

# The Operations Invariant Properties on Graphs

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**Abstract.** To determine whether or not a given graph has a Hamilton cycle (or is a planar graph), defined the operations invariant properties on graphs, and discussed the various forms of the invariant properties under the circumstance of Cartesian product graph operation and Tensor product graph operation. The main conclusions include: The Hamiltonicity of graph is invariant concerning the Cartesian product, and the non-planarity of the graph is invariant concerning the tensor product. Therefore, when we applied these principles into practice, we testified that Hamilton cycle does exist in hypercube and the Desargues graph is a non-planarity graph.

## INTRODUCTION

The Planarity, Eulerian feature, Hamiltonicity, bipartite property, spectrum, and so on, is often overlooked for a given graph. The traditional research method adopts the idea of direct proof. In recent years, indirect proof method is put forward, (Fang Xie & Yanzhong Hu, 2010) proves non-planarity of the Petersen graph, by use the non-planarity of  $K_{3,3}$ , i.e., it induces the properties of the graph  $G$  with the properties of the graph  $H$ , here  $H$  is a sub-graph of  $G$ . However, when it comes to induce the other properties, the result is contrary to what we expect. For example, Fig. 1(a) is a Hamiltonian graph which is a sub-graph of Fig. 1(b). If Hamiltonicity can be induced genetically, then invariant (b) should be also Hamiltonian, but obviously, invariant (b) is not a Hamiltonian graph.

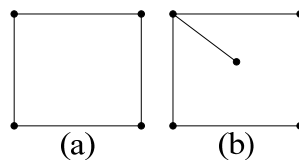


Figure 1. Only the sub-graph is a Hamilton graph

(Yanzhong Hu & Bo Ye, 2010) also uses the idea of indirect proof, but it adopts a different way. It induces properties of a graph by using sub-graph operations.

It aims at going into the idea of indirect proof further to find out which properties can be genetically preserved after special operations, such as the Cartesian product of graphs and the Tensor product of graphs.

## THE INVARIANT PROPERTIES OF OPERATIONS ON GRAPHS

**Definition 1** Let  $P$  be a property of graph (such as planarity), and  $\odot$  be a kind of operation between graphs. For the two graphs  $G_1=(V_1, E_1)$  and  $G_2=(V_2, E_2)$ , if both graph  $G_1$  and graph  $G_1 \odot G_2$  have the property  $P$ . We say that the property  $P$  is invariant concerning the operation  $\odot$ .

**Definition 2** The Cartesian product of graphs  $G_1=(V_1, E_1)$  and  $G_2=(V_2, E_2)$ , named as  $G=G_1 \otimes G_2$ , is a graph such that  $G=(V, E)$ , in which the vertexes and edges is follows:

- 1)  $V=\{(x, y) \mid x \in V_1 \text{ and } y \in V_2\} = V_1 \times V_2$ ,
- 2) any two vertexes  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent in  $G_1 \otimes G_2$  if and only if either
  - a)  $u_1=u_2$  and  $v_1 v_2 \in E_2$  or
  - b)  $v_1=v_2$  and  $u_1 u_2 \in E_1$ .

The Cartesian product of  $C_3$  and  $K_2$  is shown in Fig. 2(c).

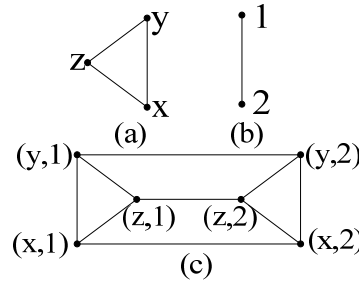


Figure 2. The Cartesian product of  $C_3$  and  $K_2$

**Definition 3** The tensor product  $G_1 \otimes G_2$  of two simple graphs  $G_1$  and  $G_2$  is the graph with the vertex set of  $G_1 \otimes G_2$  is the Cartesian product  $V(G_1) \otimes V(G_2)$ , and wherein  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent in  $G_1 \otimes G_2$  if, and only if,  $u_1$  is adjacent to  $u_2$  in  $G_1$  and  $v_1$  is adjacent to  $v_2$  in  $G_2$ .

