Journal of Nonlinear Mathematical Physics Volume 8, Number 2 (2001), 229–255 Article

Enveloping Superalgebra $U(\mathfrak{osp}(1|2))$ and Orthogonal Polynomials in Discrete Indeterminate

A SERGEEV

(Correspondence) D Leites, Department of Mathematics, University of Stockholm Roslagsv. 101, Kräftriket hus 6, SE-106 91, Stockholm, Sweden
E-mail: mleites@matematik.su.se

On leave of absence from Balakovo Institute of Technique of Technology and Control, Branch of Saratov Technical University, Balakovo, Saratov Region, Russia

Received May 6, 2000; Accepted November 18, 2000

Abstract

Let A be an associative simple (central) superalgebra over $\mathbb{C}$ and L an invariant linear functional on it (trace). Let $a \mapsto a^t$ be an antiautomorphism of A such that $(a^t)^t = (-1)^{p(a)}a$, where $p(a)$ is the parity of a, and let $L(a^t) = L(a)$. Then A admits a nondegenerate supersymmetric invariant bilinear form $\langle a, b\rangle = L(ab^t)$. For $A = U(\mathfrak{sl}(2))/\mathfrak{m}$, where $\mathfrak{m}$ is any maximal ideal of $U(\mathfrak{sl}(2))$, Leites and I have constructed orthogonal basis in A whose elements turned out to be, essentially, Chebyshev (Hahn) polynomials in one discrete variable. Here I take $A = U(\mathfrak{osp}(1|2))/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ and apply a similar procedure. As a result we obtain either Hahn polynomials over $\mathbb{C}[\tau]$, where $\tau^2 \in \mathbb{C}$, or a particular case of Meixner polynomials, or — when $A = \mathrm{Mat}(n+1|n)$ — dual Hahn polynomials of even degree, or their (hopefully, new) analogs of odd degree. Observe that the nondegenerate bilinear forms we consider for orthogonality are, as a rule, not sign definite.

1 Introduction

Classically, orthogonal polynomials were considered with respect to a sign definite bilinear form. Lately we encounter the growth of interest to the study of orthogonal polynomials relative an arbitrary (but still symmetric and nondegenerate) form, cf. [4, 7, 8] and references therein. In these approaches, however, the bilinear forms are introduced "by hands" and the differential or difference equations the orthogonal polynomials satisfy are of high degree. We would like to point out that traces and supertraces on associative algebras and superalgebras are natural sources of bilinear symmetric forms which are seldom sign-definite. The Lie structure on the algebras obtained from these associative algebras and superalgebras is more adapted to the study of orthogonal polynomials. In particular, the eigenvalue problem for the Casimir operator — the quadratic element of the center with respect to the Lie structure — naturally provides with a 2nd degree difference equation for the polynomials orthogonal relative the above (super)traces.

Let $\mathfrak{sl}(2)$ be represented as $\mathrm{Span}(Y, H, X)$ subject to relations

$$[X, Y] = H, \qquad [H, Y] = -2Y, \qquad [H, X] = 2X. \tag{1.1}$$

The quadratic Casimir operator of $\mathfrak{sl}(2)$

$$\Omega = 2YX + \frac{1}{2}H^2 + H \tag{1.2}$$

lies in the center of $U(\mathfrak{sl}(2))$. Let I_λ be the two-sided ideal in the associative algebra $U(\mathfrak{sl}(2))$ generated by $\Omega - \frac{1}{2}(\lambda^2 - 1)$. It turns out that the associative algebra $\tilde{\mathfrak{A}}_\lambda = U(\mathfrak{sl}(2))/I_\lambda$ is simple for $\lambda \notin \mathbb{Z} \setminus \{0\}$, otherwise $\tilde{\mathfrak{A}}_\lambda$ contains an ideal such that the quotient is isomorphic to the matrix algebra $\mathrm{Mat}(|\lambda|)$. Set ([1])

$$\mathfrak{A}_\lambda = \begin{cases} \tilde{\mathfrak{A}}_\lambda, & \text{if } \lambda \notin \mathbb{Z} \setminus \{0\}, \\ \mathrm{Mat}(|\lambda|), & \text{otherwise.} \end{cases} \tag{1.3}$$

Clearly, $\mathfrak{A}_{-\lambda} \simeq \mathfrak{A}_\lambda$. As associative algebra, $\mathfrak{A}_\lambda$ is generated by X, Y, and H subject to relations

$$XY = \frac{1}{4}\left(\lambda^2 - (H-1)^2\right) \tag{1.4}$$

and one more relation for integer values of λ:

$$X^{|\lambda|} = 0 \quad \text{if } \lambda \in \mathbb{Z} \setminus \{0\}. \tag{1.5}$$

It is also known that $\mathfrak{A}_\lambda$ possesses an antiautomorphism $u \mapsto u^t$ given on generators by the formula

$$X^t = Y, \qquad Y^t = X, \qquad H^t = H. \tag{1.6}$$

In [6] we have shown that on $\mathfrak{A}_\lambda$ there exists a unique, up to a constant factor, nontrivial linear functional L, which for positive integer λ's is the usual trace and which satisfies $L\left(u^t\right) = L(u)$. By means of this functional we define an invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{A}_\lambda$, by setting $\langle u, v\rangle = L\left(uv^t\right)$. The form $\langle \cdot, \cdot \rangle$ is nondegerate and symmetric.

Now, consider $\mathfrak{A}_\lambda$ as an $\mathfrak{sl}(2)$-module with respect to the adjoint representation. We have

$$\mathfrak{A}_\lambda = \begin{cases} \sum\limits_{i=0}^{\infty} L^{2i}, & \text{if } \lambda \notin \mathbb{Z} \setminus \{0\}, \\ \sum\limits_{i=0}^{|\lambda|-1} L^{2i}, & \text{otherwise.} \end{cases} \tag{1.7}$$

where L^{2i} is the irreducible finite dimensional $\mathfrak{sl}(2)$-module with highest weight $2i$ (cf. [2]). Clearly, H arranges a $\mathbb{Z}$-grading on $\mathfrak{A}_\lambda$, namely

$$(\mathfrak{A}_\lambda)_i = \{u \in \mathfrak{A}_\lambda \mid [H, u] = 2iu \text{ for } i \in \mathbb{Z}\}. \tag{1.8}$$

For any $f, g \in \mathbb{C}[H]$ and $i \geq 0$ set

$$\langle f, g\rangle_i = \langle f(H)X^i, g(H)X^i\rangle \qquad \text{and} \qquad \langle f, g\rangle_{-i} = \langle f(H)Y^i, g(H)Y^i\rangle. \tag{1.9}$$

Denote:

$$T_i(H) = \frac{1}{4}\left(n^2 - (H+2i-1)^2\right) \qquad \text{and} \qquad \alpha_i = n - 2i + 1. \tag{1.10}$$

Set

$$f_{ki}(H)X^i = (\operatorname{ad} Y)^k\left(X^{k+i}\right); \tag{1.11}$$

further set

$$(\Delta_2 f)(H) = f(H+2) - f(H) \qquad \text{and} \qquad (\nabla_2 f)(H) = f(H) - f(H-2). \tag{1.12}$$

Theorem 1.1 [6].

1) $\langle(\mathfrak{A}_\lambda)_i, (\mathfrak{A}_\lambda)_j\rangle = 0$ *for* $i \neq j$.

2) For $i \geq 0$ *the polynomials* $f_{ki}(H)$ *are of degree* k*, they are orthogonal relative to the form* $\langle\cdot,\cdot\rangle_i$.

3) For $i \geq 0$ *the polynomials* $f_{ki}(-H)$ *are of degree* k*, they are orthogonal relative to the form* $\langle\cdot,\cdot\rangle_{-i}$.

4) The polynomials $f_{ki}(H)$ *satisfy the following difference equation:*

$$\frac{1}{4}\left((H+1)^2-\lambda^2\right)(\Delta_2 f) - \frac{1}{4}\left((H-2i-1)^2-\lambda^2\right)(\nabla_2 f) = k(k+2i+1)f. \tag{1.13}$$

5) Explicitly we have

$$f_{kl}(H) = {}_3F_2\left(\begin{matrix} l-k,\ l+k+1,\ \frac{1}{2}(1-n-H) \\ l+1,\ l+1-n \end{matrix}\ \middle|\ 1\right) \times T_0((\alpha)_{l+1})\cdots T_0((\alpha)_{l+k}), \tag{1.14}$$

where

$${}_3F_2\left(\begin{matrix} \alpha_1,\ \alpha_2,\ \alpha_3 \\ \beta_1,\ \beta_2 \end{matrix}\ \middle|\ z\right) = \sum_{i=0}^{\infty} \frac{(\alpha_1)_i(\alpha_2)_i(\alpha_3)_i}{(\beta_1)_i(\beta_2)_i}\frac{z^i}{i!}$$

is a generalized hypergeometric function, $(\alpha)_0 = 1$ *and* $(\alpha)_i = \alpha(\alpha+1)\cdots(\alpha+i-1)$ *for* $i > 0$.

Our goal is to generalize this theorem by replacing $\mathfrak{sl}(2)$ with $\mathfrak{osp}(1|2)$. The main result obtained is the union of Theorems 2.2, 2.3 and 2.4.

2 Preliminaries and main result

We select the following basis in $\mathfrak{osp}(1|2) \subset \mathfrak{sl}(\bar{0}|\bar{1}|\bar{0})$:

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The defining relations (we give only the ones with nonzero values in the right hand side) are

$$\begin{aligned} &[H,F] = -F, \qquad [H,G] = G, \qquad [G,F] = H, \\ &[G,G] = 2X, \qquad [F,F] = -2Y. \end{aligned} \tag{2.1}$$

For convenience we add also the following corollaries

$$\begin{aligned} &[H,X] = 2X, \qquad [H,Y] = -2Y, \\ &[X,Y] = H, \qquad [G,Y] = F, \qquad [F,X] = G. \end{aligned} \tag{2.2}$$

These relations immediately imply that $U(\mathfrak{osp}(1|2))$ is generated, as associative superalgebra, by F, H, and G. Set

$$\begin{aligned} &\tau = 2GF - H + \frac{1}{2} = H + \frac{1}{2} - 2FG, \\ &\Omega = H^2 - H + 4XY - 2GF = H^2 + H + 4XY - 2FG, \\ &\omega = H^2 - 2H + 4XY = H^2 + 2H + 4YX. \end{aligned} \tag{2.3}$$

Lemma 2.1.

i) The element τ belongs to the supercenter of $U(\mathfrak{osp}(1|2))$, i.e., it commutes with the even elements and anticommutes with the odd ones.

ii) The centralizer of the Cartan subalgebra of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))$ is generated by H and τ.

iii) $\Omega = \tau^2 - \frac{1}{4}$ is the quadratic Casimir element of $U(\mathfrak{osp}(1|2))$.

iv) $\omega = \tau^2 + \tau - \frac{3}{4}$ is the quadratic Casimir element of $U(\mathfrak{sl}(2))$.

Proof. i) (the proof of the fact that $\tau G + G\tau = 0$ is similar):

$$\begin{aligned} \tau F + F\tau &= \left(2GF - H + \frac{1}{2}\right) F + F\left(-2FG + H + \frac{1}{2}\right) \\ &= FH - HF + F + 2GY - 2YG = 2F - 2[G,Y] = 0. \end{aligned}$$

ii) It is easy to verify that

$$\begin{aligned} G^nF^n = {} & \left(\frac{1}{2}\right)^n \left(\tau + H - \frac{1}{2}\right)\left(-\tau + H - \frac{3}{2}\right) \\ & \times \left(\tau + H - \frac{5}{2}\right) \cdots \left((-1)^n\tau + H - \frac{2n-1}{2}\right) \end{aligned} \tag{2.4}$$

Now observe that any element of the centralizer is a linear combination of the elements G^nF^n for $n \geq 0$.

Headings iii) and iv) are subject to a similar direct verification. ■

A theorem of Pinczon. Pinczon [12] described the maximal two-sided ideals of $U(\mathfrak{osp}(1|2))$. Let us formulate his results in a form convenient to us.

