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A Simple Family of Non-Local Poisson Brackets

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Abstract

A dispersionless integrable system with repeated eigenvalues is presented. For $N \geq 3$ components the system has no local Hamiltonian structure. Infinitely many simple compatible non-local Hamiltonian structures are given, using a result of Ferapontov.

1 Introduction

A *system of hydrodynamic type* is a system of N coupled quasi-linear first-order PDE's which may be written in the form

$$\frac{\partial \mathbf{q}}{\partial t} = M(\mathbf{q})\frac{\partial \mathbf{q}}{\partial x}, \tag{1}$$

where $\mathbf{q} = \left(q^1(x,t), \ldots, q^N(x,t)\right)^T$ is a vector of functions and $M(\mathbf{q})$ is an $N \times N$ matrix whose elements are functions of the dependent variables q^i. The simplest non-trivial example (for $N = 1$) is the inviscid Burgers equation $u_t = uu_x$. Another simple example is the continuum Toda equation (see [1] and references therein), $\rho_{tt} = (\mathrm{e}^\rho)_{xx}$, which may be written as two component system of hydrodynamic type by introducing the variable $u = \partial_t \int \rho \, \mathrm{d}x$. Another well-known example of an integrable system of hydrodynamic type are the Polytropic gas equations [2], which provide one of the clearest illustrations of both the Dubrovin–Novikov theorem [3] for (local) Hamiltonian structure and Dubrovin's subsequent work on Frobenius manifolds [4].

The purpose of this paper is to discuss a particular system of hydrodynamic type which, for $N \geq 3$ components, has no local Hamiltonian structure whatsoever. However, it has infinitely many simple, compatible non-local Hamiltonian operators, all of which may be determined by Ferapontov's theorem [5]: indeed, this system provides an extremely simple illustration of this elegant theorem.

Consider the system $\mathbf{q}_t = M(\mathbf{q})\mathbf{q}_x$, where M is the matrix

$$M = \begin{pmatrix} S + q^1p^{-1} & q^1p^{-2} & \cdots & q^1p^{-N} \\ q^2p^{-1} & S + q^2p^{-2} & \cdots & q^2p^{-N} \\ \vdots & \vdots & \ddots & \vdots \\ q^Np^{-1} & q^Np^{-2} & \cdots & S + q^Np^{-N} \end{pmatrix}, \tag{2}$$

where $S = \sum\limits_{j=1}^{N} q^j p^{-j}$, and $p^{-l} = q^l$ or 1. Depending on the choice of p^{-i}, the nonlinearity is either cubic or quadratic, and so the system may be thought of as either a set of dispersionless mKdV equations or as a set of dispersionless KdV equations. This is indeed so, since the system is a reduction of the dispersionless limit of the third order flow of a scattering problem associated with complex projective space $\mathbb{CP}^N$ — see [7] for details of the construction, as well as a discussion of the solutions, conservation laws, Lagrangian formulation and many other aspects of the system. Only the Hamiltonian structure will be considered in this paper.

2 Diagonalizing the system

Riemann noted that for certain systems of hydrodynamic type it is possible to diagonalize the governing matrix M: that is, there exists a set of co-ordinates, the Riemann invariants r^i, which allow the system to be written in the form

$$r^i_t = \lambda^i r^i_x \qquad \text{(no sum)}. \tag{3}$$

Each λ^i is an eigenvalue of the matrix M, and is often referred to as a *characteristic speed.* Riemann showed that for $N = 2$, it is always possible to diagonalize the system, but this is not always true for $N \geq 3$. It is a theorem (due to Tsarev [6]) that all diagonalizable systems of hydrodynamic type are completely integrable Hamiltonian systems.

It is readily found that the eigenvalues of the matrix M are $2S$ and S, with multiplicity 1 and $N - 1$ respectively. The existence of a repeated eigenvalue is unusual; indeed, the author is unaware of any examples in the literature of hydrodynamic systems with repeated eigenvalues.