**Definition 4** The bipartite double cover of an undirected graph  $G$  is a bipartite covering graph of  $G$ , with twice as many vertexes as  $G$ . It can be defined as the tensor product of graphs  $G \otimes K_2$  also.

#### THE INVARIANT PROPERTIES OF THE CARTESIAN PRODUCT GRAPH OPERATION

**Theorem 5** The non-planarity of graph is invariant concerning the Cartesian product.

**Proof** Certificate process as follows:

(1)  $K_{3,3} \otimes K_2$  is non-planar.

Since  $K_{3,3}$  is a sub-graph of the Cartesian product  $K_{3,3} \otimes K_2$ , hence  $K_{3,3} \otimes K_2$  is non-planar, by Kuratowski's theorem.

(2)  $K_5 \otimes K_2$  is non-planar.

Since  $K_5$  is a sub-graph of the Cartesian product  $K_5 \otimes K_2$ , hence  $K_5 \otimes K_2$  is non-planar, by Kuratowski's theorem.

(3) Let  $G$  be a non-planar graph, then  $G \otimes K_2$  is non-planar.

Because of  $G$  is a non-planar graph, by Kuratowski's theorem, graph  $G$  contains  $K_{3,3}$ , or  $K_5$ , or contains  $K_{3,3}$ , or  $K_5$ , a subdivision, i.e.  $K_{3,3} \otimes K_2 \subseteq G \otimes K_2$ , or  $K_5 \otimes K_2 \subseteq G \otimes K_2$ , hence  $G \otimes K_2$  is non-planar.

(4) Let  $G$  be a non-planar graph, and  $H$  be a graph which has a edge at least, then  $G \otimes H$  is non-planar.

Since  $G \otimes K_2 \subseteq G \otimes H$ , and  $G \otimes K_2$  is non-planar, i.e.  $G \otimes H$  is non-planar.

**Proposition 6** The planarity of graph is not invariant concerning the Cartesian product.

**Theorem 7** The Hamiltonicity of the graph  $G$  transmit on the Cartesian product  $G \otimes K_2$ , i.e. if  $G$  there exists a Hamilton cycle, then the Cartesian product  $G \otimes K_2$  has a Hamilton cycle also.

**Proof** For convenience, let  $G$  be a graph of order  $m$ , its Hamilton cycle is the  $C_n$ , as shown in Fig. 3. Obviously, the path:  $123... nn'.... 3'2'1'1$  is a Hamilton cycle of the  $C_n \otimes K_2$ , and is a Hamilton cycle of the  $G \otimes K_2$ , of course.

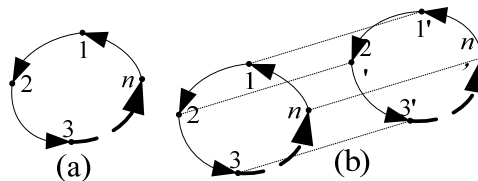


Figure 3. The Cartesian product  $C_n \otimes K_2$

**Proposition 8** The non- hamiltonicity of graph is not invariant concerning the Cartesian product.

**Proposition 9** The Euler feature of a graph is not invariant concerning the Cartesian product.

## THE HAMILTON CYCLE ABOUT THE $Q_n$

The definition of the hypercube  $Q_n$  is as followed:  $Q_1=K_2$ ,  $Q_n=Q_{n-1}\otimes K_2$  ( $n\geq 1$ ), obviously,  $Q_1=K_2$  is not a Hamiltonian graph, then for  $n\geq 2$ , is there any Hamilton cycle in  $Q_n$ ?  $Q_2$  is the  $C_4$ , itself is the Hamilton cycle, and  $Q_3$  is the cubic which have Hamilton cycle also. Whether contains a Hamilton cycle in  $Q_4$  or  $Q_5$  or not? It is not so easy to see out.

Here we show that: there exists a Hamilton cycle in  $Q_n$  ( $n\geq 2$ ).

**Proof** Because of  $Q_2$  have a Hamilton cycle, and  $Q_3=Q_2\otimes K_2$ , by theorem 7,  $Q_3$  have a Hamilton cycle also, shown as Fig. 4, and  $Q_4=Q_3\otimes K_2$ , by theorem 7,  $Q_4$  have a Hamilton cycle too, similarly,  $Q_5=Q_4\otimes K_2$  have a Hamilton cycle, and so on,  $Q_n$  ( $n\geq 2$ ) have a Hamilton cycle. Fig. 5 and Fig. 6 is the  $Q_4$  and  $Q_5$  respectively.

