Integrable Discretizations of Some Cases of the Rigid Body Dynamics

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Abstract

A heavy top with a fixed point and a rigid body in an ideal fluid are important examples of Hamiltonian systems on a dual to the semidirect product Lie algebra $e(n) = so(n) \ltimes \mathbb{R}^n$. We give a Lagrangian derivation of the corresponding equations of motion, and introduce discrete time analogs of two integrable cases of these systems: the Lagrange top and the Clebsch case, respectively. The construction of discretizations is based on the discrete time Lagrangian mechanics on Lie groups, accompanied by the discrete time Lagrangian reduction. The resulting explicit maps on $e^*(n)$ are Poisson with respect to the Lie–Poisson bracket, and are also completely integrable. Lax representations of these maps are also found.

1   Introduction

The rigid body dynamics are rich with problems interesting from the mathematical point of view, in particular, with integrable problems. Certainly, the most famous ones are the three integrable cases of the rotation of a heavy rigid body around a fixed point in a homogeneous gravity field, named after Euler, Lagrange, and Kovalevskaya. Another problem with a number of integrable cases is the Kirchhoff's one, dealing with the motion of a rigid body in an ideal fluid. Here integrable cases carry the names of Kirchhoff, Clebsch, Steklov and Lyapunov. All these classical cases were discovered in the 18th and the 19th century. However, the list of intergrable problems of the rigid body dynamics is by far not exhausted by these ones. For example, the integrablity of equations describing the rotation of a rigid body around its fixed center of mass in an arbitrary quadratic potential is a much more recent observation due to Reyman [29] and Bogoyavlensky [10]. (However, some particular case of this result was given already by Brun [12]; the equations of motion in this case are identical with those describing the integrable case of the motion of a rigid body in an ideal fluid, due to Clebsch [13]).

It was realized already in the 19th century that some of these problems admit multi-dimensional generalizations. According to bibliographical remarks in [10] and in [15], the multi-dimensional generalization of the Euler top is due to Frahm [16] and Schottky [32]. A modern period of the interest in this system started with the general theory of geodesic

flows on Lie groups, due to Arnold [3], as well as with the paper by Manakov [20], who found the n-dimensional Euler top as a reduction of the so called n wave equations, along with a spectral parameter dependent Lax representation. This was followed by a cycle of papers by Mishchenko and Fomenko and their school, generalizing the construction of a rigid body for a wide class of Lie algebras, see a review of this work in [36].

As for the multi-dimensional analogs of the Lagrange top, there exist two different versions. The first one, living on $so(n) \ltimes \mathbb{R}^n$, was proposed in [5, 29], see also [31]. The second one, living on $so(n) \ltimes so(n)$, was introduced in [27]. The multi-dimensional generalization of the Kovalevskaya top is due to [30], see also [31, 6]. For the Clebsch problem, the multi-dimensional generalization was found in [26], for the Steklov–Lyapunov case a multi-dimensional generalization, living on $so(n) \ltimes so(n)$ and not on $e(n) = so(n) \ltimes \mathbb{R}^n$, was found in [11].

In the present paper, we find *integrable discretizations* for the multi-dimensional Lagrange top and for a certain flow of the integrable hierarchy of the multi-dimensional Clebsch case. Our construction is based on the discrete time Lagrangian mechanics, as pioneered by Moser and Veselov [25], and further developed in [7, 8, 34]. In order to give the reader a flavour of our integrable discretizations, we present here the corresponding formulas. In these formulas $(M, P) \in e^*(n)$, i.e. $M \in so(n)$, $P \in \mathbb{R}^n$. The equations of motion of the totally symmetric Lagrange top in the moving frame (which are also equivalent to the equations of motion of an arbitrary Lagrange top in the rest frame) and their integrable discretization are ($A \in \mathbb{R}^n$ is a constant vector):

$$\begin{cases} \dot{M} = A \wedge P, \\ \dot{P} = -MP, \end{cases} \qquad \begin{cases} M_{k+1} = M_k + \epsilon A \wedge P_k, \\ P_{k+1} = \dfrac{I - (\epsilon/2)M_{k+1}}{I + (\epsilon/2)M_{k+1}} P_k. \end{cases}$$

Similarly, the equations of the Clebsch case of the rigid body motion in an ideal fluid and their integrable discretization read ($B = \text{diag}\,(b_1, \ldots, b_n)$ is a constant matrix):

$$\begin{cases} \dot{M} = P \wedge (BP), \\ \dot{P} = -MP, \end{cases} \qquad \begin{cases} M_{k+1} = M_k + \epsilon \dfrac{P_k \wedge (BP_k)}{1 + (\epsilon^2/4)\langle P_k, BP_k \rangle}, \\ P_{k+1} = \dfrac{I - (\epsilon/2)M_{k+1}}{I + (\epsilon/2)M_{k+1}} P_k. \end{cases}$$

We recall the general theory of the Lagrangian reduction in the continuous time and in the discrete time contexts, respectively, in Section 2 and 3. Further, we give in Section 4 a Lagrangian derivation of the equations of motion of a multi-dimensional heavy top, and in Section 5 we discuss the Lagrangian theory of the multi-dimensional Lagrange top, both in the moving frame and in the rest frame. Section 6 is devoted to the construction of an integrable Lagrangian discretization of the multi-dimensional Lagrange top, again in both formulations (in the moving frame and in the rest frame). Finally, in Section 7 we discuss the Lagrangian formulation of the problem of the rigid body motion in an ideal fluid, its integrable case due to Clebsch, and construct an integrable Lagrangian discretization of this problem. Conclusions are contained in Section 8.

2   Lagrangian mechanics and Lagrangian reduction on TG

Recall that a continuous time Lagrangian system is defined by a smooth function $\mathbf{L}(g, \dot{g})$:
$TG \mapsto \mathbb{R}$ on the tangent bundle of a smooth manifold G. The function $\mathbf{L}$ is called the
Lagrange function. We will be dealing here only with the case when G carries an additional
structure of a *Lie group*, with the Lie algebra $\mathfrak{g}$. For an arbitrary function $g(t) : [t_0, t_1] \mapsto G$
one can consider the *action functional*

$$\mathbf{S} = \int_{t_0}^{t_1} \mathbf{L}(g(t), \dot{g}(t)) dt.$$

A standard argument shows that the functions $g(t)$ yielding extrema of this functional (in
the class of variations preserving $g(t_0)$ and $g(t_1)$), satisfy with necessity the *Euler–Lagrange
equations.* In local coordinates $\{g^i\}$ on G they read:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{g}^i} \right) = \frac{\partial \mathbf{L}}{\partial g^i}. \tag{2.1}$$

The action functional $\mathbf{S}$ is independent on the choice of local coordinates, and thus the
Euler–Lagrange equations are actually coordinate independent as well. For a coordinate-
free description in the language of differential geometry, see [3, 22].

Introducing the quantities

$$\Pi = \nabla_{\dot{g}} \mathbf{L} \in T_g^* G,$$

one defines the *Legendre transformation*:

$$(g, \dot{g}) \in TG \;\mapsto\; (g, \Pi) \in T^* G.$$

If it is invertible, i.e. if $\dot{g}$ can be expressed through (g, Π), then the Legendre transformation
of the Euler–Lagrange equations (2.1) yield a *Hamiltonian system* on T^*G with respect to
the standard symplectic structure on T^*G and with the Hamilton function

$$H(g, \Pi) = \langle \Pi, \dot{g} \rangle - \mathbf{L}(g, \dot{g}), \tag{2.2}$$

(where, of course, $\dot{g}$ has to be expressed through (g, Π)).

When working with the tangent bundle of a Lie group, it is convenient to trivialize it,
translating all vectors to the group unit by left or right multiplication. We consider first
the left trivialization:

$$(g, \dot{g}) \in TG \;\mapsto\; (g, \Omega) \in G \times \mathfrak{g}, \tag{2.3}$$

where

$$\Omega = L_{g^{-1}*} \dot{g} \quad \Leftrightarrow \quad \dot{g} = L_{g*} \Omega.$$

Denote the Lagrange function pushed through the above map, by $\mathbf{L}^{(l)}(g, \Omega) : G \times \mathfrak{g} \mapsto \mathbb{R}$,
so that

$$\mathbf{L}^{(l)}(g, \Omega) = \mathbf{L}(g, \dot{g}),$$

The trivialization (2.3) of the tangent bundle TG induces the following trivialization of the cotangent bundle T^*G:

$$(g, \Pi) \in T^*G \;\mapsto\; (g, M) \in G \times \mathfrak{g}^*,$$

where

$$M = L_g^* \Pi \quad \Leftrightarrow \quad \Pi = L_{g^{-1}}^* M.$$

We now consider the Lagrangian reduction procedure, in the case when the Lagrange function is symmetric with respect to the left action of a certain subgroup of G, namely an isotropy subgroup of some element in the representation space of G. So, the next ingredient of our construction is a representation $\Phi : G \times V \mapsto V$ of a Lie group G in a linear space V; we denote it by

$$\Phi(g) \cdot v \quad \text{for} \quad g \in G, \;\; v \in V.$$

We denote also by ϕ the corresponding representation of the Lie algebra $\mathfrak{g}$ in V:

$$\phi(\xi) \cdot v = \frac{d}{d\epsilon} \left(\Phi(e^{\epsilon\xi}) \cdot v \right) \bigg|_{\epsilon=0} \qquad \text{for} \quad \xi \in \mathfrak{g}, \;\; v \in V.$$

The map $\phi^* : \mathfrak{g} \times V^* \mapsto V^*$ defined by

$$\langle \phi^*(\xi) \cdot y, v \rangle = \langle y, \phi(\xi) \cdot v \rangle \qquad \forall \, v \in V, \; y \in V^*, \; \xi \in \mathfrak{g},$$

is an anti-representation of the Lie algebra $\mathfrak{g}$ in V^*. We shall use also the bilinear operation $\diamond : V^* \times V \mapsto \mathfrak{g}^*$ defined as follows: let $v \in V$, $y \in V^*$, then

$$\langle y \diamond v, \xi \rangle = -\langle y, \phi(\xi) \cdot v \rangle \qquad \forall \, \xi \in \mathfrak{g}.$$

(Notice that the pairings on the left-hand side and on the right-hand side of the latter equation are defined on different spaces. The operation $\diamond$ can be found in [19], where it plays the key role in the theory of the so called Clebsch representations; our notation follows [18, 14].)

