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Topological Solitons in Discrete Space-Time as the Model of Fermions

A I MUSIENKO

N N Semenov Institute of Chemical Physics, Russian Academy of Sciences
ul. Kosygina 4, Moscow, 117977 Russia
E-mail: andrew@polymer.chph.ras.ru

Abstract

In the present work we discuss arguments in favour of the view that massive fermions represent dislocations (i.e. topological solitons) in discrete space-time, with Burgers vectors parallel to the axis of time. If we assume that the symmetrical parts of tensors of distortions (i.e. derivatives of atomic displacements on coordinates) and the mechanical stresses are equal to zero, then the equations of the field theory of dislocations assume the form of the Maxwell equations. If we consider these tensors as symmetrical, we obtain the equations of the theory of gravitation, and it appears that the sum of the tensor of distortions and of the pseudo-Euclidean metrical tensor is the analogue of the metrical tensor. It is shown that we can also get Dirac equation with four-fermion interaction in the framework of the field theory of dislocations. This model explains quantization of electrical charge: this is proportional to the topological charge of dislocation, and this charge accepts quantized values because of the discrete structure of the 4-dimensional lattice.

The concept of discrete (in other words quantized) space-time has been discussed in physics of elementary particles already for a long time. One of variants of this approach is the lattice quantum field theory, within the framework of which significant success was already achieved. This theory is based on approximation of space-time by a discrete lattice, which is actually 4-dimensional analogue of crystal lattices. But from the theory of solid state it is well known that the ideal crystal lattices do not exist in a nature. The real lattices always contain defects, in particular, dislocations. From the geometrical point of view the dislocations are the one-dimensional topological solitons. The first homotopy group which classifies them is the group of translations. Therefore it is quite natural to assume that similar defects should exist in 4-dimensional space-time lattice. Dislocation in $(2 + 1)$-dimensional discrete space-time with a Burgers vector, parallel to the axis of time, is shown in Fig. 1. The dislocation line in the figure is parallel to a Burgers vector.

As is known, solitons are particle-like objects; various soliton models of elementary particles were offered by many authors. Ross [1] has assumed, that fermions represent topological defects in space-time, namely, dislocations with Burgers vectors, parallel to the axis of time and taking only quantized values. Unzicker [2] offered another topological model of electron: a disclination in space-time. The author of work [2] also used the analogy between elementary particles and dislocations in crystals. Our approach is closer to the hypothesis by Ross [1]. But the consideration by Ross was based solely on the relation between fermionic spin density and space-time torsion. In the present work we bring a number of other arguments, favouring the view that fermions represent dislocations

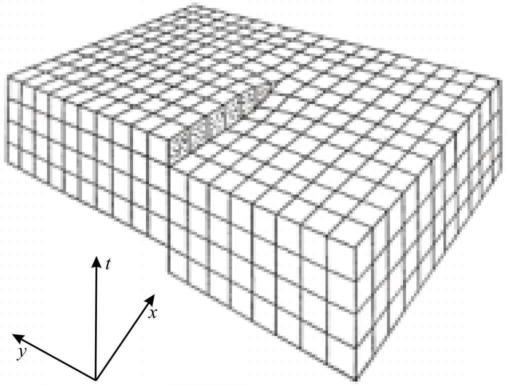

Figure 1. Dislocation in (2+1)-dimensional discrete space-time with a Burgers vector, parallel to an axis of time. The dislocation line on the figure is parallel to a Burgers vector.

in discrete space-time with Burgers vectors, parallel to an axis of time. We assume that the discrete space-time represents a 4-dimensional lattice. There are some objects (we shall call them "primary atoms"), which can be displaced from equilibrium positions, in lattice sites. We suppose that the movement of these "primary atoms" obeys only the laws of classical Newton mechanics. This naive assumption is not absolutely justified. It is conceivable that the movement of these "primary atoms" can be described by the much more complex laws. But it turns out that this "naive" assumption allows us to obtain, in a rather simple way, several of the important formulae of electrodynamics, gravitation theory and quantum field theory. Probably, accounting for deviations from the laws of "primary atoms" behaviour from the laws of the classical mechanics will allow to describe other phenomena (for example, strong or weak interactions).

