

where $\{A,B\} = AB + BA$, the linear differential operator L is given by

$$L = -\left(3+\lambda^2\right)\frac{\partial^2}{\partial\xi^2} + 2\lambda\frac{\partial^2}{\partial\xi\partial\eta} - \frac{\partial^2}{\partial\eta^2}, \tag{1.4}$$

and $\Psi = \Psi(\xi,\eta,\tau)$, $\Phi = \Phi(\xi,\eta,\tau)$, $\Lambda = \Lambda(\xi,\eta,\tau)$ and $\Omega = \Omega(\xi,\eta,\tau)$are $N \otimes N$ complex matrices.

The paper is organized as follows. In the next section we apply the reduction method to the starting equation (1.2) and obtain the new system of matrix equations (1.3)–(1.4). Moreover, we reduce the matrix system of equations to a new integrable two-component complex fields system of nonlinear equations, which, in the one-component case, reduces to the standard Davey–Stewartson equation. In Section 3 we discuss in some detail how the reduction method can be applied to the Lax pair of the equation (1.2) and we derive the Lax pair of the system of matrix equations (1.3)–(1.4). Finally in the last section we recapitulate the most important results and indicate some possible extensions.

2 A new integrable matrix system in 2 + 1 dimensions

Lemma 2.1. *The linear dispersive part of the starting equation (1.2) admits as a solution a Fourier mode, with a group velocity* $\underline{V}(\underline{K}) = (V_1(K_1,K_2), V_2(K_1,K_2))$,

$$V_1(K_1,K_2) = -3K_1^2 + \frac{K_2^2}{K_1^2}, \qquad V_2(K_1,K_2) = -2\frac{K_2}{K_1},$$

where

$$*\underline{V}(\underline{K}) = \frac{\partial\omega}{\partial\underline{K}}$$

and $\omega = \omega(K_1,K_2) = -K_1^3 - \frac{K_2^2}{K_1}$ *is the dispersion relation.*

Proof. It is sufficient to substitute the plane wave into the linear part of the matrix KP equation. ■

We use the transformation (1.1) and introduce the following formal asymptotic Fourier expansion

$$U(x,y,t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\xi,\eta,\tau;\varepsilon)\exp\{i(nz)\}, \tag{2.1}$$

where $z = K_1 x + K_2 y - \omega t$, $\gamma_n = |n|$ for $n \neq 0$, and $\gamma_0 = r$ is a non negative rational number which will be fixed later. The unknown functions ψ_n depend on ε and it is supposed that their limit for $\varepsilon \to 0$ exists and is finite; in the following this limit will be denoted with $\psi_n(\xi,\eta,\tau)$. Moreover we suppose that they can be expanded in power series of , i.e.

$$\psi_n(\xi,\eta,\tau;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \psi_n^{(i)}(\xi,\eta,\tau), \qquad \psi_n(\xi,\eta,\tau) = \psi_n^{(0)}(\xi,\eta,\tau).$$

We now introduce an analogous Fourier expansion

$$W(x,y,t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\tilde{\gamma}_n} \varphi_n(\xi,\eta,\tau;\varepsilon) \exp\{i(nz)\} \tag{2.2}$$

and obtain

$$\varphi_n = (K_2)(K_1)^{-1}\psi_n + O(\varepsilon^p).$$

In the following for simplicity we use the abbreviations $\psi_1^{(0)} = \Psi$, $\psi_{-1}^{(0)} = \Phi$, $\psi_0^{(0)} = \Lambda$ (and $\phi_n^{(0)} = \phi_n$, $\phi_0^{(0)} = \Omega$).

The final goal is to obtain the evolution equation satisfied by the modulation amplitudes $\Psi = \Psi(\xi,\eta,\tau)$ and $\Phi(\xi,\eta,\tau)$ and to understand how it is modified by choosing different wave numbers.

Proposition 2.1. *The matrix system (1.3)–(1.4) can be obtained applying the reduction technique to the matrix Kadomtsev-Petviashvili system (1.2).*

Proof. We insert the expansions (2.1) and (2.2) into the equation (1.2) and consider the different equations obtained by considering the coefficients of the Fourier modes.

