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The Shapovalov Determinant for the Poisson Superalgebras

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Abstract

Among simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth, there are several which have no Cartan matrix but, nevertheless, have a quadratic even Casimir element C_2: these are the Lie superalgebra $\mathfrak{k}^L(1|6)$ of vector fields on the $(1|6)$-dimensional supercircle preserving the contact form, and the series: the finite dimensional Lie superalgebra $\mathfrak{sh}(0|2k)$ of special Hamiltonian fields in $2k$ odd indeterminates, and the Kac–Moody version of $\mathfrak{sh}(0|2k)$. Using C_2 we compute N. Shapovalov determinant for $\mathfrak{k}^L(1|6)$ and $\mathfrak{sh}(0|2k)$, and for the Poisson superalgebras $\mathfrak{po}(0|2k)$ associated with $\mathfrak{sh}(0|2k)$. A. Shapovalov described irreducible finite dimensional representations of $\mathfrak{po}(0|n)$ and $\mathfrak{sh}(0|n)$; we generalize his result for Verma modules: give criteria for irreducibility of the Verma modules over $\mathfrak{po}(0|2k)$ and $\mathfrak{sh}(0|2k)$.

Introduction

Every simple finite dimensional Lie algebra has a symmetrizable Cartan matrix. Moreover, if the simple $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of polynomial growth (SZGLAPG, for short) has a Cartan matrix, i.e., $\mathfrak{g} = \mathfrak{g}(A)$, then A is symmetrizable. These Cartan matrices correspond to Dynkin diagrams and extended Dynkin diagrams. More exactly, the algebras corresponding to extended diagrams are not simple, they are certain "relatives", called Kac–Moody algebras, of central extensions of simple ones; in applications Kac–Moody algebras are even more interesting than simple ones, cf. [14].

For finite dimensional simple Lie algebras $\mathfrak{g}(A)$ N Shapovalov [22] suggested a powerful method for description of irreducible highest weight $\mathfrak{g}(A)$-modules (Verma modules and their quotients). The method (a development of an idea due to Gelfand and Kirillov cf. [5] with [22]) was to consider what is now called the *Shapovalov determinant.*

The Cartan matrices A corresponding to finite dimensional Lie algebras and to those of class SZGLAPG are very special. Kac and Kazhdan [11] extended Shapovalov's result to the Lie algebras with *any* symmetrizable Cartan matrix A. Namely, if A is symmetrizable, then $\mathfrak{g}(A)$ possesses a nondegenerate invariant bilinear form B and with the help of the associated with B quadratic Casimir element C_2 they computed the Shapovalov determinant for all such Lie algebras.

Using absence of zero divisors in the enveloping algebra of any Lie algebra, they further obtained a description of irreducible modules that occur in the Jordan–Hölder series of an arbitrary Verma module over these algebras.

Kac later conjectured a formula for the Shapovalov determinant for the Lie *super*algebras with symmetrizable Cartan matrix [12]; the formula had an obvious mistake and a correction was offered in [13], for the proof of the corrected formula see [21]. As shown in [11], the formula for the determinant is a corollary of an explicit form of the quadratic Casimir element. Moreover, thanks to the existence of the Casimir element, the Shapovalov determinant turns out to be equal to the product of *linear* functions. For Lie superalgebras with symmetrizable Cartan matrix this element is even, so the argument of [11] are applicable literally. Still, the direct analogy soon stops: (a) to describe irreducible modules that occur in the Jordan–Hölder series of an arbitrary Verma module over Lie superalgebras we need new ideas due to the presence of zero divisors (cf. [21]), (b) the form of the product of linear factors depends on properties of odd roots involved and is more complicated, see Th. 2.4 [21]. Still, one obtains a criterion for irreducibility of Verma modules.

Our result: calculation of the even quadratic Casimir element for simple finite dimensional Lie superalgebras without symmetrizable Cartan matrix. Such are only the Lie superalgebras $\mathfrak{sh}(0|2n)$ of special hamiltonian vector fields. We also consider a "relative" of $\mathfrak{sh}(0|2n)$, the Poisson superalgebra $\mathfrak{po}(0|2n)$. Actually we derive the result for $\mathfrak{sh}(0|2n)$ from that for $\mathfrak{po}(0|2n)$.

