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A System of Four ODEs: The Singularity Analysis

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Abstract

The singularity analysis is carried out for a system of four first-order quadratic ODEs with a parameter, which was proposed recently by Golubchik and Sokolov. A transformation of dependent variables is revealed by the analysis, after which the transformed system possesses the Painlevé property and does not contain the parameter.

Recently, Golubchik and Sokolov [1] proposed the following system of four first-order quadratic ODEs:

$$\begin{aligned} p_t &= p^2 - pr - qs, \\ q_t &= apq + (a-2)rq, \\ r_t &= r^2 - pr - qs, \\ s_t &= (1-a)ps + (3-a)rs, \end{aligned} \tag{1}$$

where a is a parameter. It was pointed out in [1] that the system (1), though integrable by quadratures, probably does not pass the Painlevé–Kovalevskaya test for generic a. More recently, Leach, Cotsakis and Flessas [2] drew a conclusion that the system (1) does not possess the Painlevé property for any value of the parameter a.

In the present note, we give our version of the singularity analysis of the system (1). The analysis reveals a very simple transformation of the variables q and s, after which the transformed system possesses the Painlevé property and does not contain the parameter a.

Let us carry out the singularity analysis for the system (1), following the Ablowitz–Ramani–Segur algorithm [3] (see also [4]). Substituting into (1) the expansions of p, q, r, s near $\phi(t) = 0$, $\phi_t = 1$,

$$\begin{aligned} p &= p_0\phi^{\alpha} + \cdots + p_n\phi^{n+\alpha} + \cdots, \qquad & q &= q_0\phi^{\beta} + \cdots + q_n\phi^{n+\beta} + \cdots, \\ r &= r_0\phi^{\gamma} + \cdots + r_n\phi^{n+\gamma} + \cdots, \qquad & s &= s_0\phi^{\delta} + \cdots + s_n\phi^{n+\delta} + \cdots, \end{aligned} \tag{2}$$

we find the following three branches (i.e. the admissible choices of α, β, γ, δ and p_0, q_0, r_0, s_0 with the corresponding positions n of resonances), besides the evident branch governed by the Cauchy theorem ($\alpha = \beta = \gamma = \delta = 0$, $\forall\, p_0,\ q_0,\ r_0,\ s_0$):

$$\begin{aligned} &\alpha = -1, \qquad \beta = -a, \qquad \gamma = 0, \qquad \delta = a-1, \\ &p_0 = -1, \qquad r_0 = q_0 s_0, \qquad \forall\, q_0,\ s_0, \qquad (n+1)n^2(n-1) = 0; \end{aligned} \tag{3}$$

$$\begin{aligned} &\alpha = 0, \qquad \beta = 2 - a, \qquad \gamma = -1, \qquad \delta = a - 3, \\ &r_0 = -1, \qquad p_0 = q_0 s_0, \quad \forall\, q_0,\ s_0, \qquad (n+1)n^2(n-1) = 0; \end{aligned} \tag{4}$$

$$\begin{aligned} &\alpha = -1, \qquad \beta = 2 - 2a, \qquad \gamma = -1, \qquad \delta = 2a - 4, \\ &p_0 = r_0 = -1, \qquad q_0 s_0 = -1, \quad \forall\, q_0 \ \text{xor}\ \forall\, s_0, \qquad (n+1)^2 n(n-2) = 0. \end{aligned} \tag{5}$$

We see from (3), (4), (5) that the system (1) may possess the Painlevé property only if the parameter a is integer. But the positions of resonances are integer and independent of a in all the branches, and this suggests that the expansions (2) do not contain terms with noninteger n. Moreover, the expansions (2) are free from logarithmic terms, as we can prove by checking the consistency of recursion relations for p_n, q_n, r_n, s_n, $n = 0, 1, \ldots$, obtainable from (1). We have the following:

$$p_1 = 0, \qquad q_1 = (a-2) q_0^2 s_0, \qquad s_1 = (3-a) q_0 s_0^2, \quad \forall\, r_1,$$

for (3);

$$r_1 = 0, \qquad q_1 = a q_0^2 s_0, \qquad s_1 = (1-a) q_0 s_0^2, \quad \forall\, p_1,$$

for (4);

$$\begin{aligned} &p_1 = q_1 = r_1 = s_1 = 0, \qquad p_2 = r_2 = -s_0 q_2 - q_0 s_2, \\ &(a-2) s_0 q_2 + (a-1) q_0 s_2 = 0, \quad \forall\, q_2 \ \text{xor}\ \forall\, s_2, \end{aligned}$$