We recall that torsion is, by definition, an antisymmetric part of connection multiplied by two:

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}.$$

The integral of this function over any 2-dimensional surface, which intersects a dislocation, is the value referred to as Burgers vector in solid state physics:

$$b^{\lambda} = \int T^{\lambda}{}_{\mu\nu}\, df^{\mu\nu},$$

where $df^{\mu\nu} \equiv dx^{\mu}dx^{\nu} - dx^{\nu}dx^{\mu}$ is the element of area. From the geometrical point of view the space-time, containing dislocations with Burgers vectors b^i is the manifold with torsion $T^{i}{}_{ah}(x_{\zeta}) = \frac{1}{2}e_{achd}\,\tau^c b^i V^d \delta\left(x_{\zeta} - x^0_{\zeta}\right)$, where e_{cahd} is the 4-dimensional Levi–Civita symbol, τ^c is the 4-dimensional unit vector tangential to the dislocation line, V^d is the 4-dimensional velocity of dislocations, $\delta(x)$ is the Dirac delta function, x^0_{ζ} are the coordinates of the dislocation line. On the other hand in the theory of gravitation it was shown [3], that the particles with semi-integer spin are also sources of torsion. The relationship between a spin density tensor $S^{\lambda}{}_{\mu\nu}$ and a torsion is described by the formula [3]

$$T^{\lambda}{}_{\mu\nu} = \frac{16\pi G}{c^3}\left(S^{\lambda}{}_{\mu\nu} + \frac{1}{2}\delta^{\lambda}_{\mu}S_{\nu} - \frac{1}{2}\delta^{\lambda}_{\nu}S_{\mu}\right). \tag{1}$$

Here G is Newton gravitational constant, c is velocity of light, $S^{\lambda}{}_{\mu\nu} = v^{\lambda} S_{\mu\nu} S_{\mu} = S^{\lambda}{}_{\mu\lambda}$, v^{λ} is the 4-vector of the particle velocity, $S_{\mu\nu}$ is the antisymmetric tensor. Its spatial components form a 3-vector $s_k = \left(S^{23}, S^{31}, S^{12}\right)$ which is equal to 3-dimensional density of the particle spin in reference frame of rest of this particle. In equations (1) and thereafter, all the indices, unless otherwise specified, assume values from 0 to 3. Therefore, non-moving particle having spin s_k, $k = 1, 2, 3$ create the same torsion, as non-moving dislocations with tangential vector $\tau_c = (\tau_0, \tau_k)$ (vectors s_k and τ_k are parallel) and Burgers vector, parallel to an axis of time. This dislocation represents a line in the 4-dimensional space-time which is coincident with a world line of fermion. At any moment of time this line intersects the 3-dimensional physical space only in one point, therefore fermions are observed as particle objects. Continuity of a dislocation line is provided by the topological laws [4]. This fact also was proved with use of the Noether theorem [5]. In continuous space-time torsion, created by fixed particles, can accept continuous set of values and in quantized space-time it accepts discrete one. It is well known that spins of elementary particles are quantized. This fact results, in accordance with the formula (1), in discreteness of torsion, created by fixed particles. It confirms hypothesis by Ross [1] of the discrete structure of space-time and the dislocation nature of fermions.

At present,the gauge theory of dislocations [4–6] is very well developed. Many authors paid attention to the analogy between this theory and both electrodynamics [4] and the theory of gravitation [7, 8]. There are the equations of incompatibility in continuous theory of dislocations

$$\partial_{\gamma}\partial_{\varepsilon} u_n(x_{\zeta}) - \partial_{\varepsilon}\partial_{\gamma} u_n(x_{\zeta}) = \frac{2}{c} T_{\gamma n \varepsilon}(x_{\zeta}), \tag{2}$$

where u_n is the vector of displacement of the 4-dimensional continuum particles (in continuous theory the 4-dimensional lattice is approximated by continuous medium), $\partial_{\gamma} = \frac{\partial}{\partial x^{\gamma}}$. Equations (2) are a definition of dislocations and a statement that there are no disclinations in the given lattice (since the term descriptive of the contribution from disclinations is absent in the right-hand side of these equations). In other words, (2) are purely geometrical equations.