A possible set of Riemann invariants corresponding to speeds $2S$ and S are S and $R^\alpha = q^\alpha/q^1$ respectively. This permits a partial decoupling of the system by writing

$$S_t = 2SS_x \qquad \text{and} \qquad R^\alpha_t = SR^\alpha_x, \tag{4}$$

where $\alpha = 2, \ldots N$. For notational convenience, Latin indices will now range from 1 to N, to denote all co-ordinates, whereas Greek indices will indicate that only $N - 1$ variables are being used. The advantage of this co-ordinate system is that is does not distinguish between dispersionless KdV or dispersionless mKdV systems. The ubiquity of the S is striking and may be explained by the underlying algebraic structure of the system (see [7] for details). It should be noted that this variable dominates all aspects of the system, and so it is natural to seek a Hamiltonian structure with a basis with such a bias.

3 Multi-Hamiltonian structure

This is determined by appealing to the geometry of the dependent variables. In [3], Dubrovin and Novikov considered operators of the form

$$\hat{A}^{ij} = g^{ij}(\mathbf{q})\frac{\mathrm{d}}{\mathrm{d}x} - b^{ij}_k(\mathbf{q})q^k_x. \tag{5}$$

They showed that this operator is Hamiltonian if $\mathbf{g} = (g^{ij})^{-1}$ defines a non-degenerate pseudo-Riemannian metric; $b^{ij}_k = -g^{is}\Gamma^j_{sk}$, where Γ^j_{sk} are the coefficients of the Levi–Civita connection; and the Riemann curvature tensor of $\mathbf{g}$ is identically zero.

Ferapontov has studied generalizations of this operator (see [5]) by adding a non-local "tail" to (5). Explicitly, the Poisson bracket of two functionals $F = \int f(u, u_x, \ldots)\,\mathrm{d}x$ and

$G = \int g(u, u_x, \ldots)\,\mathrm{d}x$ is given by

$$\{F, G\} = \int \frac{\delta F}{\delta u^i} \hat{A}^{ij} \frac{\delta G}{\delta u^j}\,\mathrm{d}x,$$

where $\hat{A}^{ij}$ is the operator

$$\hat{A}^{ij} = g^{ij}\frac{\mathrm{d}}{\mathrm{d}x} - g^{is}\Gamma^j_{sk}u^k_x - \sum_{\alpha=1}^{c} w^i_{(\alpha)k}u^k_x \mathrm{d}^{-1} w^j_{(\alpha)l}u^l_x. \tag{6}$$

This operator is Hamiltonian if $\mathbf{g} = (g^{ij})^{-1}$ is a pseudo-Riemannian metric; the connection Γ^j_{sk} is symmetric and compatible with the metric; the $\mathbf{w}_\alpha$'s are a set of c Weingarten operators; and the metric and Weingarten maps satisfy the Gauss–Peterson–Codazzi equations for submanifolds of co-dimension c with a flat normal connection. These are:

$$\begin{aligned} &g_{ik}w^k_{(\alpha)j} = g_{jk}w^k_{(\alpha)i}, \\ &\nabla_k w^i_{(\alpha)j} = \nabla_j w^i_{(\alpha)k}, \\ &R^{ij}_{kl} = \sum_{\alpha=1}^{c}\left\{w^i_{(\alpha)k}w^j_{(\alpha)l} - w^j_{(\alpha)k}w^i_{(\alpha)l}\right\}, \\ &\mathbf{w}_\alpha\mathbf{w}_\beta = \mathbf{w}_\beta\mathbf{w}_\alpha. \end{aligned} \tag{7}$$