Theorem 2.1.

A) Every maximal two-sided ideal of $U(\mathfrak{osp}(1|2))$ *is of the form* $\mathfrak{m}_\lambda$, *where* $\mathfrak{m}_\lambda$ *is:*
generated by $\tau^2 - \left(\lambda + \frac{1}{2}\right)^2$ *for* $\lambda \notin \mathbb{Z}_{\geq 0} \cup \left\{-\frac{1}{2}\right\}$;
the kernel of the finite dimensional representation with highest weight λ *for* $\lambda \in \mathbb{Z}_{\geq 0}$;
generated by τ *for* $\lambda = -\frac{1}{2}$.

B) Let $\mathfrak{B}_\lambda = U(\mathfrak{osp}(1|2))/\mathfrak{m}_\lambda$. *Then*

i) If $\lambda \notin \mathbb{Z}_{\geq 0} \cup \{-\frac{1}{2}\}$, *then* $\mathfrak{B}_\lambda$ *is generated by* G, H, *and* F *subject to the relations*

$$\begin{aligned} &[H,G] = G, \qquad [H,F] = -F; \qquad [G,F] = H; \\ &\tau = H + \frac{1}{2} - 2FG, \qquad \tau^2 = \left(\lambda + \frac{1}{2}\right)^2. \end{aligned} \tag{2.5}$$

ii) The superalgebra $\mathfrak{B}_{-\frac{1}{2}}$ *is isomorphic to the Weil algebra* $A_1 = \mathrm{Diff}\,(1) = \mathbb{C}[P,Q]$ *considered as superalgebra when generators are considered to be odd (recall that the defining relations in* $\mathrm{Diff}\,(1)$ *are* $PQ - QP = 1$.

iii) $\mathfrak{B}_\lambda \simeq \mathrm{Mat}(\lambda+1|\lambda)$ *for* $\lambda \in \mathbb{Z}_{\geq 0}$.

On the structure of $\mathfrak{B}_\lambda$. Recall that an *antiautomorphism of super*algebra A is an even linear map $a \mapsto a^t$ for $a \in A$ such that $(ab)^t = (-1)^{p(a)p(b)} b^t a^t$. Define an antiautomorphism of $U(\mathfrak{osp}(1|2))$ by setting $H^t = H$, $F^t = G$, $G^t = -F$. Clearly, this antiautomorphism induces an antiautomorphism of $\mathfrak{B}_\lambda$ for every λ.

Later on, I will show that on $\mathfrak{B}_\lambda$ exists a unique, up to a scalar factor, nontrivial invariant linear functional — the supertrace str. So the form $\langle u, v\rangle = \mathrm{str}\left(uv^t\right)$ determines an invariant supersymmetric bilinear form on $\mathfrak{B}_\lambda$. The lack of nonzero two-sided ideals guarantees the non-degeneracy of the form.

For $\lambda \notin \mathbb{Z}_+ \cup \left\{-\frac{1}{2}\right\}$ the algebra $\mathfrak{B}_\lambda$ possesses a $\mathbb{Z}$-grading of the form $\mathfrak{B}_\lambda = \underset{i\in\mathbb{Z}}{\oplus} (\mathfrak{B}_\lambda)_i$, where

$$(\mathfrak{B}_\lambda)_i = \{u \in \mathfrak{B}_\lambda \mid [H,u] = iu \quad \text{for} \quad i \in \mathbb{Z}\}.$$

For $f, g \in \mathbb{C}[H,\tau]$ and $i \geq 0$ set

$$\langle f, g\rangle_i = \langle fG^i, gG^i\rangle \qquad \text{and} \qquad \langle f, g\rangle_{-i} = \langle fF^i, gF^i\rangle. \tag{2.6}$$

Recall that $\tau^2 = \left(\lambda + \frac{1}{2}\right)^2$ and introduce a $\mathbb{Z}$-grading in $\mathbb{C}[H,\tau]$ by setting $\deg H = 2$, $\deg \tau = 1$. Now it is not difficult to verify that for any $i \in \mathbb{Z}$ there exists a basis $\{f_k\}_{k\geq 0}$ such that $\deg f_k = k$ and $\langle f_k, f\rangle = 0$ if $\deg f < k$. In what follows the basis elements with such properties will be called *orthogonal polynomials* in H and τ.

Set also

$$\Delta(H) = \nabla(H) = 1, \qquad \Delta(\tau) = -2\tau, \qquad \nabla(\tau) = 2\tau \tag{2.7}$$

and extend the action of the operators Δ and ∇ onto $\mathbb{C}[H,\tau]$ by setting

$$\begin{aligned} \Delta(fg) &= \Delta(f)\cdot g + f\cdot\Delta(g) + \Delta(f)\Delta(g), \\ \nabla(fg) &= \nabla(f)\cdot g + f\cdot\nabla(g) + \nabla(f)\nabla(g). \end{aligned} \tag{2.8}$$

Besides, set

$$\Delta_2(\tau) = \nabla_2(\tau) = 0$$

and extent the action of the operators Δ_2 and ∇_2 onto $\mathbb{C}[H,\tau]$ by formulas similar to (2.8).

Define now polynomials $f_{k,j}$ for $j \geq 0$ by setting

$$f_{k,2i}G^{2i} = \begin{cases} (\operatorname{ad} F)^k \left(G^{2i+k}\right) & \text{for } k \text{ even,} \\ (\operatorname{ad} F)^k \left(G^{2i+k}\tau\right) & \text{for } k \text{ odd} \end{cases} \tag{2.9}$$

and

$$f_{k,2i+1}G^{2i+1} = \begin{cases} (\operatorname{ad} F)^{k+1} \left(G^{2i+k+2}\right) & \text{for } k \text{ even,} \\ (\operatorname{ad} F)^{k+1} \left(G^{2i+k}\tau\right) & \text{for } k \text{ odd.} \end{cases} \tag{2.10}$$

Theorem 2.2.

1) $\langle (\mathfrak{B}_\lambda)_i, (\mathfrak{B}_\lambda)_j \rangle = 0$ *for* $i \neq j$.

2) Polynomials $f_{k,j}(H)$ *are orthogonal relative to the form* $\langle \cdot, \cdot \rangle_i$*; the polynomial* $f_{k,j}(H)$ *is a degree* k *polynomial in* H *and* τ.

3) Polynomial $f_{k,2i}(H)$ *satisfies the difference equation*

$$\left(H - \tau + \frac{1}{2}\right) \Delta\nabla f + 2(H-i)\nabla f + \left(2i + \frac{1}{2}\right) f = (-1)^{p(k)} \left(2i + k + \frac{1}{2}\right) f.$$

4) Polynomial $f_{k,2i+1}(H)$ *satisfies the difference equation*

$$\left(H - \tau + \frac{1}{2}\right) \Delta\nabla f + 2\left(H - \tau - i + \frac{1}{2}\right) \nabla f + \left(2i + \frac{5}{2}\right) f$$
$$= (-1)^{p(k)} \left(2i + k + \frac{5}{2}\right) f.$$

5) Polynomials $f_{k,j}(-H)$ *are orthogonal relative to the form* $\langle \cdot, \cdot \rangle_{-i}$*; it is a degree* k *polynomial in* H *and* τ.

6) Polynomial $f_{k,2i}(-H)$ *satisfies the difference equation*

$$\left[(H+1)^2 - \left(\tau + \frac{1}{2}\right)^2\right] \Delta_2\nabla_2 f + 4(i+1)(H-i)\nabla_2 f + 4i(i+1)f$$
$$= \begin{cases} (2i+k)(2i+k+2)f & \textit{for } k \textit{ even,} \\ (2i+k-1)(2i+k+1)f & \textit{for } k \textit{ odd.} \end{cases}$$

7) Polynomial $f_{k,2i+1}(-H)$ *satisfies the difference equation*

$$\left[(H+1)^2 - \left(\tau + \frac{1}{2}\right)^2\right] \Delta_2\nabla_2 f + [(2i+3)(2h-2i-1) - 2\tau]\nabla_2 f$$
$$+ (2i+1)(2i+5)f = \begin{cases} (2i+k+1)(2i+k+3)f & \textit{for } k \textit{ even,} \\ (2i+k)(2i+k+2)f & \textit{for } k \textit{ odd.} \end{cases}$$

8) Polynomial $f_{k,2i}(H)$ can be expressed via Hahn polynomials with parameter τ, namely

$$f_{2k,2i}(H) = (-1)^k \frac{(i+1)_k \left(i+\frac{1}{2}-\tau\right)_k}{k!} \times {}_3F_2 \left(\left. \begin{matrix} -k,\ k+2i+1,\ \frac{1}{2}\left(2i+\frac{1}{2}-\tau-H\right) \\ i+1,\ i+\frac{1}{2}-\tau \end{matrix} \right| 1 \right),$$

$$f_{2k+1,2i}(H) = \frac{1}{2}\left[(2k+2i+1)\tau - \left(\lambda+\frac{1}{2}\right)^2\right] f_{2k,2i}(H).$$

9) Polynomials $f_{k,2i+1}(H)$ can be expressed via Hahn polynomials with parameter τ, namely

$$f_{2k,2i+1}(H) = (-1)^k \frac{(i+2)_k \left(i+\frac{3}{2}-\tau\right)_k}{k!} \times {}_3F_2 \left(\left. \begin{matrix} -k,\ k+2i+2,\ \frac{1}{2}\left(2i+\frac{5}{2}-\tau-H\right) \\ i+2,\ i+\frac{3}{2}-\tau \end{matrix} \right| 1 \right),$$

$$f_{2k+1,2i+1}(H) = -\frac{\tau}{2i+k+2} f_{2k,2i+1}(H).$$

Another theorem of Pinczon. Recall (Theorem 2.1, B)) that the superalgebra $\mathfrak{B}_{-\frac{1}{2}}$ is isomorphic to the Weyl algebra A_1 considered as *super*algebra with generators P, Q and relations $PQ - QP = 1$. The corresponding isomorphism θ is given by the formulas

$$\theta(F) = \frac{1}{\sqrt{2}}P, \qquad \theta(G) = \frac{1}{\sqrt{2}}Q, \qquad \theta(H) = \frac{1}{2}(PQ+QP).$$

As is easy to verify, $\theta(\tau) = 0$.

There is a $\mathbb{Z}$-grading of A_1 such that (having identified H with $\theta(H)$)

$$A_1 = \underset{i\in\mathbb{Z}}{\oplus}(A_1)_i, \qquad \text{where } (A_1)_i = \{u \in A_1 \mid [H,u] = iu\} \text{ for } i \in \mathbb{Z}.$$

For $f, g \in \mathbb{C}[H]$ and $i \geq 0$ set

$$\langle f, g\rangle_i = \langle fQ^i, gQ^i\rangle \qquad \text{and} \qquad \langle f, g\rangle_{-i} = \langle fP^i, gP^i\rangle. \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ is the bilinear form on A_1 defined in Section 4. Now, for $i \geq 0$ define the polynomials $f_{k,i}$ from the equations

$$\begin{aligned} f_{k,2i}Q^{2i} &= (\mathrm{ad}\ P)^{2k}\left(Q^{2i+2k}\right), \\ f_{k,2i+1}Q^{2i+1} &= (\mathrm{ad}\ P)^{2k+1}\left(Q^{2i+2k+2}\right). \end{aligned} \tag{2.12}$$

Let us endow the algebra $\mathbb{C}[H]$ with a grading by setting $\deg H = 1$.

Theorem 2.3.

1) $\langle (A_1)_i (A_1)_j\rangle = 0$ for $i \neq j$.

2) $f_{k,i}(H)$ *are polynomials in* H *and* τ *of degree* k *orthogonal with respect to the form* $\langle\cdot,\cdot\rangle_i$.

3) $f_{k,i}(-H)$ *are polynomials in* H *and* τ *of degree* k *orthogonal with respect to the form* $\langle\cdot,\cdot\rangle_{-i}$.

