dimensional $\mathcal{A}$ − invariant subspace, and $\mathcal{A}|_W$ has matrix representation $\boldsymbol{F}_{p(\lambda)}$. □

Remark W is a minimal nonzero $\mathcal{A}$ − invariant subspace. In fact, $p(\lambda)$ is the characteristic polynomial of $\mathcal{A}|_W$. Assume $U \subset W$ is a smaller dimensional $\mathcal{A}$ − invariant subspace, $q(\lambda)$ is the characteristic polynomial of $\mathcal{A}|_U$, then $q(\lambda) \mid p(\lambda)$. This contradicts the fact that $p(\lambda)$ is irreducible.

Theorem 4 Let $p(\lambda) = \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0 \in \mathbb{F}[\lambda]$ be irreducible, $\mathcal{A} \in \mathrm{End}V$ has matrix representation $\boldsymbol{J}_{p^k(\lambda)}$ or $\boldsymbol{T}_{p^k(\lambda)}$, then $\mathcal{A}$ has exactly $k+1$ invariant subspaces.

Proof: Suppose W is an $\mathcal{A}$ − invariant subspace, then

$f_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{E} - \mathcal{A}) = p^k(\lambda)$, $f_{\mathcal{A}|_W}(\lambda) \mid f_{\mathcal{A}}(\lambda)$, $f_{\mathcal{A}|_W}(\lambda) = \det(\lambda \mathcal{E}|_W - \mathcal{A}|_W) = p^i(\lambda)$, $i = 0, 1, \cdots, k$.

Denote by $W_i = \mathrm{Ker}\, p^i(\mathcal{A})$, then

$$\{\mathbf{0}\} = W_0 < W_1 < \cdots < W_k = V, \quad p^i(\mathcal{A})W = \{\mathbf{0}\} \Rightarrow W \subset W_i, \quad \dim W = im, \quad i = 0, 1, \cdots, k.$$

From the proof of theorem 1 we see that both $\boldsymbol{J}_{p^k(\lambda)}$, $\boldsymbol{T}_{p^k(\lambda)}$ are similar to

$$\boldsymbol{J} = \mathrm{diag}(\boldsymbol{J}_k(\lambda_1),\ \boldsymbol{J}_k(\lambda_2),\ \cdots,\ \boldsymbol{J}_k(\lambda_m))$$

over $\mathbb{C}$, where $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{C}$ are all the roots of $p(\lambda)$. Hence

$$\dim W_i = km - r[p^i(\boldsymbol{J})] = im = \dim W, \quad W = W_i, \quad i = 0, 1, \cdots, k.$$

Therefore $\mathcal{A}$ has $k+1$ invariant subspaces: $W_0 = \{\mathbf{0}\}, W_1, \cdots, W_k = V$. □

By theorem 2 and 4 we have the following result.

Theorem 5 Ler V be a n dimensional vector space over $\mathbb{F}$, the characteristic polynomial of $\mathcal{A} \in \mathrm{End}V$ has the standard factorization $f_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{E} - \mathcal{A}) = p_1^{k_1}(\lambda) p_2^{k_2}(\lambda) \cdots p_s^{k_s}(\lambda)$, then

$$\{\det(\lambda \mathcal{E}|_W - \mathcal{A}|_W):\ W < V,\ \mathcal{A}W \subset W\} = \{p_1^{l_1}(\lambda) p_2^{l_2}(\lambda) \cdots p_s^{l_s}(\lambda):\ 0 \leqslant l_i \leqslant k_i\},$$

$$\{\dim W:\ W < V,\ \mathcal{A}W \subset W\} = \left\{\sum_{i=1}^{s} l_i \deg p_i(\lambda):\ 0 \leqslant l_i \leqslant k_i\right\}. \quad \square$$

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