What is the metric? Tsarev [6] showed that, in Riemann invariant co-ordinates, the metric is diagonal and has coefficients g_{jj} given by the equations

$$\frac{1}{2}\frac{\partial}{\partial r^i}\log g_{jj} = \frac{1}{\lambda^i - \lambda^j}\frac{\partial \lambda^j}{\partial r^i} \tag{8}$$

a set of $N(N-1)$ equations. For system (4), this is potentially problematic, since the above formula involves a denominator $(\lambda^i - \lambda^j)$ for eigenvalues λ^i, λ^j, where $i \neq j$, and $(N-1)$ of the N eigenvalues are identical. However $\partial\lambda^\alpha/\partial r^\beta = 0$ for all $\alpha, \beta = 2, \ldots, N$, and so an elementary solution of (8) is

$$\mathbf{g}(\varphi) = \varphi(S)\mathrm{d}S^2 + kS^2\left[\left(\mathrm{d}R^1\right)^2 + \cdots + \left(\mathrm{d}R^{N-1}\right)^2\right], \tag{9}$$

where $\varphi(S)$ is some smooth function and k is some constant. Let $\left(\mathcal{M}^N, \varphi(S), k\right)$ be the N dimensional manifold with this metric.

- $(\mathcal{M}^2, 1, 1)$ is the plane $\mathbb{R}^2$ equipped with the polar co-ordinates metric. Since this is a flat Euclidean space, the Weingarten maps are identically zero and the Hamiltonian operator becomes local. *Remark*: This is the only flat manifold within the class given above, and so the Dubrovin–Novikov theorem will only produce a single (local) Hamiltonian structure for $N = 2$ components, and no (local) Hamiltonian structure for $N \geq 3$ components.
- $(\mathcal{M}^N, 1/\kappa^2 S^2, 1)$ is a surface of constant curvature κ and so has Weingarten map $w^i_j = \kappa\delta^i_j$.

- $(\mathcal{M}^N, \varphi(S), k)$ may be viewed as a submanifold of co-dimension one with Weingarten map $\mathbf{w} = \mathrm{diag}\{f(S), g(S), \ldots, g(S)\}$. The functions $f(S)$ and $g(S)$ are readily determined from the Gauss–Peterson–Codazzi equations (7): these state that

$$
\begin{aligned}
&Sg'(S) = f(S) - g(S),\\
&f(S)g(S) = -\varphi'(S)/2S\varphi(S)^2 \qquad \left(= R^{1\alpha}_{1\alpha}\right),\\
&g(S)^2 = 1/S^2\varphi(S) \qquad \left(= R^{\alpha\beta}_{\alpha\beta},\ \alpha \neq \beta,\ N \geq 3\right).
\end{aligned} \tag{10}
$$

The solution for $f(S)$ and $g(S)$ is immediate from the last two equations: $f(S) = -\varphi'(S)/2\varphi(S)\sqrt{\varphi(S)}$ and $g(S) = 1/S\sqrt{\varphi(S)}$. It is readily verified that these are consistent with the first equation. The Hamiltonian operator is therefore given by:

$$
\begin{aligned}
\hat{A}^{ij} = &\begin{pmatrix} 1/\varphi(S) & 0 & \cdots & 0 \\ 0 & 1/kS^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/kS^2 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} \\
&- \begin{pmatrix} \varphi'(S)S_x/2\varphi(S)^2 & R^1_x/S\varphi(S) & \cdots & R^{N-1}_x/S\varphi(S) \\ -R^1_x/S\varphi(S) & S_x/kS^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -R^{N-1}_x/S\varphi(S) & 0 & \cdots & S_x/kS^3 \end{pmatrix} \\
&- \begin{pmatrix} f(S)S_x\mathrm{d}^{-1}f(S)S_x & f(S)S_x\mathrm{d}^{-1}g(S)R^1_x & \cdots & f(S)S_x\mathrm{d}^{-1}g(S)R^{N-1}_x \\ g(S)R^1_x\mathrm{d}^{-1}f(S)S_x & g(S)R^1_x\mathrm{d}^{-1}g(S)R^1_x & \cdots & g(S)R^1_x\mathrm{d}^{-1}g(S)R^{N-1}_x \\ \vdots & \vdots & \ddots & \vdots \\ g(S)R^{N-1}_x\mathrm{d}^{-1}f(S)S_x & g(S)R^{N-1}_x\mathrm{d}^{-1}g(S)R^1_x & \cdots & g(S)R^{N-1}_x\mathrm{d}^{-1}g(S)R^{N-1}_x \end{pmatrix}.
\end{aligned} \tag{11}
$$