4) $f_{k,2i}(H)$ *satisfies the difference equation*

$$\left(H+\frac{1}{2}\right)\Delta f+\left(H-2i-\frac{1}{2}\right)\nabla f=2kf.$$

5) $f_{k,2i+1}(H)$ *satisfies the difference equation*

$$\left(H+\frac{1}{2}\right)\Delta f+\left(H-2i-\frac{3}{2}\right)\nabla f=2kf.$$

6) The polynomials $f_{k,2i}(H)$ *can be expressed via Meixner polynomials:*

$$f_{k,2i}(H)=\frac{(2i+1)_k(2i+k+1)_k}{k!}\cdot {}_2F_1\left(\begin{array}{c}-k,\ 2i-H+\frac{1}{2}\\ 2i+1\end{array}\middle|\ 2\right),$$

where

$${}_2F_1\left(\begin{array}{c}a_1,\ a_2\\ b\end{array}\middle|\ z\right)=\sum_{i=0}^{\infty}\frac{(a_1)_i(a_2)_i}{(b)_i}\,z^i.$$

7) The polynomials $f_{k,2i+1}(H)$ *can be expressed via Meixner polynomials:*

$$f_{k,2i+1}(H)=\frac{(2i+2)_k(2i+k+2)_{k+1}}{k!}\cdot {}_2F_1\left(\begin{array}{c}-k,\ 2i-H+\frac{3}{2}\\ 2i+2\end{array}\middle|\ 2\right).$$

The case of $\mathfrak{B}_\lambda$ for $\lambda\in\mathbb{Z}_{\geq 0}$ In this case $\mathfrak{B}_\lambda=\mathrm{Mat}(\lambda+1|\lambda)$ and the image of τ under the natural homomorphism $U(\mathfrak{osp}(1|2))\longrightarrow\mathfrak{B}_\lambda$ is a polynomial in H. Therefore, having applied the arguments after Theorem 2.1 (on the structure of $\mathfrak{B}_\lambda$) we obtain an orthogonal basis distinct from the basis of orthogonal polynomials.

To construct orthogonal polynomials, set

$$U=F,\qquad V=\left(\tau-H+\frac{1}{2}\right)G.$$

It is easy to verify that

$$HU-UH=-U,\qquad HV-VH=V,\qquad VU-UV=H. \tag{2.13}$$

Relations (2.13) mean that U, V, H generate in $U(\mathfrak{osp}(1|2))$ a subalgebra isomorphic to $U(\mathfrak{sl}(2))$ considered as a superalgebra such that $p(U)=p(V)=\overline{1}$, $p(H)=\overline{0}$. Observe also that the images of U, V, and H in $\mathfrak{B}_\lambda$ generate $\mathfrak{B}_\lambda$ and are subject to relations

$$VU=\frac{1}{2}\left(\lambda(\lambda+1)-H^2+H\right),\qquad UV=\frac{1}{2}\left(\lambda(\lambda+1)-H^2-H\right) \tag{2.14}$$

(we have identified U, V, and H with their images in $\mathfrak{B}_\lambda$). The superalgebra $\mathfrak{B}_\lambda$ is $\mathbb{Z}$-graded $\mathfrak{B}_\lambda = \bigoplus_{i\in\mathbb{Z}} (\mathfrak{B}_\lambda)_i$, where

$$(\mathfrak{B}_\lambda)_i = \{u \in \mathfrak{B}_\lambda \mid [H, u] = iu\}.$$

Being a matrix superalgebra, $\mathfrak{B}_\lambda$ possesses an antiautomorphism, the supertransposition, which in terms of the generators is given by the formula

$$H^t = H, \qquad U^t = -V, \qquad V^t = U.$$

The supertrace gives rise to a bilinear form $\langle u, v\rangle = \operatorname{str}\left(uv^t\right)$ on $\mathfrak{B}_\lambda$.

For $f, g \in \mathbb{C}[H]$ and $i \geq 0$ define the bilinear forms

$$\langle f, g\rangle_i = \langle fV^i, gV^i\rangle \qquad \text{and} \qquad \langle f, g\rangle_{-i} = \langle fU^i, gU^i\rangle. \tag{2.15}$$

Further on, for $i \in \mathbb{Z}_{\geq 0}$ set

$$\begin{aligned} &f_{0,2i}V^{2i} = V^{2i}, \quad f_{2,2i}V^{2i} = \left[U, V^{2i+1}\right], \quad \dots, \\ &f_{2k,2i}V^{2i} = \left[U, \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, \left[U, \frac{1}{H-\left(i+\frac{3}{2}\right)}\cdots\right.\right.\right. \\ &\qquad\left.\left.\left.\frac{1}{H-\left(i+k-\frac{3}{2}\right)}\left[U, \left[U, V^{2i+2k-1}\right]\cdots\right]\right]\right]\right]. \end{aligned} \tag{2.16}$$

For $u, v \in \mathfrak{B}_\lambda$ set

$$\{u, v\} = uv - (-1)^{p(u)(p(v)+1)}vu$$

and define:

$$\begin{aligned} &f_{1,2i}V^{2i} = \left\{U, V^{2i+1}\right\}, \quad f_{3,2i}V^{2i} = \left\{U, V^{2i+1}\right\}, \quad \dots, \\ &f_{2k+1,2i}V^{2i} = \left\{U, \frac{1}{H-\left(i+\frac{1}{2}\right)}\left\{U, \left\{U, \frac{1}{H-\left(i+\frac{3}{2}\right)}\cdots\right.\right.\right. \\ &\qquad\left.\left.\left.\frac{1}{H-\left(i+k-\frac{3}{2}\right)}\left\{U, \left\{U, V^{2i+2k-1}\right\}\cdots\right\}\right\}\right\}\right\}. \end{aligned} \tag{2.17}$$

Further on, set

$$\begin{aligned} &f_{0,2i+1}V^{2i+1} = V^{2i+1}, \quad f_{2,2i+1}V^{2i+1} = \left[U, V^{2i+2}\right], \quad \dots, \\ &f_{2k,2i+1}V^{2i+1} = \left[U, \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, \left[U, \frac{1}{H-\left(i+\frac{3}{2}\right)}\cdots\right.\right.\right. \\ &\qquad\left.\left.\left.\frac{1}{H-\left(i+k-\frac{3}{2}\right)}\left[U, \left[U, V^{2i+2k}\right]\cdots\right]\right]\right]\right], \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &f_{1,2i+1}V^{2i+1} = \left\{U, V^{2i+2}\right\}, \quad f_{3,2i+1}V^{2i+1} = \left\{U, V^{2i+2}\right\}, \quad \dots, \\ &f_{2k+1,2i+1}V^{2i+1} = \left(H - \left(i + \frac{1}{2}\right)\right) f_{2k,2i+1}. \end{aligned} \tag{2.19}$$

Theorem 2.4.

1) $\langle(\mathfrak{B}_\lambda)_i, (\mathfrak{B}_\lambda)_j\rangle = 0$ *for* $i \neq j$.

2) $f_{l,2i}$ *are orthogonal polynomials of degree* l *with respect to the form* $\langle\cdot,\cdot\rangle_{2i}$.

3) *Polynomials* $f_{2k,2i}(H)$ *satisfy the difference equation*

$$\frac{(H-\lambda)(H+\lambda+1)}{2H-2i+1}\Delta f + \frac{(H-2i-\lambda-1)(H-2i+\lambda)}{2H-2i-1}\nabla f = 2kf.$$

4) *Polynomials* $f_{2k+1,2i}(H)$ *satisfy the difference equation*

$$\frac{(H-\lambda)(H+\lambda+1)}{2H-2i+1}\Delta f + \frac{(H-2i-\lambda-1)(H-2i+\lambda)}{2H-2i-1}\nabla f + \frac{\left(\lambda+\frac{1}{2}\right)^2 - i^2}{(H-i)^2-\frac{1}{4}} f = (2k+1)f.$$

5) *Polynomials* $f_{l,2i}(-H)$ *are orthogonal with respect to the form* $\langle\cdot,\cdot\rangle_{-2i}$.

6) *Polynomials* $f_{l,2i+1}(H)$ *are of degree* l *and satisfy the following relations*

$$\langle f_{l,2i+1}, f_{m,2i+1}\rangle_{2i+1} \neq 0 \text{ only if } \{l,m\} = \{2k, 2k+1\}_{k\in\mathbb{Z}_{\geq 0}}.$$

7) *Polynomials* $f_{l,2i+1}(-H)$ *are of degree* l *and satisfy the following relations*

$$\langle f_{l,2i+1}(-H), f_{m,2i+1}(-H)\rangle_{-(2i+1)} \neq 0 \text{ only if } \{l,m\} = \{2k, 2k+1\}_{k\in\mathbb{Z}_{\geq 0}}.$$

8) *Polynomials* $f_{2k+1,2i}(-H)$ *satisfy the difference equation*

$$\frac{(H-\lambda)(H+\lambda+1)}{2H-2i-1}\Delta f + \frac{(H-2i-1+\lambda-1)(H-2i-2+\lambda)}{2H-2i-1}\nabla f = 2kf.$$

9) *Polynomials* $f_{2k,2i}$ *can be expressed via the dual Hahn polynomials:*

$$f_{2k,2i}(H) = (-1)^k \frac{(2i+2)_k(\lambda+i+2)_k(i-\lambda)_k}{k!} \cdot {}_3F_2\left(\begin{matrix} -k,\ i-H,\ H+i \\ \lambda+i+2,\ i-\lambda \end{matrix}\ \middle|\ 1\right).$$

3 Proof: the case $\boldsymbol{\lambda \notin \left\{-\frac{1}{2}\right\} \cup \mathbb{Z}_{\geq 0}}$

Lemma 3.1. *Let* A *be an associative superalgebra generated by a set* X. *Denote by* $[X,A]$ *the set of linear combinations of the form* $\sum[x_i, a_i]$, *where* $x_i \in X$, $a_i \in A$. *Then* $[A,A] = [X,A]$.

Proof. Let us apply the identity [11, p. 561]

$$[ab,c] = [a,bc] + \varepsilon(a,bc)[b,ca], \tag{3.1}$$

where $\varepsilon(a,bc) = (-1)^{p(a)(p(b)+p(c))}$.

Namely, let $a = x_1 \cdots x_n$; let us perform induction on n to prove that $[a, A] \subset [X, A]$. For $n = 1$ the statement is obvious. If $n > 1$, then $a = xa_1$, where $x \in X$ and due to (3.1) we have

$$[a, c] = [xa_1, c] = [x, a_1 c] + \varepsilon(x, a_1 c)[a_1, cx]. \qquad \blacksquare$$

Lemma 3.2. *Let A be an associative superalgebra and $a \mapsto a^t$ be its antiautomorphism (supertransposition, i.e., it satisfies $(ab)^t = (-1)^{p(a)p(b)} b^t a^t$ and $(a^t)^t = (-1)^{p(a)} a$). Let L be an even invariant functional on A (like supertrace, i.e., $L([A, A]) = 0$) such that $L(a^t) = L(a)$ for any $a \in A$. Define the bilinear form on A by setting*

$$\langle u, v\rangle = L\left(uv^t\right) \quad \text{for any} \quad u, v \in A. \tag{3.2}$$

Then $\langle u, v\rangle = \langle v, u\rangle$ and

$$\langle [w, u], v\rangle = (-1)^{p(w)(p(u)+1)} \langle u, [w^t, v]\rangle. \tag{3.3}$$

Proof. Observe first that

$$\begin{aligned}\langle u, v\rangle &= L\left(uv^t\right) = L\left(\left(uv^t\right)^t\right) = (-1)^{p(u)p(v)} L\left(\left(v^t\right)^t u^t\right) \\ &= (-1)^{p(v)(p(u)+1)} L\left(vu^t\right) (-1)^{p(v)(p(u)+1)} \langle v, u\rangle.\end{aligned}$$

Since L is even, we see that $\langle u, v\rangle \neq 0$ only if $p(v) = p(u)$. But in this case $(-1)^{p(v)(p(u)+1)} = 1$.