Fix an element $p \in V$, and consider the isotropy subgroup $G^{[p]}$ of p, i.e.

$$G^{[p]} = \{ h : \Phi(h) \cdot p = p \} \subset G.$$

Suppose that the Lagrange function $\mathbf{L}(g, \dot{g})$ is invariant under the action of $G^{[p]}$ on TG induced by left translations on G:

$$\mathbf{L}(hg, L_{h*}\dot{g}) = \mathbf{L}(g, \dot{g}), \qquad h \in G^{[p]}.$$

The corresponding invariance property of $\mathbf{L}^{(l)}(g, \Omega)$ is expressed as:

$$\mathbf{L}^{(l)}(hg, \Omega) = \mathbf{L}^{(l)}(g, \Omega), \qquad h \in G^{[p]}.$$

We want to reduce the Euler–Lagrange equations with respect to this left action. As a section $(G \times \mathfrak{g})/G^{[p]}$ we choose the set $\mathfrak{g} \times O_p$, where O_p is the orbit of p under the action Φ:

$$O_p = \{ \Phi(g) \cdot p, \; g \in G \} \subset V.$$

The reduction map is

$$(g, \Omega) \in G \times \mathfrak{g} \;\mapsto\; (\Omega, P) \in \mathfrak{g} \times O_p, \qquad \text{where} \quad P = \Phi(g^{-1}) \cdot p,$$

so that the reduced Lagrange function $\mathcal{L}^{(l)} : \mathfrak{g} \times O_p \mapsto \mathbb{R}$ is defined as

$$\mathcal{L}^{(l)}(\Omega, P) = \mathbf{L}^{(l)}(g, \Omega), \qquad \text{where} \quad P = \Phi(g^{-1}) \cdot p.$$

The reduced Lagrangian $\mathcal{L}^{(l)}(\Omega, P)$ is well defined, because from

$$P = \Phi\left(g_1^{-1}\right) \cdot p = \Phi\left(g_2^{-1}\right) \cdot p$$

there follows $\Phi\left(g_2 g_1^{-1}\right) \cdot p = p$, so that $g_2 g_1^{-1} \in G^{[p]}$, and $\mathbf{L}^{(l)}(g_1, \Omega) = \mathbf{L}^{(l)}(g_2, \Omega)$.

Theorem 2.1. [18, 14]

a) *Under the left trivialization $(g, \dot{g}) \mapsto (g, \Omega)$ and the subsequent reduction $(g, \Omega) \mapsto (\Omega, P)$, the Euler–Lagrange equations (2.1) become the following* **left Euler–Poincaré equations***:*

$$\begin{cases} \dot{M} = \mathrm{ad}^* \Omega \cdot M + \nabla_P \mathcal{L}^{(l)} \diamond P, \\ \dot{P} = -\phi(\Omega) \cdot P, \end{cases} \tag{2.4}$$

where

$$M = L_g^* \Pi = \nabla_\Omega \mathcal{L}^{(l)}.$$

b) *If the "Legendre transformation"*

$$(\Omega, P) \in \mathfrak{g} \times O_p \mapsto (M, P) \in \mathfrak{g}^* \times O_p,$$

is invertible, then (2.4) is a Hamiltonian system on $\mathfrak{g}^* \times O_p$ with the Hamilton function

$$H(M, P) = \langle M, \Omega \rangle - \mathcal{L}^{(l)}(\Omega, P),$$

with respect to the Poisson bracket given by

$$\begin{aligned} \{F_1, F_2\} = &\langle M, [\nabla_M F_1, \nabla_M F_2] \rangle \\ &+ \langle \nabla_P F_1, \phi(\nabla_M F_2) \cdot P \rangle - \langle \nabla_P F_2, \phi(\nabla_M F_1) \cdot P \rangle \end{aligned} \tag{2.5}$$

for two arbitrary functions $F_{1,2}(M, P) : \mathfrak{g}^* \times O_p \mapsto \mathbb{R}$.

Remark 2.1. The formula (2.5) defines a Poisson bracket not only on $\mathfrak{g}^* \times O_p$, but on all of $\mathfrak{g}^* \times V$. Rewriting this formula as

$$\begin{aligned} \{F_1, F_2\} = &\langle M, [\nabla_M F_1, \nabla_M F_2] \rangle \\ &+ \langle P, \phi^*(\nabla_M F_2) \cdot \nabla_P F_1 - \phi^*(\nabla_M F_1) \cdot \nabla_P F_2 \rangle \end{aligned}$$

one immediately identifies this bracket with the Lie–Poisson bracket of the semiproduct Lie algebra $\mathfrak{g} \ltimes V^*$ corresponding to the representation $-\phi^*$ of $\mathfrak{g}$ in V^*.

We shall also need a version of the above theorem for the case when the Lagrange function is invariant with respect to the *right action* of some isotropy subgroup $G^{[A]}$ ($A \in V$), i.e.

$$\mathbf{L}(gh, R_{h*}\dot{g}) = \mathbf{L}(g, \dot{g}), \qquad h \in G^{[A]}.$$

In this case it is convenient to consider the right trivialization of the tangent bundle:

$$(g, \dot{g}) \in TG \;\mapsto\; (g, \omega) \in G \times \mathfrak{g},$$

where

$$\omega = R_{g^{-1}*}\dot{g} \quad \Leftrightarrow \quad \dot{g} = R_{g*}\omega.$$

Denote the Lagrange function pushed through the trivialization map by $\mathbf{L}^{(r)}(g, \omega) : G \times \mathfrak{g} \mapsto \mathbb{R}$, so that

$$\mathbf{L}^{(r)}(g, \omega) = \mathbf{L}(g, \dot{g}).$$

The corresponding invariance property of the Lagrange function is expressed as

$$\mathbf{L}^{(r)}(gh, \omega) = \mathbf{L}^{(r)}(g, \omega), \qquad h \in G^{[A]}.$$

The formulas for the reduction of the Euler–Lagrange equations are also slightly modified. As a section $(G \times \mathfrak{g})/G^{[A]}$ we choose the set $\mathfrak{g} \times O_A$, the reduction map being

$$(g, \omega) \in G \times \mathfrak{g} \;\mapsto\; (\omega, a) \in \mathfrak{g} \times O_A, \qquad \text{where} \quad a = \Phi(g) \cdot A \,,$$

so that the reduced Lagrange function $\mathcal{L}^{(r)} : \mathfrak{g} \times O_A \mapsto \mathbb{R}$ is defined as

$$\mathcal{L}^{(r)}(\omega, a) = \mathbf{L}^{(r)}(g, \omega), \qquad \text{where} \quad a = \Phi(g) \cdot A.$$

There holds an analog of Theorem 2.1, stating, in particular, that under the right trivialization $(g, \dot{g}) \mapsto (g, \omega)$ and the subsequent reduction $(g, \omega) \mapsto (\omega, a)$, the Euler–Lagrange equations (2.1) become the following **right Euler–Poincaré equations**:

$$\begin{cases} \dot{m} = -\text{ad}^* \, \omega \cdot m - \nabla_a \mathcal{L}^{(r)} \diamond a, \\ \dot{a} = \phi(\omega) \cdot a, \end{cases} \tag{2.6}$$

where

$$m = R_g^* \Pi = \nabla_\omega \mathcal{L}^{(r)}.$$

3   Lagrangian mechanics and Lagrangian reduction on $G \times G$

We now turn to the discrete time analog of these constructions. Our presentation of the general discrete time Lagrangian mechanics is an adaptation of the Moser–Veselov construction [37, 25] for the case when the basic manifold is a Lie group. The presentation of the discrete time Lagrangian reduction follows [7, 8]. Almost all constructions and

results of the continuous time Lagrangian mechanics have their discrete time analogs. The only exception is the existence of the "energy" integral (2.2).

Let $\mathbb{L}(g, \hat{g}) : G \times G \mapsto \mathbb{R}$ be a smooth function, called the (discrete time) *Lagrange function*. For an arbitrary sequence $\{g_k \in G, \ k = k_0, k_0 + 1, \dots, k_1\}$ one can consider the *action functional*

$$\mathbb{S} = \sum_{k=k_0}^{k_1-1} \mathbb{L}(g_k, g_{k+1}). \tag{3.1}$$

Obviously, the sequences $\{g_k\}$ delivering extrema of this functional (in the class of variations preserving g_{k_0} and g_{k_1}), satisfy with necessity the *discrete Euler–Lagrange equations*[1]:

$$\nabla_1 \mathbb{L}(g_k, g_{k+1}) + \nabla_2 \mathbb{L}(g_{k-1}, g_k) = 0. \tag{3.2}$$

Here $\nabla_1 \mathbb{L}(g, \hat{g})$ ($\nabla_2 \mathbb{L}(g, \hat{g})$) denotes the gradient of $\mathbb{L}(g, \hat{g})$ with respect to the first argument g (resp. the second argument $\hat{g}$). So, in our case, when G is a Lie group and not just a general smooth manifold, the equation (3.2) is written in a coordinate free form, using the intrinsic notions of the Lie theory. As pointed out above, an invariant formulation of the Euler–Lagrange equations in the continuous time case is more sophisticated. This seems to underline the fundamental character of discrete Euler–Lagrange equations.

