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Whitham–Toda Hierarchy in the Laplacian Growth Problem

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Abstract

The Laplacian growth problem in the limit of zero surface tension is proved to be equivalent to finding a particular solution to the dispersionless Toda lattice hierarchy. The hierarchical times are harmonic moments of the growing domain. The Laplacian growth equation itself is the quasiclassical version of the string equation that selects the solution to the hierarchy.

The Laplacian growth problem is one of the central problems in the theory of pattern formation. It has many different faces and a lot of important applications. In general words, this is about dynamics of moving front (interface) between two different phases. In many cases the dynamics is governed by a scalar field that obeys the Laplace equation; that is why this class of growth problems is called Laplacian. Here we shall confine ourselves to the two-dimensional (2D) case only. To be definite, we shall speak about two incompressible fluids with different viscosities on the plane. In practice, the 2D geometry is realized in the narrow gap between two plates. In this version, this is known as the Saffman–Taylor problem or viscous fingering in the Hele–Shaw cell. For a review, see [1].

We shall mostly concentrate on the *external radial problem* for it turns out to be the simplest case in the frame of the suggested approach. Let the exterior of a simply connected domain on the plane be occupied by a viscous fluid (oil) while the interior be occupied by a fluid with small viscosity (water). The oil/water interface is assumed to be a simple analytic curve. Other versions such as internal radial problem, wedge or channel geometry are briefly discussed at the end of the paper. Basically, they allow for the same approach.

Let $p(x, y)$ be the pressure, then p is constant in the water domain. We set it equal to zero. In the case of zero surface tension p is a continuous function across the interface, so $p = 0$ on the interface. In the oil domain the gradient of p is proportional to local velocity $\vec{V} = (V_x, V_y)$ of the fluid (Darcy's law):

$$\vec{V} = -\kappa \,\mathrm{grad}\, p, \tag{1}$$

where κ is called the filtration coefficient[1]. In particular, this law holds on the interface thus governing its dynamics:

$$V_n = -\kappa \frac{\partial p}{\partial n}. \tag{2}$$

[1]This coefficient is inversely proportional to the viscosity, so the Darcy law is formally valid in the water, too.

Here V_n is the component of the velocity normal to the interface and $\partial p/\partial n$ is normal derivative. Since the fluid is incompressible ($\mathrm{div}\,\vec{V} = 0$), the Darcy law implies that the potential $\Phi(x, y) = -\kappa p(x, y)$ is a harmonic function in the exterior (oil) domain:

$$\Delta\Phi(x, y) = 0, \tag{3}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator on the plane. The asymptotic behaviour of the function Φ very far away from the interface (at infinity) is determined by the physical condition that there is a sink with constant capacity q placed at infinity. This means that

$$\oint_\gamma (\vec{V}, \vec{n})dl = q,$$

where γ is any closed contour encircling the water domain, $\vec{n}$ is the unit vector normal to γ. So, we require $\Phi = 0$ on the interface and $\Phi = \frac{q}{2\pi}\log|z|$ as $|z| \to \infty$. The goal is to describe the interface motion subject to the local dynamical law $V_n = \partial\Phi/\partial n$.

An effective tool for dealing with this problem is the time dependent conformal mapping technique (see e.g. [1]). Passing to the complex coordinates $z = x + iy$, $\bar{z} = x - iy$ on the physical plane, we bring into play a conformal map from a reference domain on the mathematical plane w to the growing domain on the physical plane. By the Riemann mapping theorem, such a map does exist and, under some conditions, is unique. More precisely, let $z = \lambda(w)$ be the univalent conformal map from the exterior of the unit circle to the exterior of the interface (i.e., to the oil domain) such that ∞ is mapped to ∞ and the derivative $\lambda'(\infty)$ is a positive real number r. Under these conditions the map is known to be unique. The Laurent expansion of the $\lambda(w)$ around ∞ has then the following general form:

$$\lambda(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j}. \tag{4}$$

If the interface moves, the conformal map $z(w, t) = \lambda(w, t)$ becomes time-dependent. The interface itself is the image of the unit circle $|w| = 1$: as $w = e^{i\phi}$, $0 \le \phi \le 2\pi$, sweeps over the unit circle, $z = \lambda(e^{i\phi}, t)$ sweeps over the interface at the moment t.