Further on:

$$\begin{aligned}\langle [w, u], v\rangle &= L\left([w, u]v^t\right) = L\left(\left[w, uv^t\right] - (-1)^{p(w)p(u)} u\left[w, v^t\right]\right) \\ &= (-1)^{p(w)(p(u)+1)} L\left(u\left[w, v^t\right]\right).\end{aligned}$$

But $[w, v^t] = (-1)^{p(w)+1}\left(\left[w^t, v\right]\right)^t$. Therefore,

$$\langle [w, u], v\rangle = (-1)^{p(w)(p(u)+1)} L\left(u\left(\left[w^t, v\right]\right)^t\right) = (-1)^{p(w)(p(u)+1)} \langle u, \left[w^t, v\right]\rangle. \qquad \blacksquare$$

Lemma 3.3. *Set $\{u, v\} = uv - (-1)^{p(u)(p(v)+1)} vu$ and let $\langle\cdot,\cdot\rangle$ be the bilinear form as in Lemma 3.2. Then for $u, v, w \in A$ we have*

$$\langle \{w, u\}, v\rangle = (-1)^{p(w)(p(u)+1)} \langle u, \left\{w^t, v\right\}\rangle.$$

Proof. It is not difficult to verify the following identities:

$$\begin{aligned}&\{w, u\}v = [w, uv] - (-1)^{p(w)(p(u)+1)} u\{w, v\}, \\ &\left\{w, v^t\right\} = -\left(\left\{w^t, v\right\}\right)^t.\end{aligned} \tag{3.4}$$

They imply

$$\begin{aligned}\langle \{w, u\}, v\rangle &= L\left(\{w, u\}v^t\right) = L\left(\left[w, uv^t\right] - (-1)^{p(u)(p(v)+1)} u\left\{w, v^t\right\}\right) \\ &= -(-1)^{p(u)(p(v)+1)} L\left(u\left\{w, v^t\right\}\right) = -(-1)^{p(u)(p(v)+1)} L\left(u\left(\left\{w^t, v\right\}\right)^t\right) \\ &= -(-1)^{p(u)(p(v)+1)} \langle u, \left\{w^t, v\right\}\rangle. \qquad \blacksquare\end{aligned}$$

Lemma 3.4. *On $\mathfrak{B}_\lambda$, there exists a unique, up to a scalar factor, invariant linear functional, L. It is uniquely determined by its restriction onto $\mathbb{C}[H]$. To every functional L on $\mathbb{C}[H]$ assign its generating function $\varphi_L(t) = \sum\limits_{k=0}^{\infty} \frac{L(H^k)}{k!} t^k$. Then for a constant $C \in \mathbb{C}$ we have*

$$\varphi_L(t) = C\frac{e^{(\lambda+1)t} + e^{-\lambda t}}{e^t + 1}.$$

Proof. By Theorem 2.1 the superalgebra $\mathfrak{B}_\lambda$ is generated by G, H, and F subject to relations

$$[H,G] = G, \qquad [H,F] = -F, \qquad [G,F] = H,$$
$$\tau = H + \frac{1}{2} - 2FG, \qquad \tau^2 = \left(\lambda + \frac{1}{2}\right)^2.$$

Recall that $(\mathfrak{B}_\lambda)_i = \{u \in \mathfrak{B}_\lambda \mid [H,u] = iu\}$ for $i \in \mathbb{Z}$. Then for $i \in \mathbb{Z}_{\geq 0}$ we have

$$(\mathfrak{B}_\lambda)_i = \{fG^i \mid f \in \mathbb{C}[H,\tau]\}, \qquad (\mathfrak{B}_\lambda)_{-i} = \{fF^i \mid f \in \mathbb{C}[H,\tau]\}.$$

Therefore, $L((\mathfrak{B}_\lambda)_i) = 0$ if $i \neq 0$, so any trace L is only nonzero on $(\mathfrak{B}_\lambda)_0 = \mathbb{C}[H,\tau]$. To this restriction assign the generating function

$$\varphi_L(t) = \sum_{k=0}^{\infty} \frac{L\left(H^k\right)}{k!} t^k + \frac{\varepsilon}{\lambda + \frac{1}{2}} \sum_{k=0}^{\infty} \frac{L\left(\tau H^k\right)}{k!} t^k$$

where $\varepsilon^2 = 1$ and $\varepsilon t = t\varepsilon$. The following statements are easy to verify:

i) If θ is an automorphism of $\mathbb{C}[H,\tau]$ such that $\theta(H) = H + 1$, $\theta(\tau) = -\tau$, and $L_1(f) = L(\theta(f))$, then $\varphi_{L_1} = e^t \overline{\varphi_L}$, where $\overline{\varphi_L} = \varphi_{L,\bar{0}} - \varepsilon\varphi_{L,\bar{1}}$ for each $\varphi_L = \varphi_{L,\bar{0}} + \varepsilon\varphi_{L,\bar{1}}$, where $\varphi_{L,\bar{0}}$ and $\varphi_{L,\bar{1}}$ are formal power series in t.

ii) If $L_2(f) = L(\tau f)$, where $f \in \mathbb{C}[H,\tau]$, then

$$\varphi_{L_2} = \left(\lambda + \frac{1}{2}\right) \varepsilon\varphi_L.$$

iii) If $P \in \mathbb{C}[H]$, and $L_3(f) = L(Pf)$ for $f \in \mathbb{C}[H,\tau]$, then

$$\varphi_{L_3} = P\left(\frac{d}{dt}\right)\varphi_L.$$

Making use of these statements, let us calculate the generating function for the restriction of the functional L onto $\mathbb{C}[H,\tau]$. For $f \in \mathbb{C}[H,\tau]$ we have

$$[F, fG] = FfG + fGF = \theta(f)FG + fGF = \theta(f)\frac{1}{2}\left(H - \tau + \frac{1}{2}\right)$$
$$+ \frac{1}{2}f\left(H - \tau + \frac{1}{2}\right) = \theta\left(\frac{1}{2}f\left(H - \tau + \frac{1}{2}\right)\right) + \frac{1}{2}f\left(H - \tau + \frac{1}{2}\right).$$

Hence, $L\left(\theta\left(\frac{1}{2}f\left(H-\tau+\frac{1}{2}\right)\right)+\frac{1}{2}f\left(H-\tau+\frac{1}{2}\right)\right)=0$. Therefore, thanks to i)–iii) we have

$$\left(\frac{d}{dt}-\frac{1}{2}+\left(\lambda+\frac{1}{2}\right)\varepsilon\right)\left(e^t\bar{\varphi}_L+\varphi_L\right)=0. \tag{3.5}$$

For $\varphi_L=\varphi_{L,\bar{0}}+\varepsilon\varphi_{L,\bar{1}}$ we obtain a system:

$$\left(\frac{d}{dt}-\frac{1}{2}\right)\left(e^t+1\right)\varphi_{L,\bar{0}}+\left(\lambda+\frac{1}{2}\right)\left(e^t-1\right)\varphi_{L,\bar{1}}=0$$
$$\left(\frac{d}{dt}-\frac{1}{2}\right)\left(e^t-1\right)\varphi_{L,\bar{1}}+\left(\lambda+\frac{1}{2}\right)\left(e^t+1\right)\varphi_{L,\bar{0}}=0.$$

Hence,

$$\left(\frac{d}{dt}-\frac{1}{2}\right)^2\left(e^t-1\right)\varphi_{L,\bar{1}}+\left(\lambda+\frac{1}{2}\right)^2\left(e^t-1\right)\varphi_{L,\bar{1}}=0$$

or, even simpler,

$$\left(\frac{d}{dt}-\lambda-1\right)\left(\frac{d}{dt}+\lambda\right)\left(e^t-1\right)\varphi_{L,\bar{1}}=0.$$

This implies that

$$\varphi_{L,\bar{1}}=c\frac{e^{(\lambda+1)t}-e^{-\lambda t}}{e^t-1} \qquad \text{and} \qquad \varphi_{L,\bar{0}}=c\frac{e^{(\lambda+1)t}+e^{-\lambda t}}{e^t+1}.$$

Since $\varphi_{L,\bar{1}}$ is uniquely recovered from $\varphi_{L,\bar{0}}$, we see that L is uniquely recovered by its restriction onto $\mathbb{C}[H]$. This proves uniqueness.

Let us prove existence of L. It suffices to prove that $1\notin[\mathfrak{B}_\lambda,\mathfrak{B}_\lambda]$. Indeed, by Lemma 3.1

$$[\mathfrak{B}_\lambda,\mathfrak{B}_\lambda]=[F,\mathfrak{B}_\lambda]+[G,\mathfrak{B}_\lambda].$$

Hence,

$$\begin{aligned}[\mathfrak{B}_\lambda,\mathfrak{B}_\lambda]\cap\mathbb{C}[H,\tau]&=([F,\mathfrak{B}_\lambda]+[G,\mathfrak{B}_\lambda])\cap\mathbb{C}[H,\tau]\\&=[F,(\mathfrak{B}_\lambda)_1]\cap\mathbb{C}[H,\tau]=\operatorname{Span}\left(\theta\left(f\left(H+\tau-\frac{1}{2}\right)\right)+f\left(H+\tau-\frac{1}{2}\right)\right)\end{aligned}$$

for any $f\in\mathbb{C}[H,\tau]$.

But $\deg[F,fG]\geq 1$ (recall that $\deg H=2$, $\deg\tau=1$), so $1\notin[\mathfrak{B}_\lambda,\mathfrak{B}_\lambda]$. ■

Lemma 3.5. *Let L be a linear functional on $\mathfrak{B}_\lambda$ determined in Lemma 3.4 and normed so that $L(1)=1$. Define an antiautomorphism of $\mathfrak{B}_\lambda$ by setting*

$$H^t=H, \qquad G^t=-F, \qquad F^t=G.$$

Then

i) $\left(u^t\right)^t=(-1)^{p(u)}u$ *for any* $u\in\mathfrak{B}_\lambda$;

ii) $L\left(u^t\right)=L(u)$ *for any* $u\in\mathfrak{B}_\lambda$.

Proof. i) Induction on n, where $u = x_1 \dots x_n$ and $x_i \in \mathrm{Span}\,(H, G, F)$ for each i.

For $n = 1$ the statement is obvious. Let $u = u_1 u_2$ and let for u_1 and for u_2 the statement be true. Then

$$\begin{aligned}\left(u^t\right)^t &= \left((u_1u_2)^t\right)^t = \left((-1)^{p(u_1)p(u_2)} u_2^t u_1^t\right)^t \\ &= \left(u_1^t\right)^t \left(u_2^t\right)^t = (-1)^{p(u_1)+p(u_2)} u_1 u_2 = (-1)^{p(u)} u.\end{aligned}$$

ii) Let us represent $u \in \mathfrak{B}_\lambda$ in the form $u = u_0 + L(u)$, where $u_0 \in \mathrm{Ker}\, L$. But we know that $\mathrm{Ker}\, L = [\mathfrak{B}_\lambda, \mathfrak{B}_\lambda]$. So

$$(\mathrm{Ker}\, L)^t = ([\mathfrak{B}_\lambda, \mathfrak{B}_\lambda])^t = [\mathfrak{B}_\lambda, \mathfrak{B}_\lambda] = \mathrm{Ker}\, L.$$

Hence, $u_0^t \in \mathrm{Ker}\, L$ and $u^t = u_0^t + L(u)$. Thus, $L\left(u^t\right) = L(u)$. ■

Therefore, we can define a bilinear form $\langle u, v\rangle = L\left(u^t v\right)$ on $\mathfrak{B}_\lambda$. By Lemma 3.2 we have: $\langle u, v\rangle = (-1)^{p(u)p(v)}\langle v, u\rangle$ and

$$\langle [w, u], v\rangle = (-1)^{(p(u)+1)p(w)}\langle u, \left[w^t, v\right]\rangle.$$

Proof of heading 1 of Theorem 2.2. Let $u \in (\mathfrak{B}_\lambda)_i$, $u \in (\mathfrak{B}_\lambda)_j$. Then

$$i\langle u, v\rangle = \langle [H, u], v\rangle = \langle u, [H, v]\rangle = j\langle u, v\rangle.$$

Therefore, if $i \neq j$, then $\langle u, v\rangle = 0$.