Corollaries: a criterion for irreducibility of Verma modules over all these algebras and calculation of the Shapovalov determinant: it is the product of the linear terms-constituents of the above criterion.

Quantization sends $\mathfrak{sh}(0|2k)$ and $\mathfrak{po}(0|2k)$ into Lie superalgebras with Cartan matrix [17]; so our result can be read as a "dequantization" of the Shapovalov determinant for $\mathfrak{psl}\left(2^{k-1}|2^{k-1}\right)$ and $\mathfrak{gl}\left(2^{k-1}|2^{k-1}\right)$.

Another corollary related with existence of the quadratic Casimir element is an explicit solution to the classical Yang-Baxter equation with values in the above Lie superalgebras; for finite dimensional simple Lie superalgebras these solutions are given in [19].

Since it is natural to describe Poisson Lie superalgebras as subalgebras of the Lie superalgebra of contact vector fields, we describe them and recall our earlier result on the Shapovalov determinant for $\mathfrak{k}(1|6)$.

Related open problems. 1) Extension of our result to loop algebras is straightforward, elsewhere we will consider their "Kac–Moody" versions.

2) Even in the absence of the even quadratic Casimir element one can define the Shapovalov determinant provided the algebra possesses an involutive antiautomorphism which sends positive roots into negative ones. Among Lie superalgebras of type SZGLAPG only stringy or "superconformal" Lie superalgebras (for their complete list see [10]) possess this property together with relatives of the "queer" series and $\mathfrak{sh}(0|2n-1)$ as well as *its* relative, $\mathfrak{po}(0|2n-1)$, together with loop algebras and "Kac–Moody" versions thereof. For the queer Lie superalgebras, even finite dimensional ones, nobody had yet computed Shapovalov determinant. For some (but not all!) of the *distinguished*, see [10], stringy superalgebras the Shapovalov determinant is computed. In these cases it turned out to be the product of indecomposable *quadratic* polynomials, cf. [3, 15, 16] and refs. therein.

3) After graded algebras it is natural to consider filtered ones, prime examples being the Lie algebras of (a) differential operators with polynomial coefficients and of (b) "com-

plex size matrices"; for a description of a large class of them see [8]; most of them have nondegenerate invariant symmetric bilinear forms. Though even for the simplest of these algebras the Casimir element is not calculated yet, Shoikhet [23] calculated the Shapovalov determinant for some modules; conjecturally it is possible to calculate Casimir element on a wider class of modules, cf. [20].

1 A description of $\mathfrak{k}^L(1|6)$

Supercircle. A *supercircle* or (for a physicist) a closed *superstring* of dimension $1|m$ is the real supermanifold $S^{1|m}$ associated with the rank m trivial vector bundle over the circle. Let $t = e^{i\varphi}$, where φ is the angle parameter on the circle, be the even indeterminate of the Fourier transforms; let $\theta = (\theta_1, \dots, \theta_m)$, be the odd coordinates on the supercircle formed by a basis of the fiber of the trivial bundle over the circle. Then (t, θ) are the coordinates on $(\mathbb{C}^*)^{1|m}$, the complexification of $S^{1|m}$.

Let $m = 2k$ and the contact form be

$$\alpha = dt - \sum_{1\leq i\leq k} (\xi_i d\eta_i + \eta_i d\xi_i).$$

For the classification of simple "stringy" Lie superalgebras of vector fields and their nontrivial central extensions see [10]. Among the "main" series are: $\mathfrak{vect}^L(1|n) = \mathfrak{der}\,\mathbb{C}[t^{-1}, t, \theta]$, of all vector fields and $\mathfrak{k}^L(1|n)$ that preserves the Pfaff equation $\alpha = 0$. The superscript L indicates that we consider vector fields with Laurent coefficients, not polynomial ones.