Having defined the conformal map, we immediately see that the real part of the logarithm of the inverse conformal map, $w(z)$, provides the solution to the Laplace equation in the oil domain with the required asymptotics: $\Phi(x, y) = \frac{q}{2\pi}\mathrm{Re}\,\log w(z)$. Let us introduce the complex velocity $V = V_x - iV_y$. The obvious formula

$$\frac{q}{2\pi}\frac{\partial \log w(z, t)}{\partial z} = \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y}$$

allows one to represent the Darcy law (1) as follows:

$$V(z) = \frac{q}{2\pi}\frac{\partial \log w}{\partial z}, \tag{5}$$

where the derivative is taken at constant t.

In terms of the time-dependent conformal map, the Darcy law is equivalent to the following relation referred to as the Laplacian growth equation (LGE):

$$\mathrm{Im}\left(\frac{\partial z}{\partial\phi}\frac{\partial\bar{z}}{\partial t}\right) = \frac{q}{2\pi}. \tag{6}$$

It first appeared in 1945 [2] in the works on the mathematical theory of oil production. From now on we set $q = \pi$ without loss of generality. (This amounts to a proper rescaling of t.) Introducing the Poisson bracket notation

$$\{f, g\} = w\frac{\partial f}{\partial w}\frac{\partial g}{\partial t} - w\frac{\partial g}{\partial w}\frac{\partial f}{\partial t}, \tag{7}$$

for functions $f = f(w, t)$, $g = g(w, t)$ of w, t, we rewrite the LGE in the suggestive form[2]

$$\left\{z(w, t), \bar{z}\left(w^{-1}, t\right)\right\} = 1. \tag{8}$$

The LGE thus means that the transformation from $\log w$, t to z, $\bar{z}$ is canonical.

For technical simplicity, we assume that the point $z = 0$ lies in the water domain. The interface dynamics given by the Darcy law (or, equivalently, by the LGE) implies that if a point (x, y) is in the water (interior) domain at the initial moment, then it remains there for all values of time. In particular, our assumption that the point $z = 0$ belongs to the water domain means that zeros of the function $\lambda(w)$ are inside the unit circle for any t.

The Laplacian growth is a particular case of the 2D inverse potential problem. The shape of the interface can be characterized by the harmonic moments C_k of the oil domain and the area C_0 of the water domain:

$$C_k = -\int\!\!\int_{\text{exterior}} z^{-k} dxdy, \qquad k \geq 1, \qquad C_0 = \int\!\!\int_{\text{interior}} dxdy. \tag{9}$$

(The integrals at $k = 1, 2$ are assumed to be properly regularized.) A remarkable result of Richardson [3] shows that the LGE (8) implies conservation of the harmonic moments C_k when the interface moves, $dC_k/dt = 0$, while area of the water domain grows linearly in time: $C_0 = \pi t$. Therefore, the problem can be posed as follows: to find the shape of the domain as a function of its area provided all the harmonic moments of the exterior are kept fixed. Since the harmonic moments are coefficients in the expansion of the Coulomb potential created by a homogeneously distributed charge in the oil domain, this is just a specification of the inverse potential problem. We remind that to know the shape of the domain is the same as to know the coefficients r, u_j of the conformal map (4).

A good deal of hints that the LGE has much to do with integrable systems have been known for quite a long time [3]–[6]. However, its status in the realm of integrability was obscure until very recently. The nature of the LGE and its relation to integrability are clarified in the work [7]. The idea is to treat the LGE *not as a dynamical equation* but *as a constraint* in a bigger integrable hierarchy. The latter turns out to be an infinite Whitham hierarchy of the type first introduced in [8]. This hierarchy is a multi-dimensional generalization of integrable hierarchies of hydrodinamic type [9]. It naturally incorporates the general inverse potential problem as well. Namely, the coefficients of the conformal map as functions of all the harmonic moments are given by a particular solution to the dispersionless Toda lattice hierarchy (see [10] for a detailed study of the latter). The hierarchical evolution times are just the harmonic moments and their complex conjugates. In the Laplacian growth all of them but the area C_0 are frozen. Making them alive, one moves over the space of initial data for the LGE, and recovers the Whitham hierarchy.

[2]Given a Laurent series $f(z) = \sum_j f_j z^j$, we set $\bar{f}(z) = \sum_j \bar{f}_j z^j$, so $z(w)$ and $\bar{z}\left(w^{-1}\right)$ are complex conjugate only if $|w| = 1$.

For a more convenient formulation of the result, let us rescale the harmonic moments and introduce the new notation for them:

$$t \equiv t_0 = \frac{C_0}{\pi}, \qquad t_k = \frac{C_k}{\pi k}, \qquad \bar{t}_k = \frac{\bar{C}_k}{\pi k}, \qquad k \geq 1. \tag{10}$$

The symbols $(f(w))_\pm$ below mean a truncated Laurent series, where only terms with positive (negative) powers of w are kept, $(f(w))_0$ is a constant part $\left(w^0\right)$ of the series.