It is convenient to separate the contributions of the linear and nonlinear parts by writing

$$\varepsilon^{\gamma_n} D_n \psi_n = \varepsilon^2 F_n,$$

where D_n is a linear differential operator acting on $\psi_n(\xi,\eta,\tau)$ and F_n is the contribution of the nonlinear part. The operator D_n is

$$\begin{aligned} D_n = {} & (-in\omega + \varepsilon^q \partial_\tau - V_1 \varepsilon^p \partial_\xi - V_2 \varepsilon^p \partial_\eta) + (inK_1 + \varepsilon^p \partial_\xi)^3 - (inK_2 + \varepsilon^p \partial_\eta) \\ & - (i/K_1)\left(\varepsilon^p \partial_\eta - (K_2/K_1)\partial_\xi\right) + \left(1/K_1^2\right) \varepsilon^{2p} \left(\partial_{\xi\eta} - (K_2/K_1)\partial_{\xi\xi}\right). \end{aligned}$$

F_n can be derived, by assessing the importance of the different terms, which originate from the nonlinear interaction of the Fourier amplitudes $\psi_n(\xi,\eta,\tau)$:

$$\begin{aligned} & F_2 = 6iK_1\Psi^2 + O(\varepsilon^p), \\ & F_0 = \varepsilon^p \left(3\{\Psi,\Phi\}_\xi - i\sqrt{3}\left[\varphi_1^{(p)},\Phi\right] - i\sqrt{3}\left[\varphi_{-1}^{(p)},\Psi\right]\right) + O\left(\varepsilon^{2p},\varepsilon^2\right), \\ & F_1 = \varepsilon^{r-1}\left(-i\sqrt{3}[\Omega,\Psi] - i\sqrt{3}\frac{K_2}{K_1}[\Psi,\Lambda] + 3iK_1\{\Lambda,\Psi\}\right) \\ & \qquad + 3i\varepsilon K_1\{\psi_2,\Phi\} + O\left(\varepsilon^{r+p-1},\varepsilon^3\right), \end{aligned}$$

and so on.

By setting $q = 2$, $p = 1$, $r = 2$ for the proper balance of terms, we obtain the equations for the Fourier modes at the lowest order for $n = 0$, $n = 1$ and $n = 2$:

$$\begin{aligned}
&\psi_2 = -\frac{1}{K_1^2}\Psi^2,\\
&\left(3K_1^2 - \frac{K_2^2}{K_1^2}\right)\Lambda_\xi + 2\frac{K_2}{K_1}\Lambda_\eta - \Omega_\eta - 3\{\Psi,\Phi\}_\xi\\
&\qquad + i\sqrt{3}\left[\varphi_1^{(p)},\Phi\right] + i\sqrt{3}\left[\varphi_{-1}^{(p)},\Psi\right] = 0,\\
&\Psi_\tau + i\left(3K_1 + \frac{K_2^2}{K_1^3}\right)\Psi_{\xi\xi} + \frac{i}{K_1}\Psi_{\eta\eta} - 2i\frac{K_2}{K_1^2}\Psi_{\xi\eta}\\
&\qquad + i\sqrt{3}[\Omega,\Psi] + i\sqrt{3}\frac{K_2}{K_1}[\Psi,\Lambda] - 3iK_1\{\Lambda,\Psi\} - 3iK_1\{\psi_2,\Phi\} = 0.
\end{aligned}$$

Finally, after the cosmetic rescaling

$$\begin{aligned}
&\sqrt{3}K_1\Lambda \to \Lambda, \qquad \sqrt{3}\Omega \to \Omega, \qquad \sqrt{\frac{3}{K_1}}\Psi \to \Psi,\\
&\sqrt{\frac{3}{K_1}}\Phi \to \Phi, \qquad \lambda = \frac{K_2}{K_1^2}, \qquad \xi' = \xi/\sqrt{K_1}, \qquad \eta' = K_1\eta,
\end{aligned} \tag{2.3}$$

we arrive at the matrix system of nonlinear evolution equations (1.3)–(1.4). ■

This matrix system must be integrable by the spectral transform, because it has been derived from an S-integrable equation. This is explicitly demonstrated in the next section.

Let us now look in more detail at the integrable NPDEs implied these results. If we take $\Phi = \Psi^*$, $N = 1$, we obtain the equation

$$i\Psi_\tau + L_1\Psi + \chi\Psi = 0, \qquad L_2\chi = 2L_1|\Psi|^2, \tag{2.4}$$

with

$$\begin{aligned}
&\chi_\eta = 2|\Psi|^2_\eta + 2\sqrt{3}\Omega_\xi,\\
&L_1 = -\left(3+\lambda^2\right)\frac{\partial^2}{\partial\xi^2} + 2\lambda\frac{\partial^2}{\partial\xi\partial\eta} - \frac{\partial^2}{\partial\eta^2},\\
&L_2 = \left(\lambda^2 - 3\right)\frac{\partial^2}{\partial\xi^2} - 2\lambda\frac{\partial^2}{\partial\xi\partial\eta} - \frac{\partial^2}{\partial\eta^2}.
\end{aligned}$$

The NPDE (2.4), up to trivial rescalings, coincides with the Davey–Stewartson equation [6], whose integrability is well known [7]. Note that the S-integrable equations found in [10–11] are different from the standard Davey–Stewartson equation and then not connected with the S-integrable system (1.3)–(1.4).