The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over the general vectorial Lie superalgebra $\mathfrak{vect}(m|n)$ and its subalgebras, see [1]. Let $\mathfrak{g} = \mathfrak{vect}(m|n)$ realized by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m|n}$ with coordinates $x = (u, \xi)$ with the standard grading ($\deg x_i = 1$ for any $i = 1, \dots, n+m$) and $\mathfrak{g}_{\geq 0} = \underset{i\geq 0}{\oplus} \mathfrak{g}_i$. Clearly, $\mathfrak{g}_0 \cong \mathfrak{gl}(m|n)$. Let V be the $\mathfrak{gl}(m|n)$-module with the *lowest* weight $\lambda = \operatorname{lwt}(V)$. Make V into a $\mathfrak{g}_{\geq 0}$-module by setting $\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \underset{i>0}{\oplus} \mathfrak{g}_i$. The superspace $T(V) = \operatorname{Hom}_{U(\mathfrak{g}_{\geq 0})}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[[x]] \otimes V$. Its elements have a natural interpretation as formal *tensor fields of type* V (or λ). When $\lambda = (a, \dots, a)$ we will simply write $T(\vec{a})$ instead of $T(\lambda)$. In what follows we consider irreducible $\mathfrak{g}_0$-modules; for any other $\mathbb{Z}$-graded vectorial Lie superlagebra construction of modules with lowest weight is identical.

Examples. $T(\vec{0})$ is the superspace of functions; $\operatorname{Vol}(m|n) = T(1, \dots, 1; -1, \dots, -1)$ (the semicolon separates the first m coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*. We denote the generator of $\operatorname{Vol}(m|n)$ corresponding to the ordered set of indeterminates x by $\operatorname{vol}(x)$. The space of λ-densities is $\operatorname{Vol}^\lambda(m|n) = T(\lambda, \dots, \lambda; -\lambda, \dots, -\lambda)$. In particular, $\operatorname{Vol}^\lambda(m|0) = T(\vec{\lambda})$ while $\operatorname{Vol}^\lambda(0|n) = T(\overrightarrow{-\lambda})$.

Modules of tensor fields over stringy superalgebras. Denote by $T^L(V) = \mathbb{C}[t^{-1}, t] \otimes V$ the $\mathfrak{vect}(1|n)$-module that differs from $T(V)$ by allowing the *Laurent* polynomials as coefficients of its elements instead of just polynomials. Clearly, $T^L(V)$ is a $\mathfrak{vect}^L(1|n)$-module. Define the *twisted with weight* μ version of $T^L(V)$ by setting:

$$T^L_\mu(V) = \mathbb{C}[t^{-1}, t]t^\mu \otimes V.$$

The "simplest" modules. These are analogs of the *standard* or *identity* representation of the matrix algebras. The simplest modules over the Lie superalgebras of series $\mathfrak{vect}$ are, clearly, the modules Vol^λ. These modules are characterized by the fact that over $\mathcal{F}$, the algebra of functions, they are of rank 1, i.e., have only one generator. Over stringy superalgebras, we can as well twist these modules and consider Vol^λ_μ. Observe that for $\mu \notin \mathbb{Z}$ this module has only one submodule, the image of the exterior differential d, see [1]; for $\mu \in \mathbb{Z}$ this submodule coincides with the kernel of the residue:

$$\begin{aligned} &\mathrm{Res} : \mathrm{Vol}^L \longrightarrow \mathbb{C}, \\ &f\mathrm{vol}(t,\xi) \mapsto \text{ the coeff. of } \frac{\xi_1 \dots \xi_n}{t} \text{ in the power series expansion of } f. \end{aligned}$$

Over contact superalgebras $\mathfrak{k}(2n+1|m)$, it is more natural to express the simplest modules not in terms of λ-densities but in terms of powers of α:

$$\mathcal{F}_\lambda = \begin{cases} \mathcal{F}\alpha^\lambda & \text{for } n=m=0, \\ \mathcal{F}\alpha^{\lambda/2} & \text{otherwise.} \end{cases}$$