The equation (3.2) is an implicit equation for g_{k+1}. In general, it has more than one solution, and therefore defines a correspondence (multi-valued map) $(g_{k-1}, g_k) \mapsto (g_k, g_{k+1})$. To discuss symplectic properties of this correspondence, one defines:

$$\Pi_k = \nabla_2 \mathbb{L}(g_{k-1}, g_k) \in T^*_{g_k} G.$$

Then (3.2) may be rewritten as the following system:

$$\begin{cases} \Pi_k = -\nabla_1 \mathbb{L}(g_k, g_{k+1}), \\ \Pi_{k+1} = \nabla_2 \mathbb{L}(g_k, g_{k+1}). \end{cases} \tag{3.3}$$

This system defines a (multivalued) map $(g_k, \Pi_k) \mapsto (g_{k+1}, \Pi_{k+1})$ of T^*G into itself. More precisely, the first equation in (3.3) is an implicit equation for g_{k+1}, while the second one allows for the explicit and unique calculation of Π_{k+1}, knowing g_k and g_{k+1}. As demonstrated in [37, 25], this map $T^*G \mapsto T^*G$ is symplectic with respect to the standard symplectic structure on T^*G.

The tangent bundle TG does not appear in the discrete time context at all. On the contrary, the cotangent bundle T^*G still plays an important role in the discrete time

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[1]For the notations from the Lie groups theory used in this and subsequent sections see, e.g., [7]. In particular, for an arbitrary smooth function $f : G \mapsto \mathbb{R}$, its right Lie derivative $d'f$ and left Lie derivative df are functions from G into $\mathfrak{g}^*$ defined via the formulas

$$\langle df(g), \eta \rangle = \left. \frac{d}{d\epsilon} f\left(e^{\epsilon\eta} g\right) \right|_{\epsilon=0}, \qquad \langle d'f(g), \eta \rangle = \left. \frac{d}{d\epsilon} f\left(g e^{\epsilon\eta}\right) \right|_{\epsilon=0}, \qquad \forall \, \eta \in \mathfrak{g},$$

and the gradient $\nabla f(g) \in T^*_g G$ is defined as

$$\nabla f(g) = R^*_{g^{-1}} \, df(g) = L^*_{g^{-1}} \, d'f(g).$$

theory, as the phase space with the canonical invariant symplectic structure. The left trivialization of T^*G is same as in the continuous time case:

$$(g_k, \Pi_k) \in T^*G \ \mapsto \ (g_k, M_k) \in G \times \mathfrak{g}^*,$$

where

$$M_k = L^*_{g_k} \Pi_k \quad \Leftrightarrow \quad \Pi_k = L^*_{g_k^{-1}} M_k.$$

Consider also the map

$$(g_k, g_{k+1}) \in G \times G \ \mapsto \ (g_k, W_k) \in G \times G, \tag{3.4}$$

where

$$W_k = g_k^{-1} g_{k+1} \quad \Leftrightarrow \quad g_{k+1} = g_k W_k.$$

Denote the Lagrange function pushed through (3.4) by

$$\mathbb{L}^{(l)}(g_k, W_k) = \mathbb{L}(g_k, g_{k+1}).$$

Suppose that the Lagrange function $\mathbb{L}(g, \hat{g})$ is invariant under the action of $G^{[p]}$ on $G \times G$ induced by left translations on G:

$$\mathbb{L}(hg, h\hat{g}) = \mathbb{L}(g, \hat{g}), \qquad h \in G^{[p]}.$$

The corresponding invariance property of $\mathbb{L}^{(l)}(g, W)$ is expressed as:

$$\mathbb{L}^{(l)}(hg, W) = \mathbb{L}^{(l)}(g, W), \qquad h \in G^{[p]}.$$

We want to reduce the Euler–Lagrange equations with respect to this left action. As a section $(G \times G)/G^{[p]}$ we choose the set $G \times O_p$. The reduction map is

$$(g, W) \in G \times G \ \mapsto \ (W, P) \in G \times O_p, \qquad \text{where} \quad P = \Phi(g^{-1}) \cdot p,$$

so that the reduced Lagrange function $\Lambda^{(l)} : G \times O_p \mapsto \mathbb{R}$ is defined as

$$\Lambda^{(l)}(W, P) = \mathbb{L}^{(l)}(g, W), \qquad \text{where} \quad P = \Phi(g^{-1}) \cdot p.$$

Theorem 3.1. [7, 8]

   a) *Under the left trivialization $(g, \hat{g}) \mapsto (g, W)$ and the subsequent reduction $(g, W) \mapsto (W, P)$, the Euler–Lagrange equations (3.2) become the following* **left discrete time Euler–Poincaré equations***:*

$$\begin{cases} \mathrm{Ad}^* W_k^{-1} \cdot M_{k+1} = M_k + \nabla_P \Lambda^{(l)}(W_k, P_k) \diamond P_k, \\ P_{k+1} = \Phi(W_k^{-1}) \cdot P_k, \end{cases} \tag{3.5}$$

where

$$M_k = d'_W \Lambda^{(l)}(W_{k-1}, P_{k-1}) \in \mathfrak{g}^*.$$

   b) *If the "Legendre transformation"*

$$(W_{k-1}, P_{k-1}) \in G \times O_p \mapsto (M_k, P_k) \in \mathfrak{g}^* \times O_p,$$

where $P_k = \Phi(W_{k-1}^{-1}) \cdot P_{k-1}$, is invertible, then (3.5) define a map $(M_k, P_k) \mapsto (M_{k+1}, P_{k+1})$ of $\mathfrak{g}^* \times O_p$ which is Poisson with respect to the Poisson bracket (2.5).

We shall also need a version of the above theorem for the case when the Lagrange function is invariant with respect to the *right action* of some isotropy subgroup $G^{[A]}$ ($A \in V$), i.e.

$$\mathbb{L}(gh, \hat{g}h) = \mathbb{L}(g, \hat{g}), \qquad h \in G^{[A]}.$$

In this case it is convenient to consider the following map analogous to the right trivialization of the tangent bundle:

$$(g_k, g_{k+1}) \in G \times G \;\mapsto\; (g_k, w_k) \in G \times G,$$

where

$$w_k = g_{k+1} g_k^{-1} \quad \Leftrightarrow \quad g_{k+1} = w_k g_k.$$

Denote the Lagrange function pushed through this map by $\mathbb{L}^{(r)}(g_k, w_k) : G \times G \mapsto \mathbb{R}$, so that

$$\mathbb{L}^{(r)}(g_k, w_k) = \mathbb{L}(g_k, g_{k+1}).$$

The corresponding invariance property of the Lagrange function is expressed as

$$\mathbb{L}^{(r)}(gh, w) = \mathbb{L}^{(r)}(g, w), \qquad h \in G^{[A]}.$$

The formulas for the reduction of the Euler–Lagrange equations are also slightly modified. As a section $(G \times \mathfrak{g})/G^{[A]}$ we choose the set $\mathfrak{g} \times O_A$, the reduction map being

$$(g, w) \in G \times G \;\mapsto\; (w, a) \in G \times O_A, \qquad \text{where} \quad a = \Phi(g) \cdot A,$$

so that the reduced Lagrange function $\Lambda^{(r)} : G \times O_A \mapsto \mathbb{R}$ is defined as

$$\Lambda^{(r)}(w, a) = \mathbb{L}^{(r)}(g, w), \qquad \text{where} \quad a = \Phi(g) \cdot A.$$

There holds an analog of Theorem 3.1, stating, in particular, that under the right trivialization $(g_k, g_{k+1}) \mapsto (g_k, w_k)$ and the subsequent reduction $(g_k, w_k) \mapsto (w_k, a_k)$, the Euler–Lagrange equations (3.3) become the following **right discrete time Euler–Poincaré equations**:

$$\begin{cases} \mathrm{Ad}^* w_k \cdot m_{k+1} = m_k - \nabla_a \Lambda^{(r)}(w_k, a_k) \diamond a_k, \\ a_{k+1} = \Phi(w_k) \cdot a_k, \end{cases} \tag{3.6}$$

where

$$m_k = d_w \Lambda^{(r)}(w_{k-1}, a_{k-1}) \in \mathfrak{g}^*.$$

The relation between the continuous time and the discrete time equations is established, if we set

$$\begin{aligned} g_k &= g, & g_{k+1} &= g + \epsilon \dot{g} + O\left(\epsilon^2\right), & \mathbb{L}(g_k, g_{k+1}) &= \epsilon \mathbf{L}(g, \dot{g}) + O\left(\epsilon^2\right); \\ P_k &= P, & W_k &= I + \epsilon \Omega + O\left(\epsilon^2\right), & \Lambda^{(l)}(W_k, P_k) &= \epsilon \mathcal{L}^{(l)}(\Omega, P) + O\left(\epsilon^2\right); \\ a_k &= a, & w_k &= I + \epsilon \omega + O\left(\epsilon^2\right), & \Lambda^{(r)}(w_k, a_k) &= \epsilon \mathcal{L}^{(r)}(\omega, a) + O\left(\epsilon^2\right). \end{aligned}$$

4 A multi-dimensional heavy top

The basic Lie group relevant for our main examples is

$$G = \mathrm{SO}(n), \qquad \text{so that} \quad \mathfrak{g} = \mathrm{so}(n)$$

(the "physical" rigid body corresponds to $n = 3$). The scalar product on $\mathfrak{g}$ is defined as

$$\langle \xi, \eta \rangle = -\frac{1}{2} \operatorname{tr}(\xi\eta), \qquad \xi, \eta \in \mathfrak{g}.$$

This scalar product is used also to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, so that the previous formula can be considered also as a pairing between the elements $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$.