Tensor $\beta_{\varepsilon n} = \partial_{\varepsilon} u_n$ refer to as tensor of distortions. Let us introduce tensor $F_{\varepsilon n}$ which equals an antisymmetric part of a tensor of distortions multiplied by two:

$$F_{\varepsilon n} = \beta_{\varepsilon n} - \beta_{n \varepsilon}. \tag{3}$$

By summing each three equations from equations (2), we obtain the following system of the equations

$$\partial_{\mu} F_{\alpha\beta} + \partial_{\alpha} F_{\beta\mu} + \partial_{\beta} F_{\mu\alpha} = \frac{2}{c}(T_{\mu\beta\alpha} + T_{\beta\alpha\mu} + T_{\alpha\mu\beta}). \tag{4}$$

Let the vector tangential to a dislocation line be equal to $\tau_{\beta} = (\tau_0, \tau_1, 0, 0)$. We suppose that the quantities b_0 and τ_1 are very small, so that modern experimental techniques do not allow nonzero members to be detected in the right side of (4). Then the equations (4) get the form of the first pair of the Maxwell equations

$$e^{\alpha\beta\gamma\delta} \partial_{\beta} F_{\gamma\delta} = 0. \tag{5}$$

Newton second law for particles of the continuum containing dislocations has the form [5]

$$\partial^{\gamma} \sigma_{g\gamma} = \frac{2}{c} C_{g\varepsilon i\lambda} T^{\varepsilon i \lambda}. \tag{6}$$

The 4-dimensional tensor of mechanical stresses $\sigma_{g\varepsilon}$ by definition is equal to

$$\sigma_{g\varepsilon} = C_{g\varepsilon i\lambda}\partial^{\lambda}u^{i}. \tag{7}$$

Here $C_{g\varepsilon i\lambda}$ is the 4-tensor of elastic modules of 4-dimensional continuum. In other words, Lagrangian of "elastic" waves in 4-dimensional medium under consideration has the form

$$\mathcal{L}_0 = -\frac{1}{2}C^{i\gamma j\varepsilon}\beta_{\gamma i}\beta_{\varepsilon j}. \tag{8}$$

By virtue of the fact that the tensor $C_{g\varepsilon i\lambda}$ is sufficiently large the distinction from zero of the right side of equations (6) can be detected in experiments. If it is possible to neglect a symmetric part of the tensor of mechanical stresses, the equations (6) get the form of the second pair of Maxwell equations. If to consider tensors of distortions and mechanical stresses as symmetric, we shall receive the equations of the gauge translation theory of gravitation (that is theory of gravitation in space-time with zero curvature and nonvanishing torsion) [7, 8]. It is appear that the sum of tensor of distortions and the Minkowskian (pseudo-Euclidean) metrical tensor is the analogue of metrical tensor in gravitation theory. In a general case of dislocations in the 4-dimensional lattice these tensors are neither symmetrical nor antisymmetrical and then we receive as a consequence of our model variant of the uniform theory of electrodynamics and gravitation, offered by Einstein [9]: tensor of electromagnetic field is an antisymmetric part of metrical tensor. It is simultaneously possible to solve a problem, which considered by Einstein as the main lack of the theory with the asymmetrical metric: to give geometrical definition of particles.

In the theory of dislocations, the Frenkel–Kontorova model of a dislocation, based on the account of only one nonlinear member in Lagrangian of an elastic field, is well-accepted [10]. In the one-dimensional case this results in the dislocation equation of motion having the form of sine-Gordon equation. The Lagrangian of an elastic field in this case has the form

$$\mathcal{L}_{SG} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + \frac{m^2}{\beta^2}[\cos(\beta\phi) - 1]. \tag{9}$$

Coleman [11] has rigorously proved equivalence of the sine-Gordon soliton and the fundamental fermion of the massive Thirring model in $(1+1)$ dimensions. The Lagrangian of the massive Thirring model has the form

$$\mathcal{L}_T = i\bar{\psi}\gamma_{\mu}\partial^{\mu}\psi - m_f\bar{\psi}\psi - \frac{1}{2}g\left(\bar{\psi}\gamma^{\mu}\psi\right)\left(\bar{\psi}\gamma_{\mu}\psi\right), \tag{10}$$

where ψ is Fermi field, γ^{μ} are Dirac matrices in $(1+1)$ dimensions. Lagrangians (9) and (10) are equivalent on condition that

$$\frac{4\pi}{\beta^2} - 1 = \frac{g}{\pi}. \tag{11}$$

The correspondence between these two models is established by bosonization relations:

$$\begin{aligned} &\frac{m^2}{\beta^2}\cos(\beta\phi) = -m_f\bar{\psi}\psi, \\ &-\frac{\beta}{2\pi}e^{\mu\nu}\partial_{\nu}\phi = \bar{\psi}\gamma^{\mu}\psi \equiv j^{\mu}. \end{aligned} \tag{12}$$

The issues of bosonization are considered in more detail in the book by Rajaraman [12]. In particular, it was shown there that the topological charge of the sine-Gordon soliton is equivalent to the fermionic charge of the particle of the massive Thirring model. Thus, the discreteness of the fermionic charge in our approach is a consequence of discreteness of the topological charge of solitons (i.e. dislocations), which in turn directly follows from discrete structure of space-time.