Observe that, as $\mathfrak{k}(2n+1|m)$-modules, $\mathrm{Vol}^\lambda \cong \mathcal{F}_{\lambda(2n+2-m)}$ and $\mathcal{F} = \mathcal{F}_0$. In particular, the Lie superalgebra of series $\mathfrak{k}$ does not distinguish between $\frac{\partial}{\partial t}$ and α^{-1}: their transformation rules are identical. Hence, $\mathfrak{k}(2n+1|m) \cong \mathcal{F}_{-1}$ if $n=m=0$ or $\mathcal{F}_{-2}$ otherwise. (Physicists usually set $\deg\theta = \frac{1}{2}$ and $\deg t = 1$, whereas we prefer, as is customary among mathematicians, the integer values of the highest weight with respect to the Cartan subalgebra of $\mathfrak{k}(1|n)_0 \cong \mathfrak{o}(2n) \oplus \mathbb{C}z$, so we use doubled physicists weights.)

Convenient formulas. A laconic way to describe the Lie superalgebras of series $\mathfrak{k}$ is via *generating functions.* For $f \in \mathbb{C}[t,\theta]$, where $\theta = (\xi,\eta)$, set:

$$K_f = (2-E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \qquad \text{where} \quad E = \sum_i \theta_i \frac{\partial}{\partial \theta_i}$$

and where H_f is the hamiltonian field with Hamiltonian f that preserves $d\alpha$:

$$H_f = -(-1)^{p(f)} \sum_{j \leq k} \left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j} \right).$$

Since

$$L_{K_f}(\alpha) = K_1(f)\alpha, \tag{1}$$

it follows that $K_f \in \mathfrak{k}(2n+1|m)$.

To the (super)commutator $[K_f, K_g]$ there corresponds the *contact bracket* of the generating functions:

$$[K_f, K_g] = K_{\{f,g\}_{k.b.}}.$$

An explicit formula for the contact brackets is as follows. Let us first define the brackets on functions that do not depend on t. The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ is given by the formula

$$\{f,g\}_{P.b.} = -(-1)^{p(f)} \left[\sum_{j \leq m} \left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial g}{\partial \xi_j} \right) \right].$$

Now, the contact bracket is

$$\{f,g\}_{k.b.} = (2-E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)(g) - \{f,g\}_{P.b.}.$$

It is not difficult to prove that $\mathfrak{k}(1|2k) \cong \operatorname{Span}(K_f : f \in \mathbb{C}[t,\theta])$ as superspaces.

The *Poisson superalgebra* is $\mathfrak{po}(0|m) = \operatorname{Span}(K_f : f \in \mathbb{C}[\theta])$. Its quotient modulo the center, $\mathfrak{z} = \mathbb{C}K_1$, is called the *Hamiltonian Lie superalgebra* $\mathfrak{h}(0|m)$; clearly, $\mathfrak{h}(0|m) \cong \operatorname{Span}(H_f : f \in \mathbb{C}[p,q,\theta])$. On $\mathfrak{po}(0|m)$ and $\mathfrak{h}(0|m)$, there are supertraces:

$$K_f, \quad H_f \mapsto \int f \cdot \operatorname{vol}(\theta).$$

The traceless elements span ideals $\mathfrak{spo}(0|m)$, *special Poisson superalgebra*, and $\mathfrak{sh}(0|m)$, *special Hamiltonian superalgebra*. For $m = 2k$ quantization sends them into $\mathfrak{gl}\left(2^{k-1}|2^{k-1}\right)$ and $\mathfrak{psl}\left(2^{k-1}|2^{k-1}\right)$, respectively, and the integral becomes the usual supertrace [17].

Roots of $\mathfrak{k}^L(1|6)$. The Cartan subalgebra of $\mathfrak{g} = \mathfrak{k}^L(1|6)$ is the span of

$$\begin{aligned} &H_1 = K_{\xi_1\eta_1}, && H_2 = K_{\xi_2\eta_2}, && H_3 = K_{\xi_3\eta_3}, && H_4 = K_t,\\ &H_5 = K_{\frac{1}{t}\xi_2\xi_3\eta_3\eta_2}, && H_6 = K_{\frac{1}{t}\xi_1\xi_3\eta_3\eta_1}, && H_7 = K_{\frac{1}{t}\xi_1\xi_2\eta_2\eta_1}, && H_8 = K_{\frac{1}{t^2}\xi_1\xi_2\xi_3\eta_3\eta_2\eta_1}. \end{aligned}$$