The group G is a natural configuration space for problems related to the rotation of a rigid body. Indeed, if $E(t) = (e_1(t), \ldots, e_n(t))$ stands for the time evolution of a certain orthonormal frame firmly attached to the rigid body (all $e_k \in \mathbb{R}^n$, $\langle e_k, e_j \rangle = \delta_{kj}$), then

$$E(t) = g^{-1}(t)E(0) \quad \Leftrightarrow \quad e_k(t) = g^{-1}(t)e_k(0) \qquad (1 \le k \le n),$$

with some $g(t) \in G$. The Lagrange function of an arbitrary rigid body rotating about a fixed point in a homogeneous field with a linear potential

$$\varphi(x) = \langle p, x \rangle,$$

is equal to

$$\mathbf{L}(g, \dot{g}) = -\frac{1}{2} \operatorname{tr}(\Omega J \Omega) - \langle p, a \rangle, \tag{4.1}$$

where

- $\Omega = g^{-1}\dot{g} = g^{\mathrm{T}}\dot{g}$ is the angular velocity of the rigid body in the body frame $E(t)$;

- J is a symmetric matrix (tensor of inertia of the rigid body); choosing the frame $E(0)$ properly, we can assure this matrix to be *diagonal*, $J = \operatorname{diag}(J_1, \ldots, J_n)$, which will be supposed from now on;

- $p \in \mathbb{R}^n$ is the constant gravity vector, calculated in the rest frame;

- $a = a(t) = g(t)A \in \mathbb{R}^n$ is the vector pointing from the fixed point to the center of mass of the rigid body, calculated in the rest frame; A is the same vector calculated in the moving frame (where it is constant).

Obviously, the function (4.1) may be rewritten also as

$$\mathbf{L}(g, \dot{g}) = \frac{1}{2} \langle \Omega, \mathcal{J}(\Omega) \rangle - \langle P, A \rangle, \tag{4.2}$$

where $\mathcal{J} : \mathrm{so}(n) \mapsto \mathrm{so}(n)$ is the symmetric operator acting as

$$\mathcal{J}(\Omega) = J\Omega + \Omega J,$$

and $P(t) = g^{-1}(t)p$ is the gravity vector p calculated in the moving frame. The Lagrange function (4.1) is in the framework of Section 2, if the following identifications are made:

- $V = V^* = \mathbb{R}^n$ with the standard Euclidean scalar product;
- The representation Φ of G in V is defined as

$$\Phi(g) \cdot v = gv \qquad \text{for} \quad g \in \mathrm{SO}(n), \ v \in \mathbb{R}^n.$$

- Therefore the representation ϕ of $\mathfrak{g}$ in V is given by

$$\phi(\xi) \cdot v = \xi v \qquad \text{for} \quad \xi \in \mathrm{so}(n), \ v \in \mathbb{R}^n,$$

  while the anti-representation ϕ^* of $\mathfrak{g}$ in V^* is given by

$$\phi^*(\xi) \cdot y = -\xi y \qquad \text{for} \quad \xi \in \mathrm{so}(n), \ y \in \mathbb{R}^n,$$

- Finally, the bilinear operation $\diamond : V^* \times V \mapsto \mathfrak{g}^*$ is given by

$$y \diamond v = vy^{\mathrm{T}} - yv^{\mathrm{T}} = v \wedge y \qquad \text{for} \quad y, v \in \mathbb{R}^n.$$

The Lagrange function (4.2) is manifestly invariant under the left action of the isotropy subgroup $G^{[p]}$. Now Theorem 2.1 is applicable, which delivers the following *equations of motion of a heavy top in the moving frame*:

$$\begin{cases} \dot{M} = [M, \Omega] + A \wedge P, \\ \dot{P} = -\Omega P, \end{cases} \tag{4.3}$$

where

$$M = \nabla_\Omega \mathcal{L}^{(l)} = \mathcal{J}(\Omega) \quad \Leftrightarrow \quad M_{jk} = (J_j + J_k)\Omega_{jk}.$$

According to the general theory, the system (4.3) is Hamiltonian on the dual of the semidirect product Lie algebra $\mathrm{e}(n) = \mathrm{so}(n) \ltimes \mathbb{R}^n$, with the Hamilton function

$$H(M, P) = \frac{1}{2}\langle M, \Omega \rangle + \langle P, A \rangle = \frac{1}{2} \sum_{j<k} \frac{M_{jk}^2}{J_j + J_k} + \sum_{k=1}^{n} P_k A_k.$$

The correspondent invariant Poisson bracket reads:

$$\{M_{ij}, M_{k\ell}\} = M_{i\ell}\delta_{jk} - M_{kj}\delta_{\ell i} - M_{ik}\delta_{j\ell} + M_{\ell j}\delta_{ki}, \tag{4.4}$$

$$\{M_{ij}, P_k\} = P_i \delta_{jk} - P_j \delta_{ik}. \tag{4.5}$$

5   The multi-dimensional Lagrange top

The *multi-dimensional Lagrange top* is characterized by the following data: $J_1 = J_2 = \cdots = J_{n-1}$, which means that the body is rotationally symmetric with respect to the nth coordinate axis, and $A_1 = A_2 = \cdots = A_{n-1} = 0$, which means that the fixed point lies on the symmetry axis. Choosing units properly, we may assume that

$$J_1 = J_2 = \cdots = J_{n-1} = \frac{\alpha}{2}, \qquad J_n = 1 - \frac{\alpha}{2}, \qquad A = (0, 0, \ldots, 0, 1)^{\mathrm{T}}.$$

The action of the operator $\mathcal{J}$ is given by

$$M_{ij} = \mathcal{J}(\Omega)_{ij} = \begin{cases} \alpha\Omega_{ij}, & 1 \leq i,j \leq n-1, \\ \Omega_{ij}, & i = n \text{ or } j = n, \end{cases}$$

or in a more invariant fashion:

$$M = \mathcal{J}(\Omega) = \alpha\Omega + (1-\alpha)\left(\Omega AA^{\mathrm{T}} + AA^{\mathrm{T}}\Omega\right) \tag{5.1}$$
$$= \alpha\Omega - (1-\alpha)A \wedge (\Omega A). \tag{5.2}$$

Therefore, the formula (4.2) may be represented as

$$\mathbf{L}(g,\dot{g}) = \frac{\alpha}{2}\langle\Omega,\Omega\rangle + \frac{1-\alpha}{2}\langle\Omega A,\Omega A\rangle - \langle P,A\rangle. \tag{5.3}$$

The "Legendre transformation" (5.1), (5.2) is easily invertible. Notice that from (5.1) there follows immediately

$$MA = \Omega A, \tag{5.4}$$

and plugging this into the second term of the right-hand side of (5.2), we find:

$$\Omega = \frac{1}{\alpha}M + \frac{1-\alpha}{\alpha}A \wedge (MA).$$

The integrability of the multi-dimensional Lagrange top was demonstrated by Belyaev [5], who constructed "by hands" the necessary number of involutive integrals. A somewhat easier proof is delivered by the Lax representation in the loop algebra $\mathrm{sl}(n+1)[\lambda,\lambda^{-1}]$ twisted by the Cartan automorphism [30, 31].

Proposition 5.1. *For the Lagrange top, the moving frame equations (4.3) are equivalent to the matrix equation*

$$\dot{L}(\lambda) = [L(\lambda),U(\lambda)],$$

where

$$L(\lambda) = \begin{pmatrix} M & \lambda A - \lambda^{-1}P \\ \lambda A^{\mathrm{T}} - \lambda^{-1}P^{\mathrm{T}} & 0 \end{pmatrix}, \qquad U(\lambda) = \begin{pmatrix} \Omega & \lambda A \\ \lambda A^{\mathrm{T}} & 0 \end{pmatrix}.$$

Proof. A direct calculation based on the formula (5.4). ∎

Integrability is not the only distinctive feature of the Lagrange top. Another one is the existence of a nice Euler–Poincaré description not only in the moving frame, but also in the rest one. Rewriting (5.3) as

$$\mathbf{L}(g,\dot{g}) = \frac{\alpha}{2}\langle\omega,\omega\rangle + \frac{1-\alpha}{2}\langle\omega a,\omega a\rangle - \langle p,a\rangle, \tag{5.5}$$

where $\omega = \dot{g}g^{-1}$, we observe that the Lagrange function of the Lagrange top is not only left-invariant (with respect to $G^{[p]}$), but also right-invariant (with respect to $G^{[A]}$). Therefore, we may apply the formula (2.6), which in the present setup reads:

$$\begin{cases} \dot{m} = [\omega, m] + \nabla_a \mathcal{L}^{(r)} \wedge a, \\ \dot{a} = \omega a, \end{cases} \tag{5.6}$$

Straightforward calculations based on (5.5) give:

$$\nabla_a \mathcal{L}^{(r)} = -(1-\alpha)\omega^2 a - p,$$
$$m = \nabla_\omega \mathcal{L}^{(r)} = \alpha\omega + (1-\alpha)\left(\omega a a^{\mathrm{T}} + a a^{\mathrm{T}}\omega\right).$$

The last formula implies, first,

$$ma = \alpha\omega a + (1-\alpha)\omega a\langle a, a\rangle = \omega a,$$

and, second,

$$[\omega, m] - (1-\alpha)\left(\omega^2 a\right) \wedge a = \left[\omega, m - (1-\alpha)\left(\omega a a^{\mathrm{T}} + a a^{\mathrm{T}}\omega\right)\right] = [\omega, \alpha\omega] = 0.$$

Plugging these results into (5.6), we finally arrive at the following nice system:

$$\begin{cases} \dot{m} = a \wedge p, \\ \dot{a} = ma, \end{cases} \tag{5.7}$$

where $m = gMg^{-1}$ is the kinetic moment in the rest frame. This is a Hamiltonian system with respect to the (minus) Lie–Poisson bracket of $\mathrm{e}(n)$:

$$\{m_{ij}, m_{k\ell}\} = -m_{i\ell}\delta_{jk} + m_{kj}\delta_{\ell i} + m_{ik}\delta_{j\ell} - m_{\ell j}\delta_{ki}, \tag{5.8}$$
$$\{m_{ij}, a_k\} = -a_i\delta_{jk} + a_j\delta_{ik}, \tag{5.9}$$

with the Hamilton function

$$H(m, a) = \frac{1}{2}\langle m, m\rangle + \langle a, p\rangle.$$

A remarkable feature of the system (5.7) is its independence on the anisotropy parameter α.