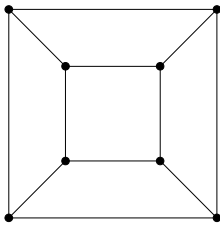


Figure 4. The cubic  $Q_3$

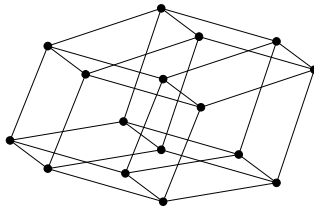


Figure 5. The cubic  $Q_4$

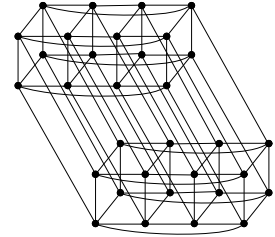


Figure 6. The cubic  $Q_5$

## THE INVARIANT PROPERTIES OF GRAPHS CONCERNING THE BIPARTITE DOUBLE COVER

**Theorem 10** Under the bipartite double cover graph operation, the bipartite graph is still a bipartite graph.

**Proof** For a graph  $G$ , let  $A=\{(x, 1)|x\in G\}$ ,  $B=\{(x, 2)|x\in G\}$ ,  $V(K_2)=\{1, 2\}$ , then  $A$  is independent, because no two edges in  $A$  are adjacent, by definition 4, so is  $B$ . And  $A\cap B=\emptyset$ ,  $V(G\odot K_2)=A\cup B$ , if  $x$  is adjacent to  $y$  in  $G$ , then  $(x, 1)$  is adjacent to  $(y, 2)$  in  $G\odot K_2$ , hence  $G\odot K_2$  is a bipartite graph with partite sets  $A$  and  $B$ .

**Proposition 11** The Hamiltonicity of a graph is not invariant concerning the bipartite double cover.

**Proposition 12** The regularity of a graph is invariant concerning the bipartite double cover.

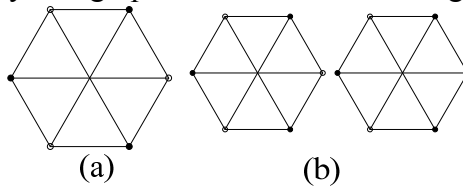


Figure 7. The tensor of  $K_{3,3}$  and  $K_2$

**Theorem 13** The non-planarity of the graph  $G$  is invariant concerning the bipartite double cover.

**Proof** (1)  $K_{3,3}\odot K_2$  is non-planar.

Shown as Fig. 7, the Fig. 7(b) is the bipartite double cover of  $K_{3,3}$ , and obviously, it is non-planar.

(2)  $K_5\odot K_2$  is non-planar.

Shown as Fig. 8. The graph Fig. 8(b) is the bipartite double cover of  $K_5$ , Let the graph shown as Figure 11.(b) be a planar graph, then  $V-E+F=2$ , according to the Euler Identity, and then  $2E=\sum l(F_i)\geq 4F$ . There exists  $4V=2E$ . Therefore, we obtain  $0=2V-E=2V-2E+E\geq 2V-E=2V-2E+2F=4$ , which is contradictory, so the graph shown as Fig. 8(b). is non-planar, and hence the tensor  $K_5\odot K_2$  is non-planar.

(3) Let  $G$  be a non-planar graph, then  $G\odot K_2$  is non-planar.

If  $G$  is a non-planar graph, then graph  $G$  contains  $K_{3,3}$ , or  $K_5$ , or contains  $K_{3,3}$ , or  $K_5$ , a subdivision, by Kuratowski's theorem, hence  $K_{3,3}\odot K_2\subseteq G\odot K_2$ . Because  $K_{3,3}\odot K_2$  is a non-planar graph, therefore graph  $G\odot K_2$  is non-planar. (or  $K_5\odot K_2\subseteq G\odot K_2$ , Because  $K_5\odot K_2$  is a non-planar graph, and hence  $G\odot K_2$  is non-planar.)