Theorem 1. *The conformal map (4) obeys the following differential equations with respect to the harmonic moments:*

$$\frac{\partial \lambda(w)}{\partial t_j} = \{H_j, \lambda(w)\}, \qquad \frac{\partial \lambda(w)}{\partial \bar{t}_j} = -\{\bar{H}_j, \lambda(w)\}, \tag{11}$$

$$\frac{\partial \bar{\lambda}\left(w^{-1}\right)}{\partial t_j} = \left\{H_j, \bar{\lambda}\left(w^{-1}\right)\right\}, \qquad \frac{\partial \bar{\lambda}\left(w^{-1}\right)}{\partial \bar{t}_j} = -\left\{\bar{H}_j, \bar{\lambda}\left(w^{-1}\right)\right\}, \tag{12}$$

where the Poisson bracket is defined in (7), and

$$H_j(w) = \left(\lambda^j(w)\right)_+ + \frac{1}{2}\left(\lambda^j(w)\right)_0, \tag{13}$$

$$\bar{H}_j(w) = \left(\bar{\lambda}^j\left(w^{-1}\right)\right)_- + \frac{1}{2}\left(\bar{\lambda}^j\left(w^{-1}\right)\right)_0. \tag{14}$$

The proof is sketched in [7] and [11]. These are the Lax–Sato equations for the dispersionless Toda lattice hierarchy of non-linear differential equations, with the $\lambda(w)$ and $\bar{\lambda}\left(w^{-1}\right)$ being the Lax functions. On comparing coefficients in front of powers of w in (11), (12), one obtains an infinite set of non-linear differential equations for the coefficients of the comformal map. Altogether, they form the hierarchy. The particular solution that solves the inverse potential problem is selected by the constraint (8), where $z(w,t) = \lambda(w,t)$, $\bar{z}(w,t) = \bar{\lambda}\left(w^{-1}, t\right)$, and all t_k are fixed. This constraint is known as (a quasiclassical version of) the *string equation*. Surprisingly, this very constraint is the key ingredient of the integrable structures in 2D gravity coupled with $c = 1$ matter [12, 13]. The mathematical theory of the dispersionless hierarchies constrained by string equations was developed in [14, 15] and extended to the Toda case in [10].

The Lax–Sato equations imply [14] the existence of the prepotential function $F(t, t_k, \bar{t}_k)$. This function solves the inverse potential problem in the following sense. Let C_{-k}, $k \geq 1$, be the complementary set of harmonic moments, i.e., the moments of the interior of the domain:

$$C_{-k} = \iint_{\text{interior}} z^k dxdy. \tag{15}$$

Were we able to find C_{-k} from a given set C_0, C_k (i.e., given t, t_k), this would yield the complete solution, alternative but equivalent to knowing the conformal map (4). Indeed, using the contour integral representation of the harmonic moments, it is easy to see that the generating function of *all* the harmonic moments, obtained as an analytic continuation of the Laurent series

$$\mu(\lambda) = \frac{1}{\pi}\sum_{k\in\mathbf{Z}} C_k \lambda^k = \sum_{k=1}^{\infty} k t_k \lambda^k + t + \frac{1}{\pi}\sum_{k=1}^{\infty} C_{-k}\lambda^{-k}, \tag{16}$$

would allow us to restore the interface curve via the equation $|z|^2 = \mu(z)$ ($S(z) = \mu(z)/z$ is what is called *Schwarz function* of the curve [16]). The function F does the job.

Theorem 2 ([7, 11]). *There exists a real function F of the (rescaled) harmonic moments t, t_k, $\bar{t}_k$ such that*

$$C_{-k} = \pi \frac{\partial F}{\partial t_k}, \qquad \bar{C}_{-k} = \pi \frac{\partial F}{\partial \bar{t}_k}, \qquad k \geq 1. \tag{17}$$

In particular, this implies the symmetry of derivatives of the inner harmonic moments with respect to the outer ones:

$$\frac{\partial C_{-k}}{\partial t_j} = \frac{\partial C_{-j}}{\partial t_k}, \qquad \frac{\partial C_{-k}}{\partial \bar{t}_j} = \frac{\partial \bar{C}_{-j}}{\partial t_k}. \tag{18}$$