Proposition 5.2. *The rest frame equations (5.7) of the Lagrange top are equivalent to the matrix equation*

$$\dot{\ell}(\lambda) = [\ell(\lambda), u(\lambda)],$$

where

$$\ell(\lambda) = \begin{pmatrix} m & \lambda a - \lambda^{-1}p \\ \lambda a^{\mathrm{T}} - \lambda^{-1}p^{\mathrm{T}} & 0 \end{pmatrix}, \qquad u(\lambda) = \begin{pmatrix} 0 & \lambda a \\ \lambda a^{\mathrm{T}} & 0 \end{pmatrix}.$$

6   A discrete time analog of the Lagrange top

An integrable discretization of the 3-dimensional Lagrange top was constructed in [7], where Lagrangian dynamics on $G = \mathrm{SU}(2)$ are used, leading to Euler–Poincaré equations on $\mathrm{su}(2) \ltimes \mathrm{su}(2)$. The multi-dimensional generalization presented here deals with Lagrangians on $G = \mathrm{SO}(n)$, leading to Euler–Poincaré equations on $\mathrm{so}(n) \ltimes \mathbb{R}^n$. This change of context (when compared with [7]) required a modification of the kinetic energy terms. Of course, this is essential only in the discrete time setting, since in the continuous time situation the kinetic energy is given purely in terms of the Lie algebra $\mathfrak{g}$ of the Lie group G, and the Lie algebras $\mathrm{su}(2)$ and $\mathrm{so}(3)$ are isomorphic.

6.1   Rest frame formulation

Consider the following discrete analog of the Lagrange function (5.5):

$$\mathbb{L}(g_k, g_{k+1}) = -\frac{\alpha}{\epsilon} \operatorname{tr} \log \left(2I + w_k + w_k^{-1} \right)$$
$$- \frac{2(1-\alpha)}{\epsilon} \log \left(1 + \langle a_k, a_{k+1} \rangle \right) - \epsilon \langle p, a_k \rangle. \tag{6.1}$$

where w_k, a_k are defined as in Section 3:

$$w_k = g_{k+1} g_k^{-1}, \qquad a_k = g_k A, \qquad \text{so that} \quad a_{k+1} = w_k a_k.$$

The powers of ϵ are introduced in a way assuring the correct asymptotics in the continuous limit $\epsilon \to 0$, namely $\mathbb{L}(g_k, g_{k+1}) \approx \epsilon \mathbf{L}(g, \dot{g})$, as $g_k = g$ and $g_{k+1} \approx g + \epsilon \dot{g}$ (see the end of Section 3). This is seen with the help of the following simple lemma.

Lemma 6.1. *Let $w(\epsilon) = I + \epsilon \omega + O\left(\epsilon^2\right) \in \mathrm{SO}(n)$ be a smooth curve, $\omega \in \mathrm{so}(n)$. Then*

$$\operatorname{tr} \log \left(2I + w(\epsilon) + w^{-1}(\epsilon) \right) = \mathrm{const} - \frac{\epsilon^2}{2} \langle \omega, \omega \rangle + O\left(\epsilon^3\right). \tag{6.2}$$

For an arbitary $a \in \mathbb{R}^n$:

$$\langle a, w(\epsilon) a \rangle = \langle a, a \rangle - \frac{\epsilon^2}{2} \langle \omega a, \omega a \rangle + O\left(\epsilon^3\right). \tag{6.3}$$

Proof. Let $w = I + \epsilon \omega + \epsilon^2 v + O\left(\epsilon^3\right)$. Then from $w w^{\mathrm{T}} = I$ we get:

$$v + v^{\mathrm{T}} + \omega \omega^{\mathrm{T}} = 0 \quad \Rightarrow \quad v = \frac{1}{2} \omega^2 + v_1, \quad v_1 \in \mathrm{so}(n). \tag{6.4}$$

Hence

$$2I + w + w^{\mathrm{T}} = 4I + \epsilon^2 \omega^2 + O\left(\epsilon^3\right)$$
$$\Rightarrow \quad \log \left(2I + w + w^{\mathrm{T}} \right) = \mathrm{const}\, I + \frac{\epsilon^2}{4} \omega^2 + O\left(\epsilon^3\right),$$

which yields (6.2). Similarly, we derive from (6.4):

$$a^{\mathrm{T}} w a = a^{\mathrm{T}} a + \frac{\epsilon^2}{2} a^{\mathrm{T}} \omega^2 a + O\left(\epsilon^3\right),$$

which implies (6.3). ∎

Writing in (6.1) $w_k a_k$ for a_{k+1}, we come to the Lagrange function $\Lambda^{(r)}(a_k, w_k) = \mathbb{L}(g_k, g_{k+1})$ depending only on a_k, w_k, hence invariant with respect to the right action of $G^{[A]}$. It may be reduced following the procedure of Section 3.

Theorem 6.1. *The Euler–Lagrange equations of motion for the Lagrange function (6.1) are equivalent to the following system:*

$$\begin{cases} m_{k+1} = m_k + \epsilon a_k \wedge p, \\ a_{k+1} = \dfrac{I + (\epsilon/2) m_{k+1}}{I - (\epsilon/2) m_{k+1}}\, a_k. \end{cases} \tag{6.5}$$

The map $(m_k, a_k) \mapsto (m_{k+1}, a_{k+1})$ *is Poisson with respect to the bracket (5.8), (5.9), and the following function is its integral of motion:*

$$H_\epsilon(m, a) = \frac{1}{2}\langle m, m \rangle + \langle a, p \rangle - \frac{\epsilon}{2}\langle m, p \wedge a \rangle. \tag{6.6}$$

Proof. The right reduced Euler–Lagrange equations (3.6) have in the present setup the following form:

$$\begin{cases} w_k^{-1} m_{k+1} w_k = m_k - a_k \wedge \nabla_a \Lambda^{(r)}(a_k, w_k), \\ a_{k+1} = w_k a_k, \end{cases} \tag{6.7}$$

where

$$m_{k+1} = d_w \Lambda^{(r)}(a_k, w_k).$$

To calculate the derivatives of $\Lambda^{(r)}$, we use the following formulas:

$$d_w \operatorname{tr} \log\left(2I + w_k + w_k^{-1}\right) = -2\,\frac{w_k - I}{w_k + I}, \tag{6.8}$$

$$d_w \langle a_k, w_k a_k \rangle = a_k \wedge (w_k a_k), \qquad \nabla_a \langle a_k, w_k a_k \rangle = w_k a_k + w_k^{\mathrm{T}} a_k. \tag{6.9}$$

Indeed, the first one of these expressions follows from:

$$\left\langle d_w \operatorname{tr} \log\left(2I + w_k + w_k^{-1}\right), \eta \right\rangle = \frac{d}{d\varepsilon} \operatorname{tr} \log\left(2I + e^{\varepsilon\eta} w_k + w_k^{-1} e^{-\varepsilon\eta}\right)\bigg|_{\varepsilon=0}$$

$$= \operatorname{tr}\left(\left(2I + w_k + w_k^{-1}\right)\left(w_k - w_k^{-1}\right)\eta\right) = \operatorname{tr}\left((w_k + I)^{-1}(w_k - I)\eta\right)$$

$$= -2\left\langle (w_k + I)^{-1}(w_k - I), \eta \right\rangle.$$

To prove the second one, proceed similarly:

$$\left\langle d_w \langle a_k, w_k a_k \rangle, \eta \right\rangle = \frac{d}{d\varepsilon} \langle a_k, e^{\varepsilon\eta} w_k a_k \rangle\bigg|_{\varepsilon=0} = \langle a_k, \eta w_k a_k \rangle$$

$$= \operatorname{tr}\left(w_k a_k a_k^{\mathrm{T}} \eta\right) = \frac{1}{2} \operatorname{tr}\left(\left(w_k a_k a_k^{\mathrm{T}} - a_k a_k^{\mathrm{T}} w_k^{\mathrm{T}}\right)\eta\right) = \langle a_k \wedge (w_k a_k), \eta \rangle.$$

Finally, for the third expression we have:

$$\left\langle \nabla_a \langle a_k, w_k a_k \rangle, \eta \right\rangle = \frac{d}{d\varepsilon} \langle a_k + \varepsilon\eta, w_k(a_k + \varepsilon\eta) \rangle\bigg|_{\varepsilon=0} = \left\langle w_k a_k + w_k^{\mathrm{T}} a_k, \eta \right\rangle.$$

With the help of (6.8), (6.9) we find:

$$m_{k+1} = \frac{2\alpha}{\epsilon} \frac{w_k - I}{w_k + I} - \frac{2(1-\alpha)}{\epsilon} \frac{a_k \wedge a_{k+1}}{1 + \langle a_k, a_{k+1} \rangle}, \tag{6.10}$$

and

$$w_k^{-1} m_{k+1} w_k + a_k \wedge \nabla_a \Lambda^{(r)}(a_k, w_k)$$
$$= \frac{2\alpha}{\epsilon} \frac{w_k - I}{w_k + I} - \frac{2(1-\alpha)}{\epsilon} \frac{a_k \wedge a_{k+1}}{1 + \langle a_k, a_{k+1} \rangle} - \epsilon a_k \wedge p = m_{k+1} - \epsilon a_k \wedge p.$$

Comparing the latter formula with the first equation of motion in (6.7), we find that it can be rewritten as

$$m_k = m_{k+1} - \epsilon a_k \wedge p,$$

which is the first equation in (6.5).