Proof of heading 2 of Theorem 2.2. By [12] (see also [3]), there is an expansion $\mathfrak{B}_\lambda = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (\Pi)^n\left(\mathcal{L}^n\right)$, where $\mathcal{L}^n$ is an irreducible highest weight module over $\mathfrak{osp}(1|2)$ with even highest weight vector, Π is the change of parity functor and where $\mathfrak{B}_\lambda$ is considered as $\mathfrak{osp}(1|2)$-module with respect to the adjoint representation. It is easy to verify that $\mathcal{L}^{2n}$ is generated by the highest weight vector G^{2n}, whereas $\Pi\left(\mathcal{L}^{2n+1}\right)$ is generated by $G^{2n+1}\tau$. Hence, $f_{k,2i}G^{2i} \in \mathcal{L}^{2i+k}$.

Making use of Lemma 3.5, it is not difficult to verify that

$$\langle \Omega * u, v\rangle = \langle u, \Omega * v\rangle,$$

where Ω is defined in Lemma 2.1 and $*$ denotes the adjoint action. This immediately implies that $\langle \mathcal{L}^p, \mathcal{L}^q\rangle = 0$ if $p \neq q$ and, therefore, $\langle f_{k,2i}, f_{l,2i}\rangle_{2i} = 0$ if $k \neq l$.

Let us show now that $\deg f_{k,2i} = k$. (Recall again that $\deg H = 2$, $\deg \tau = 1$.) It is easy to verify that

$$\begin{aligned}[F, [F, fG^p]] = [fG^p, Y] = \frac{1}{4}\Bigg\{ &\left((H+1)^2 - \left(\tau + \frac{1}{2}\right)^2\right)\Delta_2 f \\ &+ (p(2H - p + 2) - \tau + (-1)^p\tau)\Bigg\} G^{p-2} = \tilde{f}G^{p-2}.\end{aligned}$$

This implies $\deg f = \deg \tilde{f} + 2$. If $k = 0$, then $f_{0,2i}G^{2i} = G^{2i}$ and $f_{0,2i} = 1$. Formulas (2.7) imply that $f_{0,2i}G^{2i} = G^{2i} = \left[F, \left[F, f_{k,2i+2}G^{2i+2}\right]\right]$, hence, $\deg f_{k+2,2i} = \deg f_{k,2i+2} + 2$.

We similarly prove that $\langle f_{k,2i+1}, f_{l,2i+1}\rangle_{2i+1} = 0$ if $k \neq l$ and $\deg f_{k,2i+1} = k$.

Proof of heading 3 of Theorem 2.2. For any $f \in \mathbb{C}[H,\tau]$ set

$$\Delta f(H) = f(H+1) - f(H), \qquad \nabla f(H) = f(H) - f(H-1),$$
$$\Delta\tau = -2\tau, \qquad \nabla\tau = 2\tau.$$

The following identities are easy to check

$$[F, f] = \Delta f \cdot F = F\nabla f, \qquad [G, f] = -G\Delta f = -\nabla f G. \tag{3.6}$$

Moreover,

$$\begin{aligned} &\left[F, G^{2i}\right] = iG^{2i-1}, \qquad \left[F, G^{2i+1}\right] = (H-i)G^{2i},\\ &FG = \frac{1}{2}\left(H - \tau + \frac{1}{2}\right), \qquad GF = \frac{1}{2}\left(H + \tau - \frac{1}{2}\right),\\ &Ff(H) = f(H+1)F, \qquad Gf(H) = f(H-1)G,\\ &F\tau = -\tau F, \qquad G\tau = -\tau FG. \end{aligned} \tag{3.7}$$

Let us calculate the results of the adjoint action of τ on fG^{2i}, where $f \in \mathbb{C}[H,\tau]$. From the explicit expression of τ (Lemma 2.2) we deduce

$$\begin{aligned} \tau * \left(fG^{2i}\right) &= \frac{1}{2}fG^{2i} + \left[H, fG^{2i}\right] - 2\left[F\left[G, fG^{2i}\right]\right]\\ &= \left(2i + \frac{1}{2}\right) fG^{2i} + 2\left[F, \nabla f \cdot G^{2i+1}\right]\\ &= 2\Delta\nabla f \cdot FG \cdot G^{2i} + 2\nabla f \cdot \left[F, G^{2i+1}\right] + \left(2i + \frac{1}{2}\right) fG^{2i}\\ &= \left[\left(H - \tau + \frac{1}{2}\right)\Delta\nabla f + 2(H-i)\Delta f + \left(2i + \frac{1}{2}\right) f\right] G^{2i}. \end{aligned}$$

On the other hand, if $fG^{2i} \in \mathcal{L}^{2i+k}$, then $\tau * \left(fG^{2i}\right) = c \cdot fG^{2i}$ because in any irreducible $\mathfrak{osp}(1|2)$-module τ acts as a scalar multiple of the parity operator P, i.e., an operator such that $P(v) = (-1)^{p(v)}v$ for any $v \in V$.

Operator τ acts on the highest weight vector of $\mathcal{L}^{2i+k}$ as multiplication by $2i + k + \frac{1}{2}$. Observe that $p\left(f_{k,2i}G^{2i}\right) = p\left(G^{2i+k}\right)$ if k is even and $p\left(f_{k,2i}G^{2i}\right) = p\left(G^{2i+k}\tau\right) + \overline{1}$ if k is odd.

Therefore,

$$\tau * \left(f_{k,2i}G^{2i}\right) = (-1)^k \left(2i + k + \frac{1}{2}\right) f_{k,2i}G^{2i}.$$

Proof of heading 4 of Theorem 2.2. Let us calculate the results of the adjoint action of τ on fG^{2i+1}, where $f \in \mathbb{C}[H,\tau]$. We obtain

$$\begin{aligned} \tau * \left(fG^{2i}\right) &= \left(2i + \frac{3}{2}\right) fG^{2i} + 2\left[F, \nabla(fG^{2i+2}) - 2fG^{2i+2}\right]\\ &= 2\Delta\nabla f \cdot FG \cdot G^{2i+1} + 2\nabla f \cdot \left[F, G^{2i+2}\right] - 4\nabla f \cdot FG \cdot G^{2i+1} - 4f\left[F, G^{2i+2}\right] \end{aligned}$$

$$+\left(2i+\frac{3}{2}\right)fG^{2i+1} = \left(H-\tau+\frac{1}{2}\right)\Delta\nabla f\cdot G^{2i+1} + 2(i+1)\nabla f\cdot G^{2i+1}$$
$$-2\left(H-\tau+\frac{1}{2}\right)(\Delta\nabla f+\nabla f)G^{2i+1} - 4(i+1)fG^{2i+1} + \left(2i+\frac{3}{2}\right)fG^{2i+1}$$
$$= -\left\{\left(H-\tau+\frac{1}{2}\right)\Delta\nabla f + 2\left(H-\tau-i-\frac{1}{2}\right)\nabla f + \left(2i+\frac{5}{2}\right)f\right\}G^{2i+1}.$$

On the other hand, as in the proof of heading 3, we see that

$$\tau * \left(f_{k,2i+1}G^{2i+1}\right) = c\cdot f_{k,2i+1}G^{2i+1}.$$

But $f_{k,2i+1}G^{2i+1} \in \begin{cases}\mathcal{L}^{2i+k+2} & \text{for } k \text{ even}\\ \mathcal{L}^{2i+k} & \text{for } k \text{ odd}\end{cases}$ and the parity of $f_{k,2i+1}G^{2i+1}$ coincides with that of the highest weight vector of $\mathcal{L}^{2i+k+2}$ if k is even, and is opposite if k is odd; so

$$\tau * \left(f_{k,2i+1}G^{2i+1}\right) = -(-1)^{p(k)}\left(2i+k+\frac{5}{2}\right)f_{k,2i+1}G^{2i+1}.$$

Proof of heading 5 of Theorem 2.2. Let θ be an automorphism of $\mathfrak{B}_\lambda$ given on generators as follows:

$$\theta(G) = \sqrt{-1}F, \qquad \theta(F) = \sqrt{-1}G, \qquad \theta(H) = -H. \tag{3.8}$$

Let L be a functional on $\mathfrak{B}_\lambda$ defined in Lemma 3.4. Since L is unique, up to a scalar factor, invariant linear functional on $\mathfrak{B}_\lambda$, it follows that $\operatorname{Ker} L = [\mathfrak{B}_\lambda, \mathfrak{B}_\lambda]$. Hence, $\theta(\operatorname{Ker} L) = \operatorname{Ker} L$ and $L(\theta(u)) = L(u)$, if $u \in \mathfrak{B}_\lambda$. Therefore,

$$\langle f,g\rangle_{-i} = \langle fF^i, gF^i\rangle = L\left(fF^i\left(gF^i\right)^t\right) = (-1)^{i(i-1)/2}L\left(fgF^iG^i\right)$$
$$= (-1)^{i(i-1)/2}L\left(\theta\left(fgF^iG^i\right)\right) = (-1)^{i(i-1)/2}L\left(\theta(f)G^iF^i\theta(g)\right)\left(\sqrt{-1}\right)^{2i}$$
$$= (-1)^iL\left(\theta(f)G^i\right)\left(\theta(g)G^i\right)^t = (-1)^i\langle\theta(f),\theta(g)\rangle_{-i}.$$

But $\theta(H) = -H$ and

$$\theta(\tau) = \theta\left(H+\frac{1}{2}-2FG\right) = -H+\frac{1}{2}+2GF = \tau.$$

Proof of heading 6 and 7 of Theorem 2.2. Recall that $\Delta_2(H) = \nabla_2(H) = H$ and $\Delta_2(\tau) = 0$. Moreover, $X = G^2$, $Y = F^2$, and H span a Lie algebra isomorphic to $\mathfrak{sl}(2)$ and the following relations hold

$$XY = \frac{1}{4}\left(\left(\tau+\frac{1}{2}\right)^2-(H-1)^2\right), \qquad YX = \frac{1}{4}\left(\left(\tau+\frac{1}{2}\right)^2-(H+1)^2\right). \tag{3.9}$$

It is easy to verify that for $f \in \mathbb{C}[H,\tau]$ we have

$$[X,f] = -X\Delta_2 f = -\nabla_2 fX, \qquad [Y,f] = \Delta_2 fY = Y\nabla_2 f,$$
$$\left[Y,G^{2i}\right] = -i(H-i+1)G^{2i-2}, \quad \left[Y,G^{2i+1}\right] = \frac{1}{2}\left(\tau-(2i+1)\left(H-i+\frac{1}{2}\right)\right)G^{2i-1}.$$

Let us compute the result of the adjoint action of the Casimir operator $\omega = H^2+2H+4XY$ from $U(\mathfrak{sl}(2)) \subset \mathfrak{B}_\lambda$ on fG^{2i}. We have

$$\begin{aligned}\omega * \left(fG^{2i}\right) &= \left[H, \left[H, fG^{2i}\right]\right] + 2\left[H, fG^{2i}\right] + 4\left[Y, \left[X, fG^{2i}\right]\right] \\ &= (4i^2+4i)fG^{2i} - 4\left[Y, \Delta_2 fG^{2i+2}\right] \\ &= 4i(i+1)fG^{2i} - 4\Delta_2\nabla_2 fYXG^{2i} + 4i(H-i)\Delta_2 fG^{2i} \\ &= \left[(H+1)^2 - \left(\tau+\frac{1}{2}\right)^2\right]\Delta_2\nabla_2 fG^{2i} + 4(i+1)(H-i)\nabla_2 fG^{2i} + 4i(i+1)fG^{2i}.\end{aligned}$$

As $\mathfrak{sl}(2)$-module, $\mathcal{L}^{2i+k} = L^{2i+k} \oplus \Pi\left(L^{2i+k-1}\right)$, where L^m is the irreducible (finite dimensional) $\mathfrak{sl}(2)$-module with highest weight m. As is easy to calculate, ω acts on L^m as multiplication by $m(m+2)$.