The weight of the elements of $\mathfrak{g}$ is taken, however, only with respect to the *diagonalizing* part $\mathfrak{d}$ of Cartan subalgebra, namely, with respect to H_4 and, after semicolon, H_1, H_2, H_3:

$$\operatorname{wht}(K_{t^a\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}\eta_3^{\beta_3}\eta_2^{\beta_2}\eta_1^{\beta_1}}) = (2(a-1) + |\alpha| + |\beta|; \alpha_1 - \beta_1, \alpha_2 - \beta_2, \alpha_3 - \beta_3),$$

where $|\alpha| = \sum \alpha_i$, $|\beta| = \sum \beta_i$. The root vectors are ordered lexicographically.

Roots of $\mathfrak{po}(0|2k)$ and $\mathfrak{sh}(0|2k)$. The Cartan subalgebra of $\mathfrak{po}(0|2k)$ is the span of $H_\emptyset = K_1$, $H_i = K_{\xi_i\eta_i}$, $H_{i,j} = K_{\xi_i\xi_j\eta_i\eta_j}$, $\dots$, $H_{1,\dots,n} = K_{\xi_1\dots\xi_k\eta_1\dots\eta_k}$. In what follows we denote: $\mathcal{I} = (1,\dots,n)$ and let $\mathcal{P}(\mathcal{I})$ be the set of all the subsets of $\mathcal{I}$.

Observe that the diagonalizing subalgebra $\mathfrak{d}$ of the Cartan subalgebra of $\mathfrak{po}(0|2k)$ is the Cartan subalgebra of $\mathfrak{o}(2k)$ spanned by the H_i. We will call it the "small" Cartan subalgebra. The roots are given with respect to $\mathfrak{d}$ completed with (to enable a finer grading) the exterior derivation $E = \sum \theta_i \frac{\partial}{\partial\theta_i}$, where $\theta = (\xi,\eta)$. The weight with respect to E, separated by a semicolon, is given first (shifted by -2).

The root vectors and their weights are:

$$\operatorname{wht}(K_{\xi_1^{\alpha_1}\dots\xi_k^{\alpha_k}\eta_1^{\beta_1}\dots\eta_k^{\beta_k}}) = (|\alpha| + |\beta| - 2; \alpha_1 - \beta_1, \dots, \alpha_k - \beta_k)$$

and the root vectors are ordered lexicographically. A shorthand for the vector $K_{\xi_1^{\alpha_1}\dots\xi_k^{\alpha_k}\eta_1^{\beta_1}\dots\eta_k^{\beta_k}}$ is $K_{\xi^\alpha\eta^\beta}$.

The roots of $\mathfrak{sh}(0|2k)$ are similar to those of $\mathfrak{po}(0|2k)$: just delete K_1 and $K_{\xi_1\eta_1\dots\xi_k\eta_k}$ and replace all K_f with H_f.

The root vectors corresponding to simple roots are

$$X^+_{i;I} = K_{\xi_i\cdot\xi^I\eta^I} \quad \text{and} \quad X^-_{i;I} = K_{\eta_i\cdot\xi^I\eta^I} \quad \text{for any subset } I \text{ of } \mathcal{I} \text{ except } \mathcal{I} \text{ itself.}$$

Though this does not matter in this paper, observe that these root vectors $X^\pm_{i;I}$ are not the most natural generators of $\mathfrak{po}(0|2k)$; at least, a seemingly more natural and more economic set of generators is different, cf. [9].

Kac–Moody superalgebras associated with $\mathfrak{po}(0|2k)$ and $\mathfrak{sh}(0|2k)$. Recall that for each finite dimensional Lie superalgebra $\mathfrak{g}$, the *loop algebra* associated with it is denoted by $\mathfrak{g}^{(1)} = \mathbb{C}[t^{-1}, t] \otimes \mathfrak{g}$. Let B be the nondegenerate supersymmetric bilinear form on $\mathfrak{g}$. A nontrivial central extension $\widehat{\mathfrak{g}^{(1)}} = \mathfrak{g}^{(1)} \oplus \mathbb{C}z$ of $\mathfrak{g}^{(1)}$ is given by means of the cocycle

$$c(f, g) = \text{Res } B(f, dg) \quad \text{for any} \quad f, g \in \mathfrak{g}^{(1)}.$$