Hence there exist infinitely many Hamiltonian structures. It now only remains to show that these are compatible in the sense of Magri's theorem [8], that is to say, that $\{\ ,\ \}_1 + \lambda\{\ ,\ \}_2$ is itself a Hamiltonian structure for all values of λ. To do this, consider the metric

$$\mathbf{g} = \frac{\varphi_1(S)\varphi_2(S)}{\varphi_2(S) + \lambda\varphi_1(S)}\mathrm{d}S^2 + \frac{S^2}{1+\lambda}\left[\left(\mathrm{d}R^1\right)^2 + \cdots + \left(\mathrm{d}R^{N-1}\right)^2\right].$$

The contravariant form of this metric is clearly $g^{ij} = g^{ij}_{(1)} + \lambda g^{ij}_{(2)}$, where the metrics $g^{ij}_{(\alpha)}$ are defined in the obvious manner. It may be verified by direct calculation that the connection for such a metric is consistent with expectation. In order to prove that this metric gives rise to a Hamiltonian structure equal to $\{\ ,\ \}_1 + \lambda\{\ ,\ \}_2$ it is necessary to work in co-dimension two by introducing a pair of Weingarten operators $\mathbf{w}_1 = \mathrm{diag}\{f_1(S), g_1(S), \ldots, g_1(S)\}$ and $\mathbf{w}_2 = \mathrm{diag}\left\{\lambda^{1/2}f_2(S), \lambda^{1/2}g_2(S), \ldots, \lambda^{1/2}g_2(S)\right\}$, where f_i, g_i are defined in an analogue manner to f, g above. Direct calculation reveals that

$$R^{1\alpha}_{1\alpha} = -\frac{\varphi_1'(S)}{2S\phi_1(S)^2} - \frac{\lambda\varphi_2'(S)}{2S\varphi_2(S)^2}, \qquad R^{\alpha\beta}_{\alpha\beta} = \frac{1}{S^2\varphi_1(S)} + \frac{\lambda}{S^2\varphi_2(S)}. \tag{12}$$

Hence the curvature part of the Gauss–Peterson–Codazzi equations is satisfied. It is easily verified that the other equations are satisfied by these Weingarten operators. Hence $\{\ ,\ \}_1 + \lambda\{\ ,\ \}_2$ is a Hamiltonian structure (of co-dimension 2) and hence any two Hamiltonian structures (11) are compatible.

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References

[1] Fairlie D B and Strachan I A B, The Hamiltonian Structure of the Dispersionless Toda Hierarchy, *Physica D*, 1996, V.90, 1–8.

[2] Olver P J and Nutku Y, Hamiltonian Structures for Systems of Hyperbolic Conservation Laws, *J. Math. Phys.*, 1988, V.29, 1610–1619.

[3] Dubrovin B and Novikov S P, Hydrodynamical Formalism of One-Dimensional Systems of Hydrodynamic Type and the Bogolyubov–Whitham Averaging Method, *Sov. Math. Dokl.*, 1983, V.27, 665.

[4] Dubrovin B, Geometry of 2D Topological Field Theories, in Integrable Systems and Quantum Groups, Lecture Notes in Mathematics, Vol.1620, Editors M Francariglia and S Grece, Springer Verlag, 1995.

[5] Ferapontov E V, Non-Local Hamiltonian Operators of Hydrodynamic Type: Differential Geometry and Applications, *A.M.S. Translations*, 1995, V.170, 33–58.

[6] Tsarev S P, Poisson Brackets and One-Dimensional Hamiltonian Systems of Hydrodynamic Type, *Sov. Math. Dokl.*, 1985, V.31, 488–491.

[7] McCarthy O D, Dispersionless Integrable Systems of KdV Type, Ph.D. Thesis, The University of Hull, 2001.

[8] Magri F, A Simple Model of the Integrable Hamiltonian Equation, *J. Math. Phys.*, 1978, V.19, 1156–1162.