Solutions for a Class of the Higher Diophantine Equation*

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Abstract - We studied the Diophantine equation $x^2+4^n=y^9$. By using the elementary method and algebaic number theroy, we obtain the following concusions: (i) Let x be an odd number, one necessary condition which the equation has integer solutions is that $2^{8n}-1\!\!\diagup\!3$ contains some square factors. (ii) Let x be an even number, when $n=9k\,(k\geq 1)$, all integer solutions for the equation are $(x,y)=(0,4^k)$, when $n=9k+4\,(k\geq 0)$, all integer solutions are $(x,y)=(\pm 2^{9k+4},2^{2k+1})$, when $n\equiv 1,2,3,5,6,7,8 \pmod 9$ the equation has no integer solution.

Index Terms - Higher Diophantine equation, integer solutions, integer ring, algebraic number theroy

I. Introduction

Let **Z**, **N** be the sets of integers and positive integers respectively. In this paper, we deal with the solutions (x,y) of diophantine equation

$$Ax^2+B=y^m\left(x,y,m\in \mathrm{N},m\equiv 1 \pmod 2,m>1\right) \qquad (1)$$

where $A,B\in \mathbf{N}$ and A is nonsquare. Some special cases of (1) have been settled. When A=1,B=1 lebsgue [1] has proved that (1) has no integer solution, when $A=2,B=1,m=5$, Nagell [2] has proved that (1) has only integer solutions $(x,y)=(\pm 11,3)$. When $A=1,B=4^k,m=7$, and k=1, 2, 3, 4 (see [3]-[6]) it has been proved that (1) has no integer solution.

Here, we study the solution of $x^2+4^m=y^9$, and give the following conclusions:

Theorem When $A=1,B=4^n,m=9$, the conclusions will be established:

(i) Let x be an odd number, one necessary condition which the equation (1) has integer solutions is that $2^{8n}-1\!\!\diagup\!3$ contains some square factors.

(ii) Let x be an even number, when $n=9k\,(k\geq 1)$, all integer solutions for the equation are $(x,y)=(0,4^k)$, when $n=9k+4\,(k\geq 0)$, all integer solutions are $(x,y)=(\pm 2^{9k+4},2^{2k+1})$, when $n\equiv 1,2,3,5,6,7,8 \pmod 9$, the equation has no integer solution.

II. Preliminaries

Lemma 1 [7] Let M is a unique factorization domain, k is a positive integer, $k\geq 2$, and $\alpha,\beta\in M$, $(\alpha,\beta)=1$, so if $\alpha\beta=\gamma^k,\gamma\in M$, then $\alpha=\varepsilon_1\mu^k,\beta=\varepsilon_2\nu^k,\mu,\nu\in M$, and $\varepsilon_1\varepsilon_2=\varepsilon^k$, where $\varepsilon_1,\varepsilon_2$, ε are units in M.

Lemma 2 For the diophantine equation $x^2+1=2^k y^9$, there are following conclusions:

(i) If $k=0$,the equation only has integer soution $(x,y)=(0,1)$,

(ii) If $k=1$,the equation only has integer soutions $(x,y)=(\pm 1,1)$,

(iii) If $k=2,3,4,5,6,7,8$,all equations have no integer solutions.

proof: (i) (ii) By lemma 1, it is easy to prove, (iii) Obviously, x is an odd number, then $x^2\equiv 1 \pmod 4$, and $x^2+1\equiv 2 \pmod 4$. But if $k=2,3,4,5,6,7,8$ then $x^2+1=2^k y^9\equiv 0 \pmod 4$, This is a contradiction. So $x^2+1=2^k y^9\,(k=2,3,4,5,6,7,8)$ has no integer solutions.

III. Proof of Theorem

(i) First, suppose $x\equiv 1 \pmod 2$, in $Z[i]$, $x^2+4^n=y^9$ can be decomposed into as follows $(x+2^n i)(x-2^n i)=y^9,x,y\in \mathrm{Z}$. Let $\delta=(x+2^n i,x-2^n i)$, be aware of $\delta\,|\,(2x,2^{n+1}i)=2$, δ can only be $1,1+i,2$. But $x\equiv 1 \pmod 2$, so $x+2^n\equiv 1 \pmod 2$, then $\delta\neq 2$. If $\delta=1+i$, then $2=N(1+i)\,|\,N(x+2^k i)=x^2+2^{2n}$. However $x\equiv 1 \pmod 2$, So the integer x does not exist. As a result, $\delta=1$.Thus ,by lemma 1, $x+2^n i=(a+bi)^9,x,a,b\in Z$;

$$x=a^9-36a^7b^2+126a^5b^4-84a^3b^6+9ab^8,$$
$$2^n=b\left(9a^8-84a^6b^2+126a^4b^4-36a^2b^6+b^8\right).$$

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So $b=\pm 1,\pm 2^t\left(1\le t\le n-1\right),\pm 2^n$.

