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Algebraic Linearization of Hyperbolic Ruijsenaars–Schneider Systems

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Abstract

In this article, we present an explicit linearization of dynamical systems of Ruijsenaars-Schneider (RS) type and of the perturbations introduced by F Calogero [2] of these systems with all orbits periodic of the same period. The existence of this linearization and its algebraic nature relies on the dynamical equation firstly discussed in the article [3]. The notion of algebraic linearization which was first displayed in NEEDS 99 conference will be discussed further with several other examples in a forthcoming publication. A differential system is algebraically (resp. analytically) linearizable if there are n globally defined functions (rational, resp. meromorphic) which are generically independent so that the time evolution of the flow expressed in these functions is linear (in time) and algebraic in the initial coordinates.

1 Algebraic linearization of hyperbolic Ruijsenaars–Schneider systems

The dynamical systems of Ruijsenaars–Schneider (RS) type characterized by the equations of motion

$$\ddot{z}_j = \sum_{k=1,\, k\neq j}^{n} \dot{z}_j \dot{z}_k f(z_j - z_k), \qquad j = 1, \ldots, n \tag{1.1}$$

are "integrable" or "solvable" [2], if

$$1)\quad f(z) = 2/z, \tag{1.2a}$$

$$2)\quad f(z) = 2\left[z\left(1 + r^2 z^2\right)\right], \tag{1.2b}$$

$$3)\quad f(z) = 2a \text{ cotgh } (az), \tag{1.2c}$$

$$4)\quad f(z) = 2a/\sinh (az), \tag{1.2d}$$

$$5)\quad f(z) = 2a \text{ cotgh } (az)/\left[1 + r^2 \sinh^2(az)\right], \tag{1.2e}$$

$$6)\quad f(z) = -a\mathcal{P}'(az)/[\mathcal{P}(az) - \mathcal{P}(ab)]. \tag{1.2f}$$

In this first paragraph, we focus on the Hyperbolic and rational case given by (1.2e). The starting point of the analysis is the observation [1] that (1.1) with (1.2e) is equivalent to the following “Lax-type” $(n \times n)$-matrix equation:

$$\dot{L} = [L, M]_-, \tag{1.3}$$

with

$$L_{jk} = \delta_{jk}\dot{z}_j + (1-\delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}\alpha(z_j - z_k), \tag{1.4}$$

$$M_{jk} = \delta_{jk}\sum_{m=1,\, m\neq j}^{n} \dot{z}_m\beta(z_j - z_m) + (1-\delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}\gamma(z_j - z_k), \tag{1.5}$$

and

$$\alpha(z) = \sinh(a\mu)/\sinh[a(z+\mu)], \tag{1.6a}$$

$$\beta(z) = -a \text{ cotgh}(a\mu)/\left[1 + r^2\sinh^2(az)\right], \tag{1.6b}$$

$$\gamma(z) = -a \text{ cotgh}(az)\alpha(z), \tag{1.6c}$$

where

$$\sinh(a\mu) = \mathrm{i}/r. \tag{1.7}$$

It was furthermore noted [2, 4, 5, 6] that the diagonal matrix

$$X(t) = \mathrm{diag}\{\exp[2az_j(t)]\}, \tag{1.8}$$

undergoes the following time evolution:

$$\dot{X} = [X, M]_- + a[X, L]_+. \tag{1.9}$$

Let F_k and G_k be the functions defined as:

$$F_k = \mathrm{tr}\left(L^k\right), \qquad G_k = \mathrm{tr}\left(XL^k\right). \tag{1.10}$$

The functions F_k are first integrals of the dynamical system defined by (3.1). Cayley–Hamilton relate these constant of motion with the coefficients $A_0, \ldots, A_{n-1}$ of the characteristic polynomial of the matrix L,

$$L^n = A_{n-1}L^{n-1} + A_{n-2}L^{n-2} + \cdots + A_0 I. \tag{1.11}$$

The equations (1.3) and (1.9) lead to:

$$\dot{G}_k = 2a \text{ tr}\left(XL^{k+1}\right) = 2aG_{k+1}. \tag{1.12}$$

Thus,the vector $G = (G_0, \ldots, G_{n-1})$ displays the time evolution:

$$\dot{G} = AG, \tag{1.13}$$

where the matrix A is with coefficients first integrals of the differential system:

$$A_{ij} = 2a\delta_{i+1,j} + 2aA_{j-1}\delta_{i,n}. \tag{1.14}$$

2 Algebraic linearization of the Calogero extension of the Ruijsenaars–Schneider systems

F Calogero introduced the following perturbation of the trigonometric and rational Ruijsenaars–Schneider systems characterized by the equations of motion:

$$\ddot{z}_j + \mathrm{i}\Omega \dot{z}_j = \sum_{k=1,\, k\neq j}^{n} \dot{z}_j \dot{z}_k f(z_j - z_k), \qquad j = 1, \ldots, n. \tag{2.1}$$

F Calogero made the remarkable conjecture [2], now proved in the trigonometric and rational cases, that all the orbits of the dynamical system defined by (2.1) are periodic of period Ω. The equations under consideration here are modified due to the presence of the perturbation. The Lax equation (2.3) gets modified into (cf. [3]):

$$\dot{L} = [L, M]_- + \mathrm{i}\Omega L, \tag{2.2}$$

and the time evolution of the matrix X is not modified. This yields new time evolution for the functions F_k and G_k:

$$\dot{F}_k = \mathrm{i}\Omega k F_k, \tag{2.3a}$$

$$\dot{G}_k = 2a \operatorname{tr}\left(XL^{k+1}\right) + \mathrm{i}\Omega k \operatorname{tr}\left(XL^{k}\right) = 2aG_{k+1} + \mathrm{i}\Omega k G_k. \tag{2.3b}$$

3 Some general facts on the relationships between algebraic linearization, superintegrability and isochronicity

Recall that a differential system defined on a manifold of dimension n is said to be superintegrable if it displays $n-1$ first integrals. It is said to be isochronous if all its orbits, or solutions, periodic of same period. The interrelations between these notions and the notion of algebraic linearization require further developments. We like to mention here some observations in the prolongation of the examples that we discussed above. Firstly the algebraic linearization does not yield the superintegrability in general. Nevertheless the following (obvious!) fact should be noted: given an algebraic linearisable system, if the linear system obtained from the initial system in the linearizing coordinates is superintegrable, then the initial system is itself superintegrable. We have in mind the case where the linear system obtained is an harmonic oscillator. The system is then superintegrable if and only if all the ratios of frequencies of the harmonic oscillator are rational.

Let X be a vector field which displays $n-1$ generically independent integrals $F = (F_1, \ldots, F_{n-1})$. Choose $\Omega = dx_1 \wedge \ldots \wedge dx_n$ a volume form on the manifold. There is a 1-form ξ such that:

$$\xi \wedge dF_1 \wedge \ldots \wedge dF_{n-1} = \Omega. \tag{3.1}$$

Assume that each F_i satisfies the de Rham division property (this is generic). This yields a function h so that:

$$\iota_X \Omega = h\, dF_1 \wedge \ldots \wedge dF_{n-1}. \tag{3.2}$$

Assume now that the vector field X is volume preserving (Hamiltonian systems are volume preserving for instance). Then in generic cases this yields $h = H(F_1, \ldots, F_{n-1})$ and the period of the vector field X along the orbits $F^{-1}(c)$ is given as the integral:

$$\int_{F^{-1}(c)} [\xi/h]. \tag{3.3}$$

Thus the vector field X is isochronous if and only if this integral is constant independent of c.

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