To obtain the second one, derive from (6.10):

$$\frac{\epsilon}{2} m_{k+1} \, a_{k+1} = \alpha \frac{w_k - I}{w_k + I} a_{k+1} - (1-\alpha) \frac{a_k - a_{k+1}\langle a_k, a_{k+1}\rangle}{1 + \langle a_k, a_{k+1}\rangle},$$
$$\frac{\epsilon}{2} m_{k+1} \, a_k = \alpha \frac{w_k - I}{w_k + I} a_k - (1-\alpha) \frac{a_k \langle a_k, a_{k+1}\rangle - a_{k+1}}{1 + \langle a_k, a_{k+1}\rangle}.$$

Adding these two equations, we find:

$$\frac{\epsilon}{2} m_{k+1}(a_{k+1} + a_k) = \alpha \frac{w_k - I}{w_k + I}(a_{k+1} + a_k) + (1-\alpha)(a_{k+1} - a_k) = a_{k+1} - a_k.$$

(On the last step we took into account that $a_{k+1} = w_k a_k$.) This is nothing but the second equation of motion in (6.5).

The Poisson properties of the map (6.5) are assured by the version of Theorem 3.1 for right-invariant Lagrangians.

It remains to demonstrate that the function (6.6) is indeed an integral of motion. This is done by the following derivation:

$$H_\epsilon(m_{k+1}, a_{k+1}) = \frac{1}{2}\langle m_{k+1}, m_{k+1}\rangle + \langle a_{k+1} - \frac{\epsilon}{2} m_{k+1} a_{k+1}, p\rangle$$
$$= \frac{1}{2}\langle m_{k+1}, m_{k+1}\rangle + \langle a_k + \frac{\epsilon}{2} m_{k+1} a_k, p\rangle$$
$$= \frac{1}{2}\langle m_{k+1}, m_{k+1} + \epsilon p \wedge a_k\rangle + \langle a_k, p\rangle$$
$$= \frac{1}{2}\langle m_k - \epsilon p \wedge a_k, m_k\rangle + \langle a_k, p\rangle = H_\epsilon(m_k, a_k).$$

(In this calculation we used the identity $\langle ma, p\rangle = \langle m, p \wedge a\rangle$ for $m \in so(n)$, $p, a \in \mathbb{R}^n$; notice that the scalar products on the both sides of this identity are defined on different spaces!) The theorem is proved. ∎

Actually, the map (6.5) possesses not only the integral (6.6) but a full set of involutive integrals necessary for the complete integrability. The most direct way to the proof of this statement is, as usual, through the Lax representation which lives, just as in the continuous time situation, in the loop algebra $\mathrm{sl}(n+1)[\lambda, \lambda^{-1}]$ twisted by the Cartan automorphism.

Theorem 6.2. *The map (6.5) admits the following Lax representation:*

$$\ell_{k+1}(\lambda) = v_k^{-1}(\lambda)\ell_k(\lambda)v_k(\lambda), \tag{6.11}$$

with the matrices

$$\ell_k(\lambda) = \begin{pmatrix} m_k & \lambda b_k - \lambda^{-1}p \\ \lambda b_k^{\mathrm{T}} - \lambda^{-1}p^{\mathrm{T}} & 0 \end{pmatrix}, \qquad v_k(\lambda) = \frac{I + (\epsilon/2)u_k(\lambda)}{I - (\epsilon/2)u_k(\lambda)},$$

$$u_k(\lambda) = \begin{pmatrix} 0 & \lambda a_k \\ \lambda a_k^{\mathrm{T}} & 0 \end{pmatrix},$$

where the following abbreviation is used:

$$b_k = \left(I - \frac{\epsilon}{2}m_k\right)a_k + \frac{\epsilon^2}{4}p. \tag{6.12}$$

Proof. Direct verification. Notice that it is most convenient to check (6.11) in the form

$$\left(I + \frac{\epsilon}{2}u_k(\lambda)\right)\ell_{k+1}(\lambda)\left(I - \frac{\epsilon}{2}u_k(\lambda)\right) = \left(I - \frac{\epsilon}{2}u_k(\lambda)\right)\ell_k(\lambda)\left(I + \frac{\epsilon}{2}u_k(\lambda)\right),$$

when it becomes polynomial in λ.   ∎

This theorem provides us with a complete set of integrals of motion of our discrete time Lagrangian map: these are the coefficients of the characteristic polynomial $\det(\ell_k(\lambda) - \mu I)$. Notice that these integrals of motion *do not* coincide with the integrals of motion of the continuous-time problem. To be more concrete, the integrals of motion of the map (6.5) are obtained from the integrals of the continuous time problem by replacing a by $b = a + O(\epsilon)$ given in (6.12). As for the actual integration of our map in terms of theta-functions, we leave it as an open problem for the interested reader (cf., e.g., [28, 17]).

6.2   Moving frame formulation

It turns out that the equations of the discrete time Lagrange top in the moving frame formulation become a bit nicer under a little inessential change of the Lagrange function (6.1), namely under replacing $\langle p, a_k \rangle$ in the last term on the right-hand side by $\langle p, a_{k+1} \rangle$. This modification does not influence the discrete action functional (3.1), apart from the boundary terms. It is not difficult to see that this modification is equivalent to exchanging $w_k \leftrightarrow w_k^{-1}$, $a_k \leftrightarrow a_{k+1}$, which in turn is equivalent to considering the evolution backwards in time with the simultaneous change $\epsilon \mapsto -\epsilon$. Now express the discrete Lagrange function (6.1), with the above modification, in terms of $P_k = g_k^{-1}p$ and $W_k = g_k^{-1}g_{k+1}$:

$$\mathbb{L}(g_k, g_{k+1}) = \Lambda^{(l)}(P_k, W_k) = -\frac{\alpha}{\epsilon}\,\mathrm{tr}\log\left(2I + W_k + W_k^{-1}\right)$$

$$- \frac{2(1-\alpha)}{\epsilon}\,\log\left(1 + \langle A, W_k A \rangle\right) - \epsilon\langle P_k, W_k A \rangle. \tag{6.13}$$

Since $W_k = I + \epsilon\Omega + O\left(\epsilon^2\right)$, we can apply Lemma 6.1 to see that

$$\Lambda^{(l)}(P_k, W_k) = \epsilon\mathcal{L}^{(l)}(P, \Omega) + O\left(\epsilon^2\right),$$

where $\mathcal{L}^{(l)}(P, \Omega)$ is the Lagrange function (4.2) of the continuous time Lagrange top. Now, one can derive all results concerning the discrete time Lagrange top in the body frame from the ones in the rest frame by performing the change of frames so that

$$M_k = g_k^{-1} m_k g_k, \qquad P_k = g_k^{-1} p, \qquad A = g_k^{-1} a_k,$$

and taking into account the modification mentioned above. Alternatively, one can derive them independently from and similarly to the rest frame results, applying Theorem 3.1, the main result of which, the system (3.5), takes in the present setup the form

$$\begin{cases} W_k M_{k+1} W_k^{-1} = M_k + P_k \wedge \nabla_P \Lambda^{(l)}(W_k, P_k), \\ P_{k+1} = W_k^{-1} P_k, \end{cases} \tag{6.14}$$

where

$$M_{k+1} = d'_W \Lambda^{(l)}(W_k, P_k). \tag{6.15}$$

Anyway, the corresponding results read:

Theorem 6.3. *The Euler–Lagrange equations for the Lagrange function (6.13) are equivalent to the following system:*

$$\begin{cases} M_{k+1} = W_k^{-1} M_k W_k + \epsilon A \wedge P_{k+1}, \\ P_{k+1} = W_k^{-1} P_k, \end{cases} \tag{6.16}$$

where the "angular velocity" $W_k \in \mathrm{SO}(n)$ is related to the "angular momentum" $M_k \in \mathrm{so}(n)$ via the "Legendre transformation":

$$M_k = \frac{2\alpha}{\epsilon} \cdot \frac{W_k - I}{W_k + I} - \frac{2(1-\alpha)}{\epsilon} \cdot \frac{A \wedge (W_k A)}{1 + \langle A, W_k A\rangle}. \tag{6.17}$$

The map (6.16), (6.17) is Poisson with respect to the Poisson bracket (4.4), (4.5), and has a complete set of involutive integrals assuring its complete integrability. One of these integrals is

$$\bar{H}_\epsilon(M, P) = \frac{1}{2}\langle M, M\rangle + \langle P, A\rangle + \frac{\epsilon}{2}\langle M, P \wedge A\rangle.$$

Proof. The derivation of (6.16), (6.17) is straightforward, like in Theorem 6.1. Obviously, due to $W_k = I + \epsilon\Omega + O\left(\epsilon^2\right)$, the equations of motion (6.16) approximate the continuous time ones (4.3), while the "Legendre transformation" (6.17) approximates (5.2). We discuss now the inversion of the "Legendre transformation" (6.17). Obviously, it is trivially invertible if $\alpha = 1$: then

$$M_k = \frac{2}{\epsilon} \cdot \frac{W_k - I}{W_k + I} \quad \Rightarrow \quad W_k = \frac{I + (\epsilon/2)M_k}{I - (\epsilon/2)M_k}.$$