In the case $N = 2$, we get a nonlinear system for eight interacting fields. However, an interesting reduction is possible, if we set

$$\Psi = \begin{pmatrix}\psi_1 & \psi_2\\ \psi_2 & \psi_1\end{pmatrix}, \quad \Phi = \begin{pmatrix}\varphi_1 & \varphi_2\\ \varphi_2 & \varphi_1\end{pmatrix}, \quad \Lambda = \begin{pmatrix}\Lambda_1 & \Lambda_2\\ \Lambda_2 & \Lambda_1\end{pmatrix}, \quad \Omega = \begin{pmatrix}\Omega_1 & \Omega_2\\ \Omega_2 & \Omega_1\end{pmatrix}.$$

From the matrix system (1.3)–(1.4), we obtain $\Psi = \Phi^*$ and

$$i\psi_{1,\tau} + L_1\psi_1 + 2\sqrt{3}(\Lambda_1\psi_1 + \Lambda_2\psi_2) - 2\left(|\psi_1|^2\psi_1 + \psi_1^*\psi_2^2 + 2\psi_1|\psi_2|^2\right) = 0,$$
$$i\psi_{2,\tau} + L_1\psi_2 + 2\sqrt{3}(\Lambda_1\psi_2 + \Lambda_2\psi_1) - 2\left(|\psi_2|^2\psi_2 + \psi_1^2\psi_2^* + 2\psi_2|\psi_1|^2\right) = 0,$$
$$L_2\Lambda_1 = L_3\left(|\psi_1|^2 + |\psi_2|^2\right), \qquad L_2\Lambda_2 = L_3\left(\psi_1\psi_2^* + \psi_2\psi_1^*\right), \tag{2.5}$$

where

$$L_1 = -\left(3+\lambda^2\right)\partial_\xi^2 + 2\lambda\partial_{\xi\eta}^2 - \partial_\eta^2,$$
$$L_2 = \left(3-\lambda^2\right)\partial_\xi^2 + 2\lambda\partial_{\xi\eta}^2 - \partial_\eta^2, \qquad L_3 = 2\sqrt{3}\partial_\xi^2.$$

Integrable Davey–Stewartson type equations and system of equations have been extensively investigated by many authors [16–20]. A detailed list of Davey-Stewartson systems and equations integrable by the inverse scattering method has been recently given [21]. The system of equations (2.5) does not appear in these papers. We expect that this new system be integrable by the inverse scattering method, because it has been obtained from an integrable equation and the property of integrability is expected to be maintained through the application of the reduction method. The integrability of the system of equations (1.3)–(1.4), and of the system (2.5) which is a particular case, is demonstrated in the next section.

3 The Lax pair for the integrable system of equations

In this section we apply the reduction method also to the Lax pair of the starting matrix equation (1.2), to demonstrate explicitly the integrability by the spectral transform of the matrix system (1.3)–(1.4), and we thereby identify the Lax pair for the system of equations (1.3)–(1.4).

Lemma 3.1. *The Lax operators (L, A) of the matrix KP equation are*

$$L = \frac{i}{\sqrt{3}}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} - U(x,y,t), \qquad L\phi(x,y,t) = 0, \tag{3.1}$$

$$A = 4\frac{\partial^3}{\partial x^3} - 6U(x,y,t)\frac{\partial}{\partial x} - 3U_x(x,y,t) + i\sqrt{3}W, \tag{3.2}$$

with

$$\phi_t(x,y,t) + A\phi(x,y,t) = 0. \tag{3.3}$$

Proof. It can be verified by direct substitution that the operator relation

$$L_t = i[L,A] = i(LA - AL)$$

reproduces equation (1.2). ■

Proposition 3.1. *The matrix system (1.3)–(1.4) is S-integrable and its Lax pair* (L, A) *is*

$$L\hat{\phi} = 0, \tag{3.4}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \qquad L_{11} = I\left(i\partial_\eta + i(\sqrt{3}-\lambda)\partial_\xi\right), \qquad L_{12} = -\Psi,$$

$$L_{21} = -\Phi, \qquad L_{22} = I\left(i\partial_\eta - i(\sqrt{3}+\lambda)\partial_\xi\right), \qquad \hat{\phi} = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}, \tag{3.5}$$