On $\widehat{\mathfrak{g}^{(1)}}$, the form B induces the following nondegenerate invariant supersymmetric form $B^{(1)}$:

$$B^{(1)}((f, a), (g, b)) = B(f, g)(0) + ab \quad \text{for any } f, g \in \mathfrak{g}^{(1)} \text{ and } a, b \in \mathbb{C}.$$

The algebra $\widehat{\mathfrak{g}^{(1)}} \oplus \mathbb{C}t\frac{d}{dt}$ is called (affine) *Kac–Moody* Lie superalgebra, cf. [14]; for the list of simple Kac–Moody Lie superalgebras see [4].

The Cartan subalgebra of $\mathfrak{po}\widehat{(0|2k)}^{(1)}$ (resp. $\mathfrak{sh}\widehat{(0|2k)}^{(1)}$) is the span of the Cartan subalgebra of $\mathfrak{po}(0|2k)$ (resp. $\mathfrak{sh}(0|2k)$) and the central element z; hence, the roots are given with respect to $t\frac{d}{dt}$ and $E = \sum \theta_i \frac{\partial}{\partial \theta_i}$, and, after semicolon, the "small" Cartan subalgebra, spanned by the H_i.

The weights of the root vectors are

$$\text{wht}(t^n K_{\xi^\alpha \eta^\beta}) = (n; |\alpha| + |\beta| - 2; \alpha_1 - \beta_1, \ldots, \alpha_k - \beta_k)$$

and the root vectors are ordered lexicographically.

2 The bilinear forms and related Casimir elements

In what follows the root elements are normed so that $B(e_\alpha, e_\alpha^*) = 1$ for the series $\mathfrak{po}$ and $\mathfrak{sh}$ as well as for $\mathfrak{k}^L(1|6)$. The vector e_α^* is called the *right dual* of e_α, cf. [18].

In the realization of $\mathfrak{k}^L(1|6)$ by means of generating functions the invariant nondegenerate even supersymmetric bilinear form B is given by the formula

$$B(K_f, K_g) = \text{Res } fg, \quad \text{where Res}\,(f) = \text{the coefficient of } \frac{\xi_1 \ldots \xi_3 \eta_1 \ldots \eta_3}{t}. \tag{2}$$

It is easy to verify directly that

$$H_1^* = H_5, \qquad H_2^* = H_6, \qquad H_3^* = H_7, \qquad H_4^* = H_8. \tag{3}$$

Lemma 1. *The following Casimir element corresponds to the form B and, therefore, belongs to (the completion of) the center of the enveloping algebra of $\mathfrak{k}^L(1|6)$:*

$$C_2 = \sum_{\alpha > 0} e_\alpha^* e_\alpha + \sum_{i=1}^{4} H_i H_i^* + 4H_5 + 2H_6 - 4H_8. \tag{4}$$

Observe, that if in formula (4) we replace the right dual elements with the left dual ones, i.e., such that $B(e_\alpha^*, e_\alpha) = 1$, we obtain an element which does not belong to the center.

In the realization of $\mathfrak{po}(0|2k)$ by means of generating functions, the invariant nondegenerate bilinear form B is given by the formula

$$B(K_f, K_g) = \int fg \operatorname{vol}(\xi, \eta).$$

Clearly, this form induces an invariant nondegenerate form $B(H_f, H_g) = B(K_f, K_g)$ on $\mathfrak{sh}(0|2k)$. Obviously, up to a sign, $H_i^* = H_{\mathcal{I}\setminus\{i\}}$, $H_{i,j}^* = H_{\mathcal{I}\setminus\{i,j\}}$, etc.

Lemma 2. *The following Casimir element corresponds to the form B and, therefore, belongs to the center of the enveloping algebra of $\mathfrak{po}(0|2k)$:*

$$C_2 = 2\sum_{\alpha>0} e_\alpha^* e_\alpha + \sum_{J\in\mathcal{P}(\mathcal{I})} H_J H_J^* + (-2)^{k-1} H_{\mathcal{I}\setminus\{k\}}. \tag{5}$$

The Casimir element for $\mathfrak{sh}(0|2k)$ is obtained from the above one by deleting terms with $H_\emptyset$ and $H_\mathcal{I}$.