When $b=\pm 1$, $9a^8-84a^6+126a^4-36a^2+1=\pm 2^n$, that is $9a^8-84a^6+126a^4-36a^2=\pm 2^n-1$, so a is odd, Thus $x=a^9-36a^7+126a^5-84a^3+9a$ is even, this contradict with $x\equiv 1\left(\bmod 2\right)$,

When $b=\pm 2^t\left(1\le t\le n-1\right)$, $9a^8-84a^6b^2+126a^4b^4-36a^2b^6+b^8=\pm 2^{n-t}$, so a is even. Thus $x=a^9-36a^7b^2+126a^5b^4-84a^3b^6+9ab^8$ is even, this contradict with $x\equiv 1\left(\bmod 2\right)$,

When $b=2^n$, $9a^8-84a^6b^2+126a^4b^4-36a^2b^6+b^8=1$, that is $9a^8-84a^6b^2+126a^4b^4-36a^2b^6=1-2^{8n}$, so $a^2\left(3a^6-28a^4b^2+42a^2b^4-12b^6\right)=1-2^{8n}/3$, thus, only when $2^{8n}-1/3$ contains some square factors, the equation may have integer solutions,

When $b=-2^n$, $9a^8-84a^6b^2+126a^4b^4-36a^2b^6+b^8=-1$, that is $9a^8-84a^6b^2+126a^4b^4-36a^2b^6=-1-2^{8n}$, so $2^{8n}+1\equiv 0\left(\bmod 3\right)$, however, $2^{8n}+1\equiv 2\left(\bmod 3\right)$, this is a contradiction.

So, when $x\equiv 1\left(\bmod 2\right)$, one necessary condition which the equation has integer solutions is that $2^{8n}-1/3$ contains some square factors.

(ii) Second, suppose $x\equiv 0\left(\bmod 2\right)$, thus $y\equiv 0\left(\bmod 2\right)$. Now make $x=2x_1, y=2y_1$, then the equation can be turned into ${x_1}^2+4^{n-1}=2^7{y_1}^9$, obviously $x_1\equiv 0\left(\bmod 2\right)$, then make $x_1=2x_2$, it can be ${x_2}^2+4^{n-2}=2^5{y_1}^9$, also make $x_2=2x_3$ again, it can be ${x_3}^2+4^{n-3}=2^3{y_1}^9$, also make $x_3=2x_4$ again, it can be ${x_4}^2+4^{n-4}=2{y_1}^9$, now make $x_4=2x_5, y_1=2y_2$, it can be ${x_5}^2+4^{n-5}=2^8{y_2}^9$, then make $x_5=2x_6$, it can be ${x_6}^2+4^{n-6}=2^6{y_2}^9$, also make $x_6=2x_7$ again, it can be ${x_7}^2+4^{n-7}=2^4{y_2}^9$, also make $x_7=2x_8$ again, it can be ${x_8}^2+4^{n-8}=2^2{y_2}^9$ finally, make $x_8=2x_9$,it can be ${x_9}^2+4^{n-9}={y_2}^9$, where $x_1,x_2,x_3,x_4,x_5,x_6,x_7,y_1,y_2\in Z$.

According to such substituted method, it can be concluded:

When $n\equiv 1\left(\bmod 9\right)$, the original equation is equivalent to solving $x^2+4=y^9$, and according to the above-mentioned regularity, it is finally equivalent to solving $x^2+1=2^7y^9$,

When $n\equiv 2\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^2=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2^5y^9$,

When $n\equiv 3\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^3=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2^3y^9$,

When $n\equiv 4\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^4=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2y^9$,

When $n\equiv 5\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^5=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2^8y^7$,

When $n\equiv 6\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^6=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2^6y^9$,

When $n\equiv 7\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^7=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2^4y^9$,

When $n\equiv 8\left(\bmod 9\right)$, the original equation is equivalent to solving $x^2+4^8=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=2^2y^9$,

When $n\equiv 0\left(\bmod 9\right)$, it is equivalent to solving $x^2+4^9=y^9$, and according to the same regularity, it is finally equivalent to solving $x^2+1=y^9$.

Therefore, by lemma 2,

when $n\equiv 1,2,3,5,6,7,8\left(\bmod 9\right)$, the equation has no integer solutions,

when $n\equiv 0,4\left(\bmod 9\right)$, the equation has integer solutions, and when $n\equiv 0\left(\bmod 9\right)$ that is $n=9k\left(k\ge 1\right)$, solutions of the equation will must be $\left(x,y\right)=\left(0,4^k\right)$, when $n\equiv 4\left(\bmod 9\right)$ that is $n=9k+4\left(k\ge 0\right)$, solutions of the equation will must be $\left(x,y\right)=\left(\pm 2^{9k+4},2^{2k+1}\right)$.

IV. Conclusions

When x is an even number, all integer solutions of the mentioned equation are given, but when x is an odd number, the equation has not yet been thoroughly solved. And, in this paper, we only give one necessary condition which the equation has integer solutions. However, the methods using in this paper has important reference significances.

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