We have

$$f_{k,2i} \in \begin{cases} L^{2i+k} \subset \mathcal{L}^{2i+k} & \text{for } k \text{ even,} \\ L^{2i+k-1} \subset \mathcal{L}^{2i+k} & \text{for } k \text{ odd.}\end{cases}$$

Now, let us compute the action of ω on fG^{2i+1}:

$$\omega * \left(fG^{2i+1}\right) = \left\{\left[(H+1)^2 - \left(\tau+\frac{1}{2}\right)^2\right]\Delta_2\nabla_2 f \right.$$
$$\left. + \left[(2i+3)(2H-2i-1) - 2\tau\right]\nabla_2 f + (2i+1)(2i+5)f\right\}G^{2i+1}.$$

Proof of heading 8 and 9 of Theorem 2.2. Since $\Delta_2\nabla_2 = \Delta_2 - \nabla_2$, we can express the left hand side of the equation of heading 6) as

$$\left[(H+1)^2 - \left(\tau+\frac{1}{2}\right)^2\right]\Delta_2 f + \left[(2H-2i-1)^2 - \left(\tau+\frac{1}{2}\right)^2\right]\nabla_2 f.$$

Making the change $x = \frac{1}{2}H + \frac{1}{2}\tau - \frac{1}{4} - i$ we reduce the above equation to the form

$$(x-N)(x+\alpha+1)\Delta\varphi - x(x-\beta-N-1)\nabla\varphi$$
$$= \begin{cases} \dfrac{k}{2}\left(\dfrac{k}{2}+\alpha+\beta+1\right)\varphi & \text{for } k \text{ even,} \\ \dfrac{k-1}{2}\left(\dfrac{k-1}{2}+\alpha+\beta+1\right)\varphi & \text{for } k \text{ odd,}\end{cases}$$

where $N = \tau - \frac{1}{2} - i$, $\alpha = \beta = i$ and $\varphi(x) = f\left(2x + i + \frac{1}{2} - \tau\right)$.

Thanks to [5, p. 30], we know that one of the solutions of the above equation is equal to ${}_3F_2\left(\begin{matrix} -l,\ l+\alpha+\beta+1,\ -x \\ \alpha+1,\ -N\end{matrix}\ \middle|\ 1\right)$. (In [5] it is supposed that N is a positive integer and x is real, but one can clearly assume that N and x belong to any commutative ring.) Thus,

$$\varphi(x) = c\cdot {}_3F_2\left(\begin{matrix} -l,\ l+\alpha+\beta+1,\ -x \\ \alpha+1,\ -N\end{matrix}\ \middle|\ 1\right) \quad \text{for some } \ c \in \mathbb{C}[\tau].$$

To calculate the exact value of the constant c, it suffices to compute the leading coefficient of the polynomial $f_{k,2i}$. Formula (2.7) implies that

$$\begin{aligned} f_{k,2i}G^{2i} &= \left[F,\left[F, f_{k-2,2i+2}G^{2i+2}\right]\right] = \left[f_{k-2,2i+2}G^{2i+2}, Y\right] \\ &= \frac{1}{4}\left\{\left[(H+1)^2 - \left(\tau+\frac{1}{2}\right)^2\right]\Delta_2 f_{k-2,2i+2} + 2i(2H-2i+2)f_{k-2,2i+2}\right\}G^{2i+2}. \end{aligned} \quad (3.10)$$

In particular,

$$f_{k-2,2i+2} = 1, \qquad f_{1,2i+2}G^{2i} = \left[F, G^{2i+1}\tau\right] = \left(\frac{2i+1}{2}\tau - \frac{\left(\lambda+\frac{1}{2}\right)^2}{2}\right)G^{2i}.$$

Let $f_{k,2i} = a_{k,2i}H^{[k/2]} + \cdots$, where $[x]$ denotes the integer part of x. Then (3.10) implies that

$$a_{k,2i} = \left(\frac{1}{2}\left[\frac{k-2}{2}\right] + i + 1\right) a_{k-2,2i+2}. \quad (3.11)$$

Formula (3.11) implies that

$$a_{2l,2i} = \frac{(2i+l+1)_l}{2^l}, \qquad a_{2l+1,2i} = \left[(2i+l+1)\tau - \left(\lambda+\frac{1}{2}\right)^2\right] \times \frac{(2i+l+1)_l}{2^{l+1}}.$$

Since the coefficient of the leading power of x in ${}_3F_2\left(\begin{matrix} -l,\ l+\alpha+\beta+1,\ -x \\ \alpha+1,\ -N \end{matrix}\ \middle|\ 1\right)$ is equal to

$$\frac{(-1)^l l!(l+\alpha+\beta+1)_l}{(\alpha+1)_l(-N)_l},$$

we deduce that

$$c = \frac{(-1)^l(\alpha+1)_l(-N)_l}{l!(l+\alpha+\beta+1)_l} \cdot \frac{(2i+l+1)_l(-N)_l}{2^l}2^l$$

which leads to formulas of heading 8.

Similar calculations show that

$$a_{2l,2i+1} = \frac{(2i+l+2)_l}{2^l}, \qquad a_{2l+1,2i+1} = -\frac{(2i+l+1)_l}{2^l}\tau.$$

This leads to formulas of heading 9.

4 Proof for $\lambda = -\frac{1}{2}$

We will stick to notations introduced after Theorem 2.2 concerning "another theorem of Pinczon".

Lemma 4.1. *On Weyl algebra $A_1 = \mathbb{C}[P,Q]$, there exists a unique up to a constant factor invariant linear functional L. It is uniquely determined by its restriction onto $\mathbb{C}[H]$. The generating function of L is of the form*

$$\varphi_L(t) = c\frac{e^{t/2}}{1+e^t} \qquad \text{for} \ \ c \in \mathbb{C}.$$

Proof. Since $(A_1)_i = \{u \in A_1 \mid [H,u] = iu\}$ for $i \in \mathbb{Z}$, we see that $L((A_1)_i) = 0$ for $i \neq 0$. So L is uniquely determined by its restrictions onto $(A_1)_0 = \mathbb{C}[H]$.

Further,

$$[A_1, A_1] \cap \mathbb{C}[H] = [P, (A_1)_1] \cap \mathbb{C}[H] = \mathrm{Span}([P, fQ])$$

where $f \in \mathbb{C}[H]$. But

$$\begin{aligned}[P, fQ] &= PfQ + fPQ = f(H+1)PQ + f(H)QP \\ &= \left(H + \frac{1}{2}\right) f(H+1) + \left(H - \frac{1}{2}\right) f(H).\end{aligned}$$

And, since $\deg[P, fQ] \geq 1$ (here we assume that $\deg H = 1$), it follows that $1 \notin [A_1, A_1]$ which proves the existence of L. Further on,

$$L\left(\left(H + \frac{1}{2}\right) f(H+1) + \left(H - \frac{1}{2}\right) f(H)\right) = 0$$

wherefrom, as in Lemma 3.4, we deduce that

$$\left(\frac{d}{dt} - \frac{1}{2}\right)\left(1 + e^t\right)\varphi_L = 0$$

and the desired form of φ_L. ∎

Lemma 4.2. *Define an automorphism of $A_1 = \mathbb{C}[P,Q]$ by setting*

$$H^t = h, \qquad Q^t = -P, \qquad P^t = Q.$$

Then

i) $\left(u^t\right)^t = (-1)^{p(u)}u$;

ii) $L(u^t) = L(u)$, *where* $u \in A_1$.

Proof is similar to that of Lemma 3.5.

Thus, the form $\langle \cdot, \cdot \rangle$ is supersymmetric and invariant.

Proof of Theorem 2.3. Heading 1 is proved as heading 1 of Theorem 2.2. Heading 2 is proved as heading 2 of Theorem 2.2 with the help of decomposition $A_1 = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{L}^{2n}$, where $\mathcal{L}^{2n}$ is an irreducible highest weight module over $\mathfrak{osp}(1|2)$ with even highest weight vector. Heading 3 is proved as heading 5 of Theorem 2.2 with the help of automorphism (3.8), where $F = P$ and $G = Q$. The difference equations for $f_{k,j}$ follows from the study of the result of application of $\tau = H + \frac{1}{2} - 2FG$ to $f_{k,j}G^j$ under the adjoint action of $\mathfrak{osp}(1|2)$ on A_1 and arguments similar to those from the proof of heading 3 of Theorem 2.2. Statements of headings 6 and 7 are results of comparison of difference equations in headings 4 and 5 with corresponding equations in [5] and calculation of the leading terms.

5 Proof for the case $\boldsymbol{\lambda} \in \mathbb{Z}_{\geq 0}$

As was observed in proof after formula (2.13), the elements $U = F$ and $V = \left(\tau - H + \frac{1}{2}\right) G$ generate in $U(\mathfrak{osp}(1|2))$ a subalgebra isomorphic to $U(\mathfrak{sl}(2))$, considered as a superalgebra with nontrivial odd part.

It is also convenient to consider $U(\mathfrak{sl}(2))$ per se, not as a subalgebra of $U(\mathfrak{osp}(1|2))$. We mean the following.

Let $\mathfrak{sl}(2) = \mathrm{Span}(X, H, Y)$ with relations (1.1). Consider $U(\mathfrak{sl}(2))$ as a superalgebra with parity given by the formula $p(X) = p(Y) = \overline{1}$ (hence, $p(H) = \overline{0}$). Set

$$U = \frac{1}{\sqrt{2}} Y, \qquad V = \frac{1}{\sqrt{2}} X, \qquad H \mapsto \frac{1}{2} H,$$

we, clearly, have

$$HU - UH = -U, \qquad HV - VH = V, \qquad VU - UV = H. \tag{5.1}$$

The Casimir operator, being even, remains the same:

$$\Omega = H^2 - H + 2VU = H^2 + H + 2UV.$$

Therefore,

$$\begin{aligned} &VU = \frac{1}{2}\left(\Omega - \left(H^2 - H\right)\right), \qquad UV = \frac{1}{2}\left(\Omega - \left(H^2 + H\right)\right), \\ &[V, U] = \Omega - H^2. \end{aligned} \tag{5.2}$$

(Recall that $[U, V] = UV + VU$.)

Let $\mathfrak{A}_\lambda$ be a quotient of $U(\mathfrak{sl}(2))$, as in (1.2). Then $\mathfrak{B}_\lambda \simeq \mathfrak{A}_{2\lambda}$ for any $\lambda \in \mathbb{Z}_{\geq 0}$.

Set $\mathfrak{C}_\lambda := \mathfrak{A}_{2\lambda}$ for any $\lambda \in \mathbb{C}$; in other words, $\mathfrak{C}_\lambda$ is generated by odd indeterminates U and V subject to relations

$$\begin{aligned} &HU - UH = -U, \qquad HV - VH = V, \qquad VU - UV = H, \\ &VU = \frac{1}{2}\left(\lambda(\lambda+1) - \left(H^2 - H\right)\right), \qquad UV = \frac{1}{2}\left(\lambda(\lambda+1) - \left(H^2 + H\right)\right), \end{aligned} \tag{5.3}$$

and one more relation:

$$V^{2|\lambda|} = 0 \quad \text{for} \quad \lambda \in \frac{1}{2}\mathbb{Z}_{\geq 0}.$$

In what follows we will assume that $\mathfrak{C}_\lambda$ is considered for $\lambda \in \mathbb{C}$ because all the proofs hold for such λ, not only for $\lambda \in \mathbb{Z}_{\geq 0}$.

Formulas (5.3) easily imply that by setting

$$H^t = H, \qquad U^t = -V, \qquad V^t = U$$

we determine an antiautomorphism of superalgebra $\mathfrak{C}_\lambda$, i.e., $\left(u^t\right)^t = (-1)^{p(u)} u$ for any $u \in \mathfrak{C}_\lambda$.