For general Whitham hierarchies, the function F was introduced in [14]. In the dispersionless Toda case, it is the dispersionless limit of logarithm of the τ-function [10]. This function obeys the dispersionless limit of the Hirota equation (a leading term of the differential Fay identity [17, 10]):

$$\begin{aligned} &(z-\zeta)\exp\left(\sum_{n,m\geq 1} \frac{F_{nm}}{nm} z^{-n}\zeta^{-m}\right) \\ &\quad = z\exp\left(-\sum_{k\geq 1}\frac{F_{0k}}{k}z^{-k}\right) - \zeta\exp\left(-\sum_{k\geq 1}\frac{F_{0k}}{k}\zeta^{-k}\right). \end{aligned} \tag{19}$$

This is an infinite set of relations between the second derivatives $F_{nm} = \partial_{t_n}\partial_{t_m}F$, $F_{0m} = \partial_t \partial_{t_m} F$. The equations appear while expanding both sides of (19) in powers of z and ζ.

To complete the identification with the objects of the theory of Whitham hierarchies, we just mention that the function $\mu(\lambda)$ (16) is the quasiclassical limit of the Orlov–Shulman operator [18] for the 2D Toda lattice. This function obeys the conditions $\{\lambda, \mu\} = \lambda$, $\{\bar{\lambda}, \bar{\mu}\} = -\bar{\lambda}$. The string equation can be then written as a relation

$$\bar{\lambda} = \lambda^{-1}\mu \tag{20}$$

between the Laurent series, together with the reality condition $\mu(\lambda) = \bar{\mu}(\bar{\lambda})$. Written in this form, it admits generalizations which are also relevant to the Laplacian growth problem, and which we are going to discuss now.

Consider the Laplacian growth problem for domains symmetric under the $\mathbf{Z}_n$-transformations $z \to e^{2\pi i l/n} z$, $l = 1, \ldots, n-1$, where n is a positive integer. In this case $C_m = 0$ unless $m = 0 \,(\mathrm{mod}\, n)$, i.e., only the moments C_{kn} are non-zero. The conformal map from the exterior of the unit circle to the exterior of such a domain is given by $z(w) = (\lambda(w^n))^{1/n}$, where $\lambda(w)$ has the same general structure as in (4). Equivalently, the problem may be posed as the Laplacian growth in the wedge (sector) domain restricted by the rays $\arg z = 0$ and $\arg z = 2\pi/n$ with the periodic condition on the boundary. In the latter case, the conformal map is

$$z(w) = (\lambda(w))^{1/n}. \tag{21}$$

The LGE in terms of the conformal map $z(w,t)$ has the same form (8) if the time is rescaled as $t \to t/n$. In terms of the $\lambda(w,t)$, one has

$$\{\lambda, \bar{\lambda}\} = n^2(\lambda\bar{\lambda})^{\frac{n-1}{n}}. \tag{22}$$

Let us introduce the following notation for the non-zero harmonic moments:

$$t = \frac{C_0}{n\pi}, \qquad t_k = \frac{C_{kn}}{\pi k}, \qquad \bar{t}_k = \frac{\bar{C}_{kn}}{\pi k}, \qquad k \geq 1 \tag{23}$$

(cf. (10)). The more general string equation (22) amounts to the relation

$$\bar{\lambda} = n^n \lambda^{-1} \mu^n \tag{24}$$

between the Lax and Orlov–Shulman functions (note that this means $|z|^2 = n\mu(z)$). This relation is familiar from [10, 12]. It is consistent with the hierarchy and selects a solution that, as $n > 1$, is different from the one selected by (20). The Lax–Sato equations (11)–(14) for the Lax function λ and the Hirota equation (19) hold true as they stand provided the times are redefined as in (23). The solution to the Laplacian growth problem in the channel geometry (in the Hele–Shaw cell) can be obtained from the above formulas as a somewhat tricky limit $n \to \infty$. It is unclear, however, if the existing finite-dimensional solutions of the LGE [4–6] will survive after these transformations. We should also mention that inclusion of the "no-flux" boundary conditions at the walls of the wedge, which state that $\partial \log z / \partial \log w$ is real, imposes an extra symmetry in the solution and doubles the number of wedges. We will elucidate these questions in the near future.

At last we point out that the above formulas have sense for negative integer n, too. In particular, the case $n = -1$ describes the *internal* radial problem when oil is inside while water is outside. In this case the internal and external harmonic moments are interchanged in their role: the internal moments (15) together with the C_0 become independent variables (cf. (23)) while the external moments are found as derivatives of the prepotential function according to (17).

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