I is the $N \otimes N$ unit matrix and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3.6}$$

where

$$A_{11} = 6i\partial_\xi^2 I + i\Omega - i\left(\sqrt{3}+\lambda\right)\Lambda + i\left(1 + \frac{\lambda\sqrt{3}}{12}\right)\Phi\Psi,$$

$$A_{12} = -2\sqrt{3}\Psi\partial_\xi - \left(\sqrt{3}+\lambda\right)\Psi_\xi + \Psi_\eta,$$

$$A_{12} = -2\sqrt{3}\Phi\partial_\xi - \left(\sqrt{3}-\lambda\right)\Phi_\xi - \Phi_\eta,$$

$$A_{22} = -6iI\partial_\xi^2 + i\Omega + i\left(\sqrt{3}-\lambda\right)\Lambda + i\left(\frac{\lambda\sqrt{3}}{12} - 1\right)\Psi\Phi.$$

Proof. Let us apply the reduction method to the Lax pair (3.1)–(3.3) of equation (1.2).

The components $\phi_j(x, y, t)$, $j = 1, \ldots, N$, of the column vector $\phi(x, y, t)$ can be expanded in Fourier modes as follows

$$\phi_j(x, y, t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \phi_{j,n}(\xi, \eta, \tau; \varepsilon) \exp\left[i\left((\lambda_1 x + \lambda_2 y + \lambda_3 t) + \frac{n}{2}z\right)\right], \tag{3.7}$$

where $z = K_1 x + K_2 y - \omega t$, the $\phi_{j,n}(\xi, \eta, \tau; \varepsilon)$ depend parametrically on ε and remain finite when $\varepsilon \to 0$, the γ_n are non negative rational numbers and λ_m, $m = 1, \ldots, 3$, are real constants to be determined.

Inserting now the expression for $\phi_j(x, y, t)$ in (3.1), we derive a series of relations which are generated by the coefficients of the Fourier modes. Each relation must be valid for a given order of approximation in ε.

In particular, for the fundamental harmonics $n = \pm 1$,considering terms $O\left(\varepsilon^0\right)$ in (3.1) and (3.3), we obtain

$$\frac{i}{\sqrt{3}}\left(\pm\frac{iK_2}{2} + i\lambda_2\right) + \left(\pm\frac{iK_1}{2} + i\lambda_1\right)^3 = 0,$$

$$\left(\mp\frac{i\omega}{2} + i\lambda_3\right) + 4\left(\pm\frac{iK_1}{2} + i\lambda_1\right)^3 = 0,$$

and then

$$\lambda_1 = -\frac{K_2}{2K_1\sqrt{3}}, \qquad \lambda_2 = -\frac{\sqrt{3}}{4}\left(\frac{K_2^2}{3K_1^2} + K_1^2\right), \qquad \lambda_3 = -\frac{K_2^3}{6K_1^3\sqrt{3}} - \frac{\sqrt{3}}{2}K_1K_2.$$

We thereby understand that the harmonics

$$\phi_{j,1}, \qquad \phi_{j,-1}, \qquad j = 1, \ldots, N, \tag{3.8}$$

are fundamental, i.e. for them γ_n assumes the smallest value, $\gamma_n = 0$.

The successive order ε for the equation (3.1) allow us to obtain the new spectral problem, because all the $\phi_{j,n}$ may be expressed by means of the fundamental harmonics (3.8), which are connected through the relations:

$$i\phi_{+,\eta} + i\left(\sqrt{3} - \lambda\right)\phi_{+,\xi} - \Psi\phi_- = 0, \tag{3.9}$$

$$i\phi_{-,\eta} - i\left(\sqrt{3} + \lambda\right)\phi_{-,\xi} - \Phi\phi_+ = 0, \tag{3.10}$$

where we set $(\phi_{j,1};\ j = 1, \ldots, N) = \phi_+$, $(\phi_{j,-1};\ j = 1, \ldots, N) = \phi_-$.

By means of the variable rescaling (2.3), and by introducing the $2N \otimes 2N$ matrix operator L, we arrive at the final form (3.4)–(3.5).