For a general α we derive from (6.17):

$$W_k A = \frac{I + (\epsilon/2)M_k}{I - (\epsilon/2)M_k} A. \tag{6.18}$$

Indeed, we have:

$$\frac{\epsilon}{2}M_k A = \alpha \frac{W_k - I}{W_k + I} A - (1-\alpha)\frac{A\langle A, W_k A\rangle - W_k A}{1 + \langle A, W_k A\rangle},$$

$$\frac{\epsilon}{2}M_k W_k A = \alpha \frac{W_k - I}{W_k + I} W_k A - (1-\alpha)\frac{A - W_k A\langle A, W_k A\rangle}{1 + \langle A, W_k A\rangle}.$$

Adding these two equations, we find:

$$\frac{\epsilon}{2}M_k(A + W_k A) = \alpha \frac{W_k - I}{W_k + I}(A + W_k A) - (1-\alpha)(A - W_k A) = W_k A - A,$$

which yields (6.18). Now plugging the expression for $W_k A$ through M_k, A into the second term on the right-hand side of (6.17), we see that this equation is uniquely solvable for W_k. Turning to the last statement of Theorem 6.3, we have:

$$\begin{aligned}
\bar{H}_\epsilon(M_{k+1}, P_{k+1}) &= \frac{1}{2}\langle M_{k+1}, M_{k+1} + \epsilon P_{k+1} \wedge A\rangle + \langle P_{k+1}, A\rangle \\
&= \frac{1}{2}\langle W_k^{-1} M_k W_k - \epsilon P_{k+1} \wedge A, W_k^{-1} M_k W_k\rangle + \langle P_{k+1}, A\rangle \\
&= \frac{1}{2}\langle M_k, M_k\rangle + \langle P_{k+1}, A - \frac{\epsilon}{2} W_k^{-1} M_k W_k A\rangle \\
&= \frac{1}{2}\langle M_k, M_k\rangle + \langle P_k, W_k A - \frac{\epsilon}{2} M_k W_k A\rangle \\
&= \frac{1}{2}\langle M_k, M_k\rangle + \langle P_k, A + \frac{\epsilon}{2} M_k A\rangle = \bar{H}_\epsilon(M_k, P_k).
\end{aligned}$$

(On the last but one step we used the formula (6.18).)   ∎

We close this section with a Lax representation for the map (6.16), (6.17).

Theorem 6.4. *The map (6.16), (6.17) has the following Lax representation:*

$$L_{k+1}(\lambda) = V_k^{-1}(\lambda)L_k(\lambda)V_k(\lambda),$$

with the matrices

$$L_k(\lambda) = \begin{pmatrix} M_k & \lambda B_k - \lambda^{-1}P_k \\ \lambda B_k^{\mathrm{T}} - \lambda^{-1}P_k^{\mathrm{T}} & 0 \end{pmatrix},$$

$$V_k(\lambda) = \begin{pmatrix} W_k & 0 \\ 0 & 1 \end{pmatrix}\frac{I + (\epsilon/2)U(\lambda)}{I - (\epsilon/2)U(\lambda)}, \qquad U(\lambda) = \begin{pmatrix} 0 & \lambda A \\ -\lambda A^{\mathrm{T}} & 0 \end{pmatrix},$$

where the following abbreviation is used:

$$B_k = \left(I + \frac{\epsilon}{2} M_k\right) A + \frac{\epsilon^2}{4} P_k.$$

7   The Clebsch case of the rigid body motion in an ideal fluid

Clebsch [13] found an integrable case of the motion of a 3-dimensional rigid body in an ideal fluid, which was generalized in [26] for the n-dimensional situation. This problem is traditionally described by a Hamiltonian system on $\mathrm{e}^*(n)$ with the Hamilton function

$$H(M,P) = \frac{1}{2}\sum_{j<k} c_{jk}M_{jk}^2 - \frac{1}{2}\sum_{k=1}^{n} b_k P_k^2.$$

Here $(M,P) \in \mathrm{e}^*(n)$, so that $M \in \mathrm{so}(n)$, $P \in \mathbb{R}^n$, and $C = \{c_{jk}\}_{j,k=1}^n$, $B = \mathrm{diag}\,(b_k)$ are some symmetric matrices. The equations of motion read:

$$\begin{cases} \dot{M} = [M,\Omega] + P \wedge (BP), \\ \dot{P} = -\Omega P, \end{cases} \tag{7.1}$$

where the matrix $\Omega \in \mathrm{so}(n)$ is defined by the formula

$$\Omega_{jk} = c_{jk}M_{jk}.$$

The "physical" Lagrangian formulation of this problem is dealing with a Lagrangian on the group $\mathrm{E}(n)$, left-invariant under the action of a whole group. However, it may be obtained also in the framework of Section 2, from a Lagrangian on $\mathrm{SO}(n)$, left-invariant under the action of an isotropy subgroup of some element $p \in \mathbb{R}^n$. These two different settings lead to formally identical results.

So, one considers the Lagrange function

$$\mathbf{L}(g,\dot{g}) = \mathcal{L}^{(l)}(\Omega, P) = \frac{1}{2}\langle \Omega, \mathcal{J}(\Omega)\rangle + \frac{1}{2}\langle P, BP\rangle. \tag{7.2}$$

Here $(g,\dot{g}) \in T\mathrm{SO}(n)$, and $\Omega = g^{-1}\dot{g} \in \mathrm{so}(n)$, $P = g^{-1}p \in \mathbb{R}^n$. It is supposed that $\mathcal{J} : \mathrm{so}(n) \mapsto \mathrm{so}(n)$ is a linear operator acting as

$$\mathcal{J}(\Omega)_{jk} = c_{jk}^{-1}\Omega_{jk}.$$

The reduced equations of motion for this Lagrange function delivered by Theorem 2.1 coincide with (7.1).

The Clebsch case is characterized by the relations

$$\frac{b_i - b_j}{c_{ij}} + \frac{b_j - b_k}{c_{jk}} + \frac{b_k - b_i}{c_{ki}} = 0,$$

which implies that

$$c_{jk} = \frac{b_j - b_k}{a_j - a_k}$$

for some matrix $A = \mathrm{diag}\,(a_k)$. The Lax representation found in [26] reads:

$$\dot{L}(\lambda) = [L(\lambda), U(\lambda)],$$

where

$$L(\lambda) = \lambda A + M + \lambda^{-1}PP^{\mathrm{T}}, \qquad U(\lambda) = \lambda B + \Omega.$$

In [34] we found an integrable Lagrangian discretization of the flow of the Clebsch system characterized by

$$c_{jk} = \frac{1}{J_j + J_k}, \qquad b_k = J_k, \quad \text{so that} \quad a_k = J_k^2.$$

This flow may be considered as a particular case of the motion of a rigid body in a quadratic potential (notice that the kinetic energy term in (7.2) is in this case typical for the heavy top, since $\mathcal{J}(\Omega) = J\Omega + \Omega J$). Here we concentrate on another flow of the Clebsch system characterized by

$$c_{jk} = 1, \qquad \text{so that} \quad a_k = b_k.$$

The corresponding Lagrange function reads:

$$\mathbf{L}(g, \dot{g}) = \mathcal{L}^{(l)}(\Omega, P) = \frac{1}{2}\langle \Omega, \Omega \rangle + \frac{1}{2}\langle P, BP \rangle, \tag{7.3}$$

so that

$$M = \Omega,$$

and the equations of motion become

$$\begin{cases} \dot{M} = P \wedge (BP), \\ \dot{P} = -MP. \end{cases} \tag{7.4}$$

The Hamilton function of this flow is given by

$$H(M, P) = \frac{1}{2}\langle M, M \rangle - \frac{1}{2}\langle P, BP \rangle.$$

The Lax representation of this flow can be given in two equivalent forms:

$$\dot{L}(\lambda) = [L(\lambda), U_+(\lambda)] = [U_-(\lambda), L(\lambda)],$$

where

$$L(\lambda) = \lambda B + M + \lambda^{-1}PP^{\mathrm{T}}, \qquad U_+(\lambda) = \lambda B + M, \qquad U_-(\lambda) = \lambda^{-1}PP^{\mathrm{T}}.$$

Consider the following discrete time Lagrange function approximating (7.3):

$$\begin{aligned} \mathbb{L}(g_k, g_{k+1}) &= \Lambda^{(l)}(P_k, W_k) \\ &= -\frac{1}{\epsilon}\operatorname{tr}\log\left(2I + W_k + W_k^{-1}\right) + \frac{4}{\epsilon}\log\left(1 + \frac{\epsilon^2}{4}\langle P_k, BP_k \rangle\right). \end{aligned} \tag{7.5}$$

Here, as usual, $W_k = g_k^{-1}g_{k+1}$ and $P_k = g_k^{-1}p$.