Lemma 5.1. *On $\mathfrak{C}_\lambda$, there exists a unique, up to a scalar multiple, invariant linear functional L, such that $L(H) = 0$ and $L\left(u^t\right) = L(u)$ for any $u \in \mathfrak{C}_\lambda$. The functional L is uniquely determined by its restriction onto $\mathbb{C}[H]$ and its generating function is*

$$\varphi_L(t) = c\frac{e^{(\lambda+1)t} + e^{-\lambda t}}{e^t + 1}.$$

Proof. We have

$$\mathfrak{C}_\lambda = \begin{cases} \bigoplus_{i\in\mathbb{Z}} (\mathfrak{C}_\lambda)_i, & \text{if } \lambda \notin \frac{1}{2}\mathbb{Z}_{\geq 0}, \\ \bigoplus_{|i|\leq 2\lambda} (\mathfrak{C}_\lambda)_i, & \text{otherwise,} \end{cases}$$

where $(\mathfrak{C}_\lambda)_i = \{u \in \mathfrak{C}_\lambda \mid [H, u] = iu\}$ for $i \in \mathbb{Z}$. Hence, $L((\mathfrak{C}_\lambda)_i) = 0$ unless $i = 0$. Further on,

$$[U, fG] = \frac{1}{2}\left(\lambda(\lambda+1) - H^2 - H\right) f(H+1) + \frac{1}{2}\left(\lambda(\lambda+1) - H^2 + H\right) f(H)$$

and $L([U, fG]) = 0$. Therefore, as in Lemma 3.4, we obtain

$$\left(\lambda(\lambda+1) - \left(\frac{d}{dt}\right)^2 + \frac{d}{dt}\right)\left(e^t + 1\right)\varphi_L(t) = 0$$

and

$$\varphi_L(t) = \begin{cases} \dfrac{c_1 e^{(\lambda+1)t} + c_2 e^{-\lambda t}}{1 + e^t}, & \text{for } \lambda \neq -\frac{1}{2}, \\ \dfrac{(c_1 + c_2 t)e^{\frac{1}{2}t}}{1 + e^t}, & \text{otherwise.} \end{cases}$$

The condition $L(H) = 0$ implies that $c_1 = c_2$ for $\lambda \neq -\frac{1}{2}$ and $c_2 = 0$ otherwise. This proves the uniqueness.

Let us prove the existence. By Lemma 3.1 we have

$$[\mathfrak{C}_\lambda, \mathfrak{C}_\lambda] = [U, \mathfrak{C}_\lambda] + [V, \mathfrak{C}_\lambda]$$

and

$$\begin{aligned}[\mathfrak{C}_\lambda, \mathfrak{C}_\lambda] \cap \mathbb{C}[H] &= ([U, \mathfrak{C}_\lambda] + [V, \mathfrak{C}_\lambda] \cap \mathbb{C}[H] = [U, (\mathfrak{C}_\lambda)_1] \cap \mathbb{C}[H] \\ &= \operatorname{Span}([U, fV] \mid f \in \mathbb{C}[H]) = \operatorname{Span}(f(H+1)Uv + f(H)VU \mid f \in \mathbb{C}[H]).\end{aligned}$$

Hence, $\deg[U, fV] = \deg f + 2$. Hence, $1, H \notin [\mathfrak{C}_\lambda, \mathfrak{C}_\lambda]$ and $\mathfrak{C}_\lambda/[\mathfrak{C}_\lambda, \mathfrak{C}_\lambda]$ is the linear span of the images of 1 and H. This proves the existence of L.

Let $\tilde{L}(u) = L\left(u^t\right) = 0$. Then $\tilde{L}$ is invariant and $\tilde{L}(H) = L(H) = 0$. The uniqueness implies that $\tilde{L} = L$. ■

Lemma 5.2. *Let $f \in \mathbb{C}[H]$. We have*

i) if $f(2k - H) = -f(H)$, then $L\left(f(H)V^{2k}U^{2k}\right) = 0$;

ii) if $f(2k+1-H)=f(H)$, *then* $L\left(f(H)V^{2k+1}U^{2k+1}\right)=0$.

Proof. i) Determine an automorphism of $\mathfrak{C}_\lambda$ by setting

$$\theta(H)=-H, \qquad \theta(U)=V, \qquad \theta(V)=U$$

and set

$$g(H)=f(H)V^{2k}U^{2k}+U^{2k}f(H)V^{2k}.$$

Then

$$\begin{aligned} g(-H)&=\theta(g(H))=\theta\left(f(H)V^{2k}U^{2k}+U^{2k}f(H)V^{2k}\right)\\ &=f(-H)V^{2k}U^{2k}+U^{2k}f(-H)V^{2k}\\ &=U^{2k}f(2k-H)V^{2k}+f(2k-H)V^{2k}U^{2k}=-g(H). \end{aligned}$$

In other words, $g(H)$ is an odd polynomial. By Lemma 5.1 the generating function for L is an even one, so $L(g(H))=0$.

Further on,

$$L\left(f(H)V^{2k}U^{2k}\right)=L\left(\frac{1}{2}\left[f(H)V^{2k},U^{2k}\right]+\frac{1}{2}\left(f(H)V^{2k}U^{2k}+U^{2k}f(H)V^{2k}\right)\right)=0.$$

ii) Set

$$g(H)=f(H)V^{2k+1}U^{2k+1}-U^{2k+1}f(H)V^{2k+1}.$$

Then

$$\begin{aligned} g(-H)&=\theta(g(H))=\theta\left(f(H)V^{2k+1}U^{2k+1}-U^{2k+1}f(H)V^{2k+1}\right)\\ &=f(-H)V^{2k+1}U^{2k+1}-U^{2k+1}f(-H)V^{2k+1}\\ &=U^{2k+1}f(2k+1-H)V^{2k+1}+f(2k+1-H)V^{2k+1}U^{2k+1}\\ &=V^{2k+1}f(H)U^{2k+1}-f(H)V^{2k+1}U^{2k+1}=-g(H). \end{aligned}$$

Therefore, $L(g(H))=0$. Further on,

$$\begin{aligned} f(H)V^{2k+1}U^{2k+1}=&\frac{1}{2}\left[f(H)V^{2k+1},U^{2k+1}\right]\\ &+\frac{1}{2}\left(f(H)V^{2k+1}U^{2k+1}-U^{2k+1}f(H)V^{2k+1}\right); \end{aligned}$$

hence, $L\left(f(H)V^{2k+1}U^{2k+1}\right)=0$. ■

Lemma 5.3. *Let* $f,g\in\mathbb{C}[H]$, $\varepsilon=\pm1$, $i\in\mathbb{Z}_{\geq0}$. *We have:*

i) If $f(H)V^{2i+1}=\left[U,gV^{2i+2}\right]$, *then the condition* $\varepsilon g(H)=g(2i+2-H)$ *is equivalent to the condition* $f(2i+1-H)=-\varepsilon f(H)$ *and* $\deg f=\deg g+1$.

ii) If $f(H)V^{2i}=\left[U,gV^{2i+1}\right]$, *then the condition* $\varepsilon g(H)=g(2i+1-H)$ *is equivalent to the condition* $f(2i-H)=\varepsilon f(H)$ *and* $\deg f=\deg g+2$.

iii) *If* $f(H)V^{2i+1} = \{U, gV^{2i+2}\}$, *then the condition* $\varepsilon g(H) = g(2i+2-H)$ *is equivalent to the condition* $f(2i+1-H) = \varepsilon f(H)$ *and* $\deg f = \deg g + 2$.

iv) *If* $f(H)V^{2i} = \{U, gV^{2i+1}\}$, *then the condition* $\varepsilon g(H) = g(2i+1-H)$ *is equivalent to the condition* $f(2i-H) = -\varepsilon f(H)$ *and* $\deg f = \deg g + 1$.

Proof. i) The condition $f(H)V^{2i+1} = [U, gV^{2i+2}]$ is equivalent to the equation

$$f(H) = T(H+1)g(H+1) - T(H-2i-1)g(H), \tag{5.4}$$

where $T(H) = \frac{1}{2}(\lambda + H)(\lambda + 1 - H)$. This implies

$$\begin{aligned} f(2i+1-H) &= T(2i+2-H+1)g(2i+2-H) - T(-H)g(2i+1-H) \\ &= \varepsilon T(H-2i-1)g(H) - \varepsilon T(H+1)g(H+1) = \varepsilon f(H). \end{aligned}$$

Conversely, let $f(2i+1-H) = -\varepsilon f(H)$. Set $\psi(H) = g(H) - \varepsilon g(2i+2-H)$. Then (5.4) implies

$$\psi(H+1)T(H+1) - \psi(H)T(H-2i-1) = 0. \tag{5.5}$$

If $i \geq 0$, then polynomial $T(H+1)$ has a root α such that $\alpha - k$ is not a root of $T(H-2i-1)$ for any $k \in \mathbb{Z}_{\geq 0}$. Select this root α. Then equation (5.5) implies $\psi(\alpha) = 0$, but then $\psi(\alpha-1)T(\alpha-1-2i-1) = 0$ and, therefore, $\psi(\alpha-1) = 0$. So $\psi(\alpha-k) = 0$ for any $k \in \mathbb{Z}_{\geq 0}$. Thus, $\psi = 0$.

Headings ii)–iv) are similarly proved. ■

Proof of heading 2 of Theorem 2.4. Let $l = 2k$. Let us prove by induction on k that $f_{2k,2i}$ is an orthogonal polynomials of degree $2k$ with respect to the form $\langle\cdot,\cdot\rangle_{2i}$. Let

$$\mathbb{C}[H]^i_+ = \{f \in \mathbb{C}[H] \mid f(2i-H) = f(H)\} \tag{5.6}$$

and

$$\mathbb{C}[H]^i_- = \{f \in \mathbb{C}[H] \mid f(2i-H) = -f(H)\}. \tag{5.7}$$

Then Lemma 5.2 implies that the spaces $\mathbb{C}[H]^i_+$ and $\mathbb{C}[H]^i_-$ are orthogonal with respect to the form $\langle\cdot,\cdot\rangle_{2i}$. So it suffices to prove that $\langle f_{2k,2i}, g\rangle_{2i} = 0$ for $g \in \mathbb{C}[H]^i_+$ and $\deg g < 2k$.

Let us induct on k. If $k = 1$, then

$$\begin{aligned} \langle f_{2k,2i}, 1\rangle_{2i} &= \langle [U, V^{2i+1}], V^{2i}\rangle \overset{\text{Lemma 3.2}}{=} -\langle V^{2i+1}, [U, V^{2i}]\rangle \\ &= -2\langle V^{2i+1}, V^{2i+1}\rangle \overset{\text{Lemma 5.2}}{=} 0. \end{aligned}$$

Let $k > 1$; then equations (2.15) imply that

$$f_{2k,2i}V^{2i} = \left[U, \frac{1}{H - \left(i + \frac{1}{2}\right)}\left[f_{2k-2,2i+2}, V^{2i+2}\right]\right]. \tag{5.8}$$

This equation and Lemma 5.3 imply that $\deg f_{2k,2i} = 2k$. Let $g \in \mathbb{C}[H]^i_+$ and $\deg g \le 2k-2$. Then

$$\langle f_{2k,2i}, g\rangle_{2i} = \langle f_{2k,2i}V^{2i}, gV^{2i}\rangle = \langle \left[U, \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2k-2,2i+2}V^{2i+2}\right]\right], gV^{2i}\rangle$$

$$= -\langle \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2k-2,2i+2}V^{2i+2}\right], \left[V, gV^{2i}\right]\rangle$$

$$= \langle \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2k-2,2i+2}V^{2i+2}\right], \nabla g V^{2i}\rangle$$

$$= \langle \left[U, f_{2k-2,2i+2}V^{2i+2}\right], \frac{\nabla g}{H-\left(i+\frac{1}{2}\right)}V^{2i+1}\rangle$$

$$= \langle f_{2k-2,2i+2}V^{2i+2}, \left[V, \frac{\nabla g}{H-\left(i+\frac{1}{2}\right)}V^{2i+1}\right]\rangle$$

$$= \langle f_{2k-2,2i+2}V^{2i+2}, \frac{2\nabla g}{H-\left(i+\frac{1}{2}\right)} - \nabla\left(\frac{\nabla g}{H-(i+\frac{1}{2})}\right)\rangle_{2i+2} = 0$$

since $\deg\left(\frac{2\nabla g}{H-\left(i+\frac{1}{2}\right)} - \nabla\left(\frac{\nabla g}{H-\left(i+\frac{1}{2}\right)}\right)\right) \le 2k-3$.