Many authors consider the incompatibility of a space-time lattice with a condition of Lorentz invariance as traditional lack of models of discrete space-time. In the model proposed, this contradiction is eliminated. The Lorentz transformations arise in this model by a natural way as a consequence of finiteness of velocity of light. But only the values relating to properties of particles — fields, forces of interaction, and so on — depend on the velocity according to Lorentz law. This dependence is a consequence of the occurrence of Lorentz roots in expression for classical Green function of the equations (6). Therefore quantities, expressed through Green function: fields, created by particles, force of interaction between particles, and so on, depend only on the velocity through Lorentz law. All other quantities, including the parameters of the lattice, are not subjected to any transformations.

In this connection we shall notice that relativistic expressions always occur in soliton theories, in particular, in the theory of dislocations in crystals. In this theory instead of velocity of light velocities of sound appear in the formulae. As in solids even in isotropic case not only transversal, but also longitudinal sonic waves can exist, in the theory of dislocations in certain cases expressions, containing Lorentz roots of different kinds: $\sqrt{1 - v^2/c_\lambda^2}$, $\lambda = 1, 2$ can occur. For example, in case of straight dislocation in isotropic medium parallel to axis z with Burgers vector $b_i = (b, 0, 0)$, moving at the velocity v parallel to axis x, the displacements of particles of continuum are described by the following formulas [13]:

$$\begin{aligned} u_1(x,y,t) = bc_1^2/\left(\pi v^2\right) \Big[& \operatorname{arctg}\left(y\left(1 - v^2/c_2^2\right)^{1/2}/(x - vt)\right) \\ & + \left(v^2/\left(2c_1^2\right) - 1\right) \operatorname{arctg}\left(y\left(1 - v^2/c_1^2\right)^{1/2}/(x - vt)\right)\Big], \\ u_2(x,y,t) = bc_1^2/\left(2\pi v^2\right) \Big[& \left(v^2/\left(2c_1^2\right) - 1\right)\left(1 - v^2/c_1^2\right)^{-1/2} \\ & \times \ln\left((x - vt)^2/\left(1 - v^2/c_1^2\right) + y^2\right) \\ & + \left(1 - v^2/c_2^2\right)^{1/2} \ln\left((x - vt)^2/\left(1 - v^2/c_2^2\right) + y^2\right)\Big]. \end{aligned}$$

Here $c_1 = (\mu/\rho)^{1/2}$ is the speed of transversal sound waves, $c_2 = [(\lambda + 2\mu)/\rho]^{1/2}$ is the speed of longitudinal sound waves, λ and μ are Lame constants. Such relations already are not Lorentz-covariant in traditional sense. Formulae of the dislocation theory can be Lorentz-covariant only in some special cases: for example, in case of straight dislocation with a Burgers vector parallel to a dislocation line in isotropic solid. Therefore under certain conditions Lorentz invariance violation in offered model is possible. May be, it will allow to explain occurrence in the last years of a number of field theories, not satisfying to a condition of Lorentz invariance.

Thus, in the present work, we propose a model describing electromagnetic and gravitational properties of electrons and positrons. The inclusion of strong and weak interactions in the model is left for future investigation. But the model already has allowed to unify advantages of soliton models and theories with discrete (quantized) space-time by removing at the same time their lacks. The model explains the quantization of electrical charge: it is proportional to the topological charge of dislocation, and this charge accepts quantized values because of the discrete structure of the 4-dimensional lattice. At the same time the existence of the lattice allows to avoid occurrence of divergence, in particular resulting in the finite mass of a particle. Within the framework of the given model the phenomenon of annihilation of particle-antiparticle pair is easily explained. The process is similar to the annihilation of dislocation pairs in solids. From topological arguments it follows that at a meeting of two dislocations with Burgers vectors, which are equal in magnitude and opposite in direction, the ideal structure of a lattice is restored. Both dislocations (that is both particles) thus disappear, and their energy is radiated as elastic waves. As follows from what stated above, the waves of antisymmetric distortions are perceived by us as electromagnetic, and the symmetric ones as gravitational. It is important to note that the relativistic and quantum properties of particles occur in this model as consequences of the classical Newton mechanics of a 4-dimensional deformable solid.

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