Theorem 7.1. *The reduced Euler–Lagrange equations of motion for the Lagrange function (7.5) read:*

$$
\begin{cases}
M_{k+1} = M_k + \epsilon \dfrac{P_k \wedge (BP_k)}{1 + (\epsilon^2/4)\langle P_k, BP_k \rangle}, \\[4mm]
P_{k+1} = \dfrac{I - (\epsilon/2)M_{k+1}}{I + (\epsilon/2)M_{k+1}}\, P_k.
\end{cases}
\tag{7.6}
$$

The map $(M_k, P_k) \mapsto (M_{k+1}, P_{k+1})$ *is Poisson with respect to the bracket (4.4), (4.5). This map is completely integrable and admits the following Lax representation:*

$$
L_{k+1}(\lambda) = V_k(\lambda)L_k(\lambda)V_k^{-1}(\lambda),
$$

with the matrices

$$
L_k(\lambda) = \left(I + \frac{\epsilon^2}{4}\,B\right)^{-1}\left(\lambda B + M_k + \lambda^{-1}\mathcal{P}_k\right), \qquad V_k(\lambda) = \frac{I + (\epsilon\lambda^{-1}/2)\mathcal{Q}_k}{I - (\epsilon\lambda^{-1}/2)\mathcal{Q}_k},
$$

where

$$
\mathcal{P}_k = \left(I + \frac{\epsilon}{2}\,M_k\right) P_k P_k^{\mathrm{T}} \left(I - \frac{\epsilon}{2}\,M_k\right),
\tag{7.7}
$$

$$
\mathcal{Q}_k = \frac{1}{1 + (\epsilon^2/4)\,\langle P_k, BP_k \rangle}\, P_k P_k^{\mathrm{T}} \left(I + \frac{\epsilon^2}{4}\,B\right).
$$

Proof. Derivation of equations of motion is based on (6.14), (6.15). We have:

$$
M_{k+1} = \frac{2}{\epsilon} \cdot \frac{W_k - I}{W_k + I} \quad \Rightarrow \quad W_k = \frac{I + (\epsilon/2)M_{k+1}}{I - (\epsilon/2)M_{k+1}},
$$

which proves the second equation of motion in (7.6), and

$$
\nabla_P \Lambda^{(l)}(W_k, P_k) = \frac{\epsilon BP_k}{1 + (\epsilon^2/4)\,\langle P_k, BP_k \rangle},
$$

which proves the first one. As for the Lax representation, it is verified by a direct calculation. It is most convenient to check it in the form

$$
\left(I - \frac{\epsilon\lambda^{-1}}{2}\mathcal{R}_k\right)\left(\lambda B + M_{k+1} + \lambda^{-1}\mathcal{P}_{k+1}\right)\left(I + \frac{\epsilon\lambda^{-1}}{2}\mathcal{Q}_k\right)
$$

$$
= \left(I + \frac{\epsilon\lambda^{-1}}{2}\mathcal{R}_k\right)\left(\lambda B + M_k + \lambda^{-1}\mathcal{P}_k\right)\left(I - \frac{\epsilon\lambda^{-1}}{2}\mathcal{Q}_k\right),
\tag{7.8}
$$

where

$$
\mathcal{R}_k = \frac{1}{1 + (\epsilon^2/4)\langle P_k, BP_k \rangle}\left(I + \frac{\epsilon^2}{4}\,B\right) P_k P_k^{\mathrm{T}}.
$$

Let us indicate the main steps of this calculation. Expanding (7.8) in powers of λ, we come to the following equations:

$$
\lambda^0: \quad M_{k+1} - M_k = \epsilon(\mathcal{R}_k B - B\mathcal{Q}_k);
\tag{7.9}
$$

$$
\lambda^{-1}: \quad \mathcal{P}_{k+1} - \mathcal{P}_k = \frac{\epsilon}{2}\,\mathcal{R}_k(M_{k+1} + M_k) - \frac{\epsilon}{2}\,(M_{k+1} + M_k)\mathcal{Q}_k;
\tag{7.10}
$$

$$
\lambda^{-2}: \quad (\mathcal{P}_{k+1} + \mathcal{P}_k)\mathcal{Q}_k - \mathcal{R}_k(\mathcal{P}_{k+1} + \mathcal{P}_k) = \frac{\epsilon}{2}\,\mathcal{R}_k(M_{k+1} - M_k)\mathcal{Q}_k;
\tag{7.11}
$$

$$
\lambda^{-2}: \quad \mathcal{R}_k(\mathcal{P}_{k+1} - \mathcal{P}_k)\mathcal{Q}_k = 0.
\tag{7.12}
$$

Here (7.9) follows from the first equation in (7.6). Equation (7.10) yields (7.12), which follows from the skew-symmetry of M_{k+1}, M_k. To prove (7.10) we notice that the first equation of motion in (7.6) easily yields the following expressions:

$$\mathcal{Q}_k = P_k P_k^{\mathrm{T}} \left(I + \frac{\epsilon}{4}(M_{k+1} - M_k) \right), \qquad \mathcal{R}_k = \left(I - \frac{\epsilon}{4}(M_{k+1} - M_k) \right) P_k P_k^{\mathrm{T}}, \quad (7.13)$$

while the second equation of motion in (7.6) shows that

$$\mathcal{P}_{k+1} = \left(I - \frac{\epsilon}{2} M_{k+1} \right) P_k P_k^{\mathrm{T}} \left(I + \frac{\epsilon}{2} M_{k+1} \right). \tag{7.14}$$

This is a companion formula to (7.7). Now (7.10) follows directly from (7.13), (7.14), (7.7). Another consequence of these three formulas is:

$$\begin{aligned}(\mathcal{P}_{k+1} + \mathcal{P}_k)\mathcal{Q}_k &= \mathcal{R}_k(\mathcal{P}_{k+1} + \mathcal{P}_k) \\ &= \left(I - \frac{\epsilon}{4}(M_{k+1} - M_k) \right) P_k P_k^{\mathrm{T}} \left(I + \frac{\epsilon}{4}(M_{k+1} - M_k) \right),\end{aligned}$$

and this proves (7.11), since the right-hand side of this formula vanishes due to the skew-symmetry of M_{k+1}, M_k.   ■

We conclude this section with the following remarks. The discrete time system introduced here was found in the case $n = 3$ and in the su(2) framework by Adler [2], who however did not use the Lagrangian formalism. It is well known that the restriction of the flow (7.4) to the symplectic leaf in $\mathrm{e}^*(n)$ of the smallest possible dimension $2n$ is equivalent to the famous *Neumann system* which describes the motion of a particle on the surface of the sphere $S = \{x \in \mathbb{R}^n : \langle x, x \rangle = 1\}$ under the influence of the harmonic potential $(1/2)\langle x, Bx \rangle$. (The identification of the above mentioned symplectic leaf with T^*S is achieved via the formulas $P = x$, $M = x \wedge p$). It turns out that on this symplectic leaf our map (7.6) leads to *Adler's discretization of the Neumann system* [2], whose complete integrability was proved in [35]. The present construction delivers therefore a Lax representation for this discretization of the Neumann system – the problem left open in [35].

8   Conclusion

The models introduced in the present paper serve as further important examples of completely integrable Lagrangian systems with a discrete time à la Moser–Veselov. The version of the discrete Lagrangian reduction leading to systems on duals to semidirect product Lie algebras was developed in [8]. The list of relevant examples grows slowly, but now it already became quite representative: [37, 25, 7, 34], and the present work. Several remarks are in order here.

- Usually, discrete time Lagrangians lead to multi-valued maps (correspondences), cf. [25]. This is also the case for the models introduced in [37, 34] (discrete time Euler top and a discretization of the rigid body dynamics in a quadratic potential). Amasingly, the models of [7], as well as those of the present paper, lead to *single-valued*, and moreover *explicit* maps. Of course, one would like to be able to predict this outstanding property by just looking at the Lagrange function, but at present we do not know how to do this, since no deep reasons for this behavior are apparent.

- A very intriguing and still not completely understood point is the capability of the discrete Lagrangian approach to produce completely integrable systems. It should be stressed that all the models listed above were found by guess, and the same holds for their Lax representations. There seems to exist (at least, at present) no regular procedure for finding decent integrable discretizations for finite-dimensional systems of the classical mechanics, like the rigid body dynamics. This is in a sharp contrast to the area of integrable *differential-difference*, or *lattice* systems, where the problem of integrable discretization may be solved in an almost algorithmic way, preserving the Lax matrix and the Hamiltonian properties, cf. [33].

- For any completely integrable system, the involutive integrals of motion yield a set of commuting Hamiltonian flows (called a *hierarchy* in the infinite dimensional situation). These flows share the Lax matrix with the original one, and there exist well-established procedures for finding the Lax representations for them. In the discrete time, the situation seems to be different. Although there is a recipe for producing commuting Poisson maps for a given one, provided its Lax representation admits an *r*-matrix interpretation (which is the case for all the models listed above), there is no way, in general, for producing decent discretizations for concrete flows. Here "decent" means given by nice formulas in physical coordinates, or, if one prefers more precise terms, given by explicit Lagrange functions on Lie groups. In this connection it should be mentioned that in the only case when decent discretizations are known for two different flows of the same hierarchy (the Clebsch problem, cf. [34] and the present paper), these discretizations belong to *different* hierarchies and posess *different* Lax matrices and integrals of motion.

All in one, it should be said that while the Veselov's papers [37, 25] appeared more than a decade ago and happened to give a strong impetus for the development of the whole subject of discrete integrable systems, the narrower area of integrable discrete Lagrangian systems pioneered there is still at the beginning of its development.

Generally, we consider the discrete time Lagrangian mechanics as an important source of symplectic and, more general, Poisson maps. From some points of view the variational (Lagrangian) structure is even more fundamental and important than the Poisson (Hamiltonian) one. (Cf. [18, 21], where a similar viewpoint is represented. Notice that the book [19] also dealing with the omnipresence of Hamiltonian systems on the semidirect product Lie algebras, takes an opposite viewpoint: the variational (Lagrangian) principles are derived there from the postulated Hamiltonian structures.) It would be important to continue the search for integrable Lagrangian discretizations of the known integrable systems. Also generalizations to the infinite dimensional case, e.g. to a discretization of ideal compressible fluids motion (see [18, 19]), are highly desirable.

Acknowledgements

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