To calculate the temporal evolution, we must insert the expression (3.7) in (3.3) and consider the relation obtained for the different harmonics n and for a given order of approximation in ε. If we consider the first order in ε, we obtain again the spectral problem (3.4)–(3.5). Only if we take into account the next orders of approximation of equation (3.3), i.e. the order ε^2, the temporal evolution can be determined. However, new quantities, the corrections $\tilde{\phi}_\pm(\xi, \eta, \tau)$ of order ε to the fundamental harmonics $\phi_\pm(\xi, \eta, \tau)$, appear. These unknown quantities can be eliminated in the equation (3.3) by taking advantage of the relation obtained from equation (3.1), considering terms of order ε^2. This elimination is possible only because equations (3.1) and (3.3) are identical at the order ε. In particular, if we consider (3.3) calculated to the order ε^2 for $n = \pm 1$, we get

$$\begin{aligned}
&\phi_{+,\tau} + 12i\left(\frac{K_1}{2} + \lambda_1\right)\phi_{+,\xi\xi} - 6\Psi\phi_{-,\xi} - 3\Psi_\xi\phi_- + \frac{\sqrt{3}}{K_1}\left(\Psi_\eta - \frac{K_2}{K_1}\Psi_\xi\right)\phi_- \\
&\quad + i\sqrt{3}\Omega - 6i\Lambda\left(\frac{K_1}{2} + \lambda_1\right)\phi_+ + i\left(-6\lambda_1 - 6K_1 + \sqrt{3}\frac{K_2}{K_1}\right)\Phi\phi_{+3} \\
&\quad - \frac{2iK_2\sqrt{3}}{K_1}\left(\frac{i}{\sqrt{3}}\tilde{\phi}_{+,\eta} + i\left(K_1 - \frac{K_2}{K_1\sqrt{3}}\right)\tilde{\phi}_{+,\xi} - \Psi\tilde{\phi}_-\right) = 0, \\
&\phi_{-,\tau} + 12i\left(-\frac{K_1}{2} + \lambda_1\right)\phi_{-,\xi\xi} - 6\Phi\phi_{+,\xi} - 3\Phi_\xi\phi_+ - \frac{\sqrt{3}}{K_1}\left(\Phi_\eta - \frac{K_2}{K_1}\Phi_\xi\right)\phi_+ \\
&\quad + i\sqrt{3}\Omega - 6i\Lambda\left(-\frac{K_1}{2} + \lambda_1\right)\phi_- + i\left(-6\lambda_1 + 6K_1 + \sqrt{3}\frac{K_2}{K_1}\right)\Psi\phi_{-3} \\
&\quad - \frac{2iK_2\sqrt{3}}{K_1}\left(\frac{i}{\sqrt{3}}\tilde{\phi}_{-,\eta} - i\left(K_1 + \frac{K_2}{K_1\sqrt{3}}\right)\tilde{\phi}_{-,\xi} - \Phi\tilde{\phi}_+\right) = 0.
\end{aligned}$$

To evaluate this expression we took advantage of the fact that $\phi_{\pm 3}$ are connected with the fundamental harmonics (these relations are obtained from (3.1) for $n = \pm 3$ at the lower order in ε):

$$\phi_{+3} = \left(\frac{-1}{2K_1^2}\right)\Psi\phi_+, \qquad \phi_{-3} = \left(\frac{-1}{2K_1^2}\right)\Phi\phi_-. \tag{3.11}$$

We now consider the equation (3.1) at the order ε^2 for $n = \pm 1$, which provides the corrections $\tilde{\phi}_+(\xi, \eta, \tau)$, $\tilde{\phi}_-(\xi, \eta, \tau)$. Via the transformation (2.3) and after a lengthy calculation we arrive at the final form (3.6) for the $2N \otimes 2N$ matrix operator A, which satisfies the equation

$$\hat{\phi}_\tau + A\hat{\phi} = 0. \tag{3.12}$$

■

The determination of the Lax pair (3.5) and (3.6), which satisfies the equations (3.4) and (3.12), demonstrates the S-integrability of the system (1.3)–(1.4).

4 Conclusion

We have derived a new, integrable, and presumably of applicative interest, nonlinear matrix system of evolution equations of Davey–Stewartson type from the integrable matrix equation (1.2), by means of an extension of the reduction method based on Fourier expansion and space-time rescalings. It reduces to the standard Davey–Stewartson equation in the single mode case and to a new integrable system of two interacting fields in the $N = 2$ case. Moreover, we have applied the reduction method to the Lax pair (3.1)–(3.3) of the original equation and have demonstrated the integrability property of the new matrix system of equations, by exhibiting the corresponding Lax pair (3.8)–(3.9) and (3.11)–(3.12).

We have outlined the approach that permits to obtain such system of equations and the next steps will be the explicit resolution of the spectral problem and the possible identification of localized or asymptotically finite solutions.

It is also convenient to push the approach beyond its "leading order" application by considering different rescalings in the two spatial variables or looking at special cases when some key parameters vanish, in analogy to the case of the model equations treated in [8–9].

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