for (5). Consequently, in all the branches, solutions of (1) are represented by the expansions

$$p = \phi^\alpha \sum_{n=0}^{\infty} p_n \phi^n, \qquad q = \phi^\beta \sum_{n=0}^{\infty} q_n \phi^n, \qquad r = \phi^\gamma \sum_{n=0}^{\infty} r_n \phi^n, \qquad s = \phi^\delta \sum_{n=0}^{\infty} s_n \phi^n,$$

where $\alpha, \gamma = -1, 0$, and the functions $\beta(a)$ and $\delta(a)$ vary from branch to branch. Therefore we can hope to improve the analytic properties of the system (1) by an appropriate transformation of the dependent variables q and s.

Let us consider a variable $z(t)$,

$$z = q^x s^y, \tag{6}$$

where x and y are constants, and study its dominant behavior $z = z_0 \phi^\epsilon + \cdots$ near $\phi = 0$ in the branches (3), (4), (5). We have $\epsilon = i$,

$$i = -ax + (a-1)y, \tag{7}$$

in the branch (3); $\epsilon = j$,

$$j = (2-a)x + (a-3)y, \tag{8}$$

in the branch (4); $\epsilon = k$,

$$k = (2-2a)x + (2a-4)y, \tag{9}$$

in the branch (5). Since $k = i + j$ due to (7), (8), (9), k will be integer for any integer i and j. And, for any integer i and j, the variable z (6) will possess a good dominant behavior in each of the branches (3), (4), (5), if we set

$$x = \frac{1}{2}(a-3)i - \frac{1}{2}(a-1)j, \qquad y = \frac{1}{2}(a-2)i - \frac{1}{2}aj \tag{10}$$

due to (7), (8).

Which choice of the integers i and j to prefer? According to (1) and (6),

$$z_t = (-ip - jr)z,$$

and it seems natural to choose $i = -1$, $j = 0$ or $i = 0$, $j = -1$. Denoting $z|_{i=-1,j=0}$ as u, and $z|_{i=0,j=-1}$ as v, we find from (6), (10) that

$$u = q^{\frac{1}{2}(3-a)} s^{\frac{1}{2}(2-a)}, \qquad v = q^{\frac{1}{2}(a-1)} s^{\frac{1}{2}a}.$$

In the new variables p, r, u, v, the system (1) changes into

$$\begin{aligned} p_t &= p^2 - pr - uv, \\ r_t &= r^2 - pr - uv, \\ u_t &= pu, \\ v_t &= rv. \end{aligned} \tag{11}$$

The system (11) possesses the Painlevé property and does not contain the parameter a. Also, the three constants of motion of the system (1), given in [1] in the variables p, q, r, s, become more simple in the new variables p, r, u, v:

$$c_1 = pr + uv, \qquad c_2 = \frac{p-r}{uv}, \qquad c_3 = \frac{r^2 - pr - uv}{v^2}.$$

The constant of motion $c_4 = (p^2 - pr - uv)/u^2$ simply follows from c_3 via the evident symmetry $p \leftrightarrow r$, $u \leftrightarrow v$ of the system (11), and one can use any three of c_1, c_2, c_3, c_4 as mutually independent.

Obviously, the analytic properties of the Golubchik–Sokolov system (1) do not contradict to the empirically well known interrelation between the Painlevé property and the integrability of ODEs and PDEs. This system is similar to the sine-Gordon equation, which also does not possess the Painlevé property until a simple transformation is made [5].

References

[1] Golubchik I Z and Sokolov V V, Generalized Operator Yang-Baxter Equations, Integrable ODEs and Nonassociative Algebras, *J. Nonlin. Math. Phys.* **7**, No.2 (2000), 184–197.

[2] Leach P G L, Cotsakis S and Flessas G P, Symmetry, Singularities and Integrability in Complex Dynamics I: The Reduction Problem, *J. Nonlin. Math. Phys.* **7**, No.4 (2000), 445–479.

[3] Ablowitz M J, Ramani A and Segur H, A Connection between Nonlinear Evolution Equations and Ordinary Differential Equations of P-Type I, *J. Math. Phys.* **21**, No.4 (1980), 715–721.

[4] Ramani A, Grammaticos B and Bountis T, The Painlevé Property and Singularity Analysis of Integrable and Non-Integrable Systems, *Phys. Rep.* **180**, No.3 (1989), 159–245.

[5] Weiss J, The Sine-Gordon Equation: Complete and Partial Integrability, *J. Math. Phys.* **25**, No.7 (1984), 2226–2235.