Proof of orthogonality of polynomials $f_{2k+1,2i}$ is similarly performed with appellation to Lemma 3.3.

Proof of headings 3 and 4 of Theorem 2.4. Define an operator D by setting

$$Df\cdot V^{2i} = \left[V, \frac{\nabla g}{H-\left(i+\frac{1}{2}\right)}\left[V, fV^{2i}\right]\right],$$

where f satisfies $f(2i-H) = f(H)$. Let us show that D is well defined. Indeed,

$$(\nabla f)\left(i+\frac{1}{2}\right) = f\left(i+\frac{1}{2}\right) - f\left(i-\frac{1}{2}\right) = f\left(2i-\left(i-\frac{1}{2}\right)\right) - f\left(i-\frac{1}{2}\right) = 0.$$

Set $\varphi(H) = \dfrac{\nabla f}{H-(i+\frac{1}{2})}$. We have

$$\varphi(2i+1-H) = \frac{f(2i+1-H)-f(2i-H)}{(2i+1-H)-\left(i+\frac{1}{2}\right)} = \frac{f(H-1)-f(H)}{i+\frac{1}{2}-H} = \varphi(H).$$

Therefore, $(Df)(2i-H) = (Df)(H)$ by Lemma 5.3.

Let us show now that D is selfadjoint with respect to the form $\langle\cdot,\cdot\rangle_{2i}$. Indeed, let $\mathbb{C}[H]^i_+$ and $\mathbb{C}[H]^i_-$ be as in (5.6)–(5.7). We have shown that D sends $\mathbb{C}[H]^i_+$ into itself.

Let $f, g \in \mathbb{C}[H]^i_+$; then

$$\langle Df, g\rangle_{2i} = \langle Df \cdot V^{2i}, gV^{2i}\rangle = \langle \frac{1}{H - \left(i + \frac{1}{2}\right)} \left[V, fV^{2i}\right], \left[V, gV^{2i}\right]\rangle$$
$$= -\langle \left[V, fV^{2i}\right], \frac{1}{H - \left(i + \frac{1}{2}\right)} \left[V, gV^{2i}\right]\rangle$$
$$= \langle fV^{2i}, \left[U, \frac{1}{H - \left(i + \frac{1}{2}\right)} \left[V, gV^{2i}\right]\right]\rangle = \langle f, Dg\rangle_{2i}.$$

Hence, $\langle Df_{2k,2i}, g\rangle_{2i} = \langle f_{2k,2i}, Dg\rangle_{2i}$ if $g \in \mathbb{C}[H]^i_+$ and $\deg g < 2k$. The uniqueness of the orthogonal polynomial of given degree implies that $Df_{2k,2i} = \alpha_k f_{2k,2i}$. Furthermore,

$$Df \cdot V^{2i} = \left[U, \frac{1}{H - \left(i + \frac{1}{2}\right)} \left[V, fV^{2i}\right]\right] = \left[U, -\frac{\nabla f}{H - \left(i + \frac{1}{2}\right)} V^{2i+1}\right]$$
$$= -U \frac{\nabla f}{H - \left(i + \frac{1}{2}\right)} V^{2i+1} - \frac{\nabla f}{H - \left(i + \frac{1}{2}\right)} V^{2i+1} U$$
$$= -\frac{\nabla f}{H - \left(i + \frac{1}{2}\right)} UV \cdot V^{2i} - \frac{\nabla f}{H - \left(i + \frac{1}{2}\right)} V^{2i} VU$$
$$= \left(\frac{1}{2} \cdot \frac{(H-\lambda)(H+\lambda+1)}{2H - i + \frac{1}{2}} \Delta f + \frac{(H - 2i - \lambda + 1)(H - 2i + \lambda + 1)}{2H - i + \frac{1}{2}}\right) \Delta f.$$

In other words,

$$Df = \left(\frac{(H-\lambda)(H+\lambda+1)}{2H - 2i + 1} + \frac{(H - 2i - \lambda + 1)(H - 2i + \lambda + 1)}{2H - 2i + 1}\right) \Delta f.$$

By calculating the leading coefficient of Df leads us to the equation of heading 3.

Equation of heading 4 is similarly obtained by considering operator

$$Df \cdot V^{2i} = \left\{U, \frac{1}{H - \left(i + \frac{1}{2}\right)} \left\{V, fV^{2i}\right\}\right\}.$$

Proof of headings 5–7 of Theorem 2.4. Statements of heading 5 follow from the study of automorphism θ given by formulas

$$\theta(H) = -H, \qquad \theta(U) = V, \qquad \theta(V) = U$$

in the same way as in heading 2 of Theorem 2.4.

To prove statements of heading 6, consider the following subspaces of $\mathbb{C}[H]$:

$$\mathbb{C}[H]^{i+\frac{1}{2}}_+ = \{f \in \mathbb{C}[H] \mid f(2i + 1 - H) = f(H)\},$$
$$\mathbb{C}[H]^{i+\frac{1}{2}}_- = \{f \in \mathbb{C}[H] \mid f(2i + 1 - H) = -f(H)\}.$$

By Lemma 5.2 these subspaces are completely isotropic with respect to the form $\langle\cdot,\cdot\rangle_{2i+1}$. Let us show that the form $\langle f_{2k,2i+1}, f_{2k+1,2i+1}\rangle_{2i} \neq 0$, while the remaining scalar products vanish. Indeed, equations (2.15)–(2.18) imply

$$f_{2k,2i+1}V^{2i+1} = \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[V, f_{2k,2i+2}V^{2i+2}\right] \tag{5.9}$$

and

$$f_{2l+1,2i+1}\left[V, f_{2l,2i+2}V^{2i+2}\right]. \tag{5.10}$$

Hence,

$$\begin{aligned}\langle f_{2k,2i+1}, f_{2l+1,2i+1}\rangle_{2i+1} &= \langle \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2k,2i+2}V^{2i+2}\right], \left[U, f_{2l,2i+2}V^{2i+2}\right]\rangle \\ &= \langle \left[U, f_{2k,2i+2}V^{2i+2}\right], \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2l,2i+2}V^{2i+2}\right]\rangle \\ &= \left\langle f_{2k,2i+2}V^{2i+2}, \left[V, \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2l,2i+2}V^{2i+2}\right]\right]\right\rangle \\ &= \langle f_{2k,2i+2}V^{2i+2}, \tilde{D}f_{2l,2i+2}V^{2i+2}\rangle = \langle f_{2k,2i+2}, \tilde{D}f_{2l,2i+2}\rangle_{2i+2},\end{aligned}$$

where

$$\tilde{D}f\cdot V^{2i+2} = \left[V, \frac{1}{H-\left(i+\frac{1}{2}\right)}\left[U, f_{2l,2i+2}V^{2i+2}\right]\right]$$

and is well defined thanks to Lemma 5.3i).

It is easy to show that $\tilde{D}$ is selfadjoint and $f_{2k,2i+2}$ is its eigenfunction corresponding to a nonzero eigenvalue α_l. So

$$\langle f_{2k,2i+1}, f_{2l+1,2i+1}\rangle_{2i+1} = \alpha_l\langle f_{2k,2i+2}, f_{2l,2i+2}\rangle_{2i+2}.$$

This proves statement of heading 6.

Statements of heading 7 are proved similar to those of heading 5.

Proof of headings 1, 8, 9 of Theorem 2.4. Proof of heading 1 is similar to that of heading 1 of Theorem 2.2. To prove heading 8, consider the operator

$$\tilde{D}f\cdot V^{2i+1} = \left[U, \left[V, fV^{2i+1}\right]\right] \quad \text{for} \quad f \in \mathbb{C}[H]_+^{i+\frac{1}{2}}.$$

It is easy to verify that $\tilde{D}f \in \mathbb{C}[H]_-^{i+\frac{1}{2}}$ and, the other way round, if $f \in \mathbb{C}[H]_-^{i+\frac{1}{2}}$, then $\tilde{D}f \in \mathbb{C}[H]_+^{i+\frac{1}{2}}$. Moreover, $\langle Df, g\rangle_{2i+1} = \langle f, Dg\rangle_{2i+1}$, i.e., D is selfadjoint and $\deg Df = \deg f + 1$ if $\deg f > 0$. Hence,

$$\langle Df_{2k,2i+1}, f_{2l,2i+1}\rangle = \langle f_{2k,2i+1}, Df_{2l,2i+1}\rangle = 0 \quad \text{if} \quad l < k.$$

So $Df_{2k,2i+1} = \alpha\cdot Df_{2k+1,2i+1}$. Having calculated α we obtain the statement of heading 8.

Proof of heading 9 is similar to arguments from the proof of of headings 8 and 9 of Theorem 2.2.

Acknowledgements

I am thankful to D Leites for encouragement and help and to ESI, Vienna, for hospitality and support.

References

[1] Dixmier J, Algèbres Envellopentes, Gautier-Villars, Paris, 1974; Enveloping Algebras, AMS, 1996.

[2] Feigin B L, The Lie Algebras $\mathfrak{gl}(\lambda)$ and Cohomologies of Lie Algebra of Differential Operators, *Russian Math. Surveys* **43**, Nr.2 (1988), 157–158.

[3] Grozman P and Leites D, Lie Superalgebras of Supermatrices of Complex Size and Integrable Dynamical Systems, in Complex Analysis and Related Topics, Proc. of the International Symposium, Editors N Vasilevsky et. al., Cuernavaca, Mexica, November 18–22, 1996, Birkhauser, 1999, 73–105.

[4] Kwon K H, Lee D W and Park S B, New Characterization of Discrete Classical Orthogonal Polynomials, *J. Approx. Theory* **89** (1977), 156–171.

[5] Koekoek R and Swarttouw R, The Askey-Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue, math.CA/9602214.

[6] Leites D and Sergeev A, Orthogonal Polynomials of Discrete Variable and Lie Algebras of Complex Size Matrices, *Teoret. Mat. Fiz.*, (Russian) V. 123 (2000), no. 2, 205–236 translation in *Theoret. and Math. Physics* V. 123 (2000), no. 2, 582–608; preprinted in Procedings of M Saveliev Memorial Conference, MPI, Bonn, February, 1999, MPI-1999-36, 49–70.

[7] Marcellan F, Complex Path Integral Representation for Semiclassical Linear Functionals, *J. Approx. Theory* **94** (1998), 107–127.

[8] Marcellan F, Branquinho A and Petronillo T, Classical Orthogonal Polynomials: a Functional Approach, *Acta Applicandae Math.* **34** (1994), 283–303.

[9] Mingarelli A and Krall A, Jacobi-Type Polynomials under an Indefinite Inner Product, *Proc. Roy. Soc. Edinburgh, Sect. A* **90** (1981), 147–153.

[10] Nikiforov A F, Suslov S K and Uvarov V B, Classical Orthogonal Polynomials of a Discrete Variable, Springer Series in Computational Physics, Springer-Verlag - Berlin, 1991 (Translated from the Russian).

[11] Montgomery S, Constructing Simple Lie Superalgebras from Associative Graded Algebras, *J. Algebra* **1 95**, Nr.2 (1997), 558–579.

[12] Pinczon G, The Enveloping Algebra of the Lie Superalgebra $\mathfrak{osp}(1|2)$, *J. Algebra* **132** (1990), 219–242.