Observe that the third summand in (5) is precisely the element of the small Cartan subalgebra corresponding to the weight "2ρ", which is defined for any finite dimentional Lie superalgebra as the halfsum of positive even roots minus the halfsum of positive odd roots. It is remarkable that its form is so simple.

In the realization of $\widehat{\mathfrak{po}(0|2k)}^{(1)}$ with the help of generating functions the invariant nondegenerate bilinear form B is given by the formula

$$B\left((t^n K_f, a), (t^m K_g, b)\right) = \delta_{m+n,0} \int fg \operatorname{vol}(\xi, \eta) + ab.$$

Clearly, this form induces an invariant nondegenerate form on $\widehat{\mathfrak{sh}(0|2k)}^{(1)}$.

The Casimir element corresponding to the invariant form B belongs to the center of the (completed) enveloping algebra of $\widehat{\mathfrak{po}(0|2k)}^{(1)}$.

On proofs. The proof of Lemmas 1 and 2 is a direct verification: it suffices to apply root vectors corresponding to simple negative roots. This is a routine done by hand, but tests are much easier to perform with the help of Grozman's SuperLie package, see [6].

3 The Shapovalov determinant. Irreducible Verma modules over $\mathfrak{k}^L(1|6)$, and $\mathfrak{po}(0|2k)$ with its relatives

Let $a = (a_1, \dots, a_m)$ be the highest weight of the Verma module M^a over $\mathfrak{g}$ and $b = (b_1, \dots, b_m)$ (here $m = 8$ for $\mathfrak{k}^L(1|6)$, and $m = 2^{k-1}$ for $\mathfrak{po}(0|2k)$ and $m = 2^{k-1} - 2$ for $\mathfrak{sh}(0|2k)$) be one of the weights of M^a, i.e., $b = a - \sum n_i r_i$, where the r_i are positive roots and $n_i \in \mathbb{N}$ if the root r_i is even, $n_i = 0$ or 1 if the root r_i is odd, i.e., $\sum n_i r_i$ is a quasiroot. By applying C_2 to the highest weight vector of the Verma module we get, as in [11], the following theorems.

Theorem 1. *Let $\mathfrak{g} = \mathfrak{k}^L(1|6)$. The module M^a is irreducible if and only if for every quasiroot*

$$a_1a_5 + a_2a_6 + a_3a_7 + a_4a_8 \neq a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8 + b_1a_5 + b_2a_6 \\ + b_3a_7 + b_4a_8 + b_1b_5 + b_2b_6 + b_3b_7 + b_4b_8 - 4b_5 - 2b_6 + 4b_8,$$

or, in other words, if and only if $2(a+\rho,\beta) \neq (\beta,\beta)$ *for every quasiroot* β *and* $\rho = 2H_5 + H_6 - 2H_8$.

Theorem 2. *Let* $\mathfrak{g} = \mathfrak{po}(0|2k)$. *The module* M^a *is irreducible if and only if*

$$2(a+\rho,\beta) \neq (\beta,\beta) \quad \textit{for every quasiroot } \beta \textit{ and } \rho = (-2)^{k-2} H_{\mathcal{I}\setminus\{k\}}.$$

More explicitly, the above formula can be written as

$$\sum_{J\in\mathcal{P}(\mathcal{I})} a_J a_{J^*} \neq \sum_{J\in\mathcal{P}(\mathcal{I})} a_J b_{J^*} + \sum_{J\in\mathcal{P}(\mathcal{I})} b_J a_{J^*} + \sum_{J\in\mathcal{P}(\mathcal{I})} b_J b_{J^*} - (-2)^{k-1} b_{\mathcal{I}\setminus\{k\}}.$$

The description of the irreducible Verma modules M^a *over* $\mathfrak{sh}(0|2k)$ *is similar: in the above formula delete the terms with subscripts* $\emptyset$ *and* $\mathcal{I}$.

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