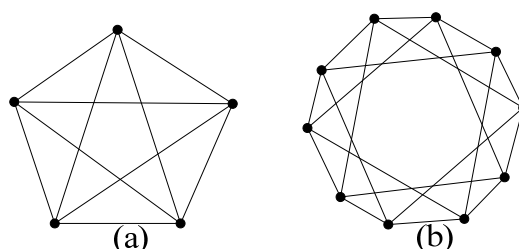


Figure 8. The tensor  $K_5 \otimes K_2$

**Corollary 14** The non-planarity of the graph  $G$  is invariant concerning the tensor product.

**Proof** Let  $G$  be a non-planar graph, and  $H$  be a non-empty finite simple graph, then  $G \otimes H \cong G$ , or  $G \otimes K_2 \subseteq G \otimes H$ , hence  $G \otimes H$  is non-planar.

**Proposition 15** The planarity of the graph  $G$  is not invariant concerning the bipartite double cover.

### THE NON-PLANARITY OF THE DESARGUES GRAPH

The Desargues graph is a distance-transitive cubic graph with 20 vertexes and 30 edges, as shown in Fig. 9. It is named after Gérard Desargues, arises from several different combinatorial constructions, has a high level of symmetry, is the only known non-planar cubic partial cube, and has been applied in chemical databases.

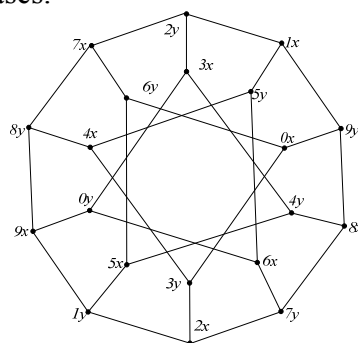
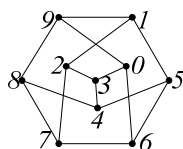


Figure 9. The Desargues graph

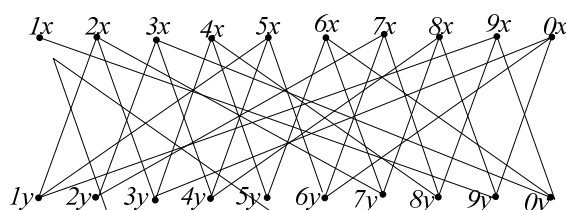
The Desargues graph is a symmetric graph: it has symmetries that take any vertex to any other vertex and any edge to any other edge. Its symmetry group has order 240, and is isomorphic to the product of a symmetric group on 5 points with a group of order 2.

On the other hand, it is the bipartite double cover of the Petersen graph, formed by replacing each Petersen graph vertex by a pair of vertexes and each Petersen graph edge by a pair of crossed edges. The bipartite double cover of the Petersen graph is shown as Figure 4., in which, let  $V(K_2) = \{x, y\}$ , and for convenience,  $(1, x)$  is denoted by  $1x$ ,  $(2, x)$  is denoted by  $2x$ , and so on.

Here we show that: the Desargues graph is non-planar.



(a) The Petersen graph  $P$



(b) The tensor of  $P$  and  $K_2$

Figure 10. The construction of the Desargues graph

**Proof** By theorem 10, the non-planarity of a graph is invariant concerning the bipartite double cover, now the Petersen graph is non-planar, hence the Fig. 10(b) is non-planar. On the other hand, Fig. 10(b) is isomorphic to Fig. 9, hence the Fig. 9 is non-planar., hence the Desargues graph is non-planar.

## CONCLUSION

Using the properties of sub-graph deduce the properties of graph  $G$  is a feasible method. This paper discussed some of the invariant properties of the Cartesian product graph operation and the Tensor product graph operation. The main conclusions include: the non-planarity and Hamiltonicity of graph are hereditary concerning the Cartesian product, the non-planarity of the graph is invariant concerning the tensor product, under the bipartite double cover, the bipartite graph is still a bipartite graph, and the regularity of a graph is invariant concerning the bipartite double cover. The method mentioned above to recognize the properties of the graph has good reference value. The invariant properties under the lexicographic product graph operation will be our focus of future research.

## ACKNOWLEDGMENT

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