

An Actuarial Approach to Reload Option Pricing in Fractional Jump-diffusion Environment

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Abstract. Assume that the underlying asset price follows the fractional jump-diffusion process, the financial market model is built by the stochastic analysis theory for fractional Brownian motion. Using physical probabilistic measure of price process and the principle of fair premium, the pricing formula for reload option is obtained.

Introduction

The reload option is a kind of exotic option. Feng, Liu and Hou (2003) [1] assumed that underlying asset price follows jump-diffusion process in which jumps are constant, and obtained the pricing formula of reload option by martingale method. Wang and Du (2007) [2] assumed that underlying asset price satisfies jump-diffusion process in which the jump process is Poisson process and the height of jump satisfies lognormal distribution, and obtained the pricing formula of reload option by martingale method.

Bladt and Rydberg (1998) [3] proposed the actuarial approach of option price. Using physical probability measure of price process and the principle of fair premium, they deal with the problems of options pricing under the unbalance, arbitrage existing and incomplete circumstance, and transform option pricing into a problem of equivalent and fair insurance premium. Jiao, Liang and Jiang (2009) [4] assumed that the underlying asset price obeys jump-diffusion process, and established the improved reload option pricing model, and obtained the pricing formula of the improved reload option by actuarial approach. Fang, He and Wang (2011) [5] assumed that underlying asset price follows the fractional jump-diffusion process, and obtained the European compound option pricing formula by actuary method.

In this paper, we assume that the underlying asset price follows the fractional jump-diffusion process, and build the financial market mathematical model. Using the stochastic analysis theory for fractional Brownian motion and method for actuarial mathematics, we obtain the reload option pricing formula.

Financial Market Model

In the financial market, there are two kinds of assets, one is risk-less asset M_t which satisfies

$$dM_t = rM_t dt \quad (1)$$

Where r is risk-less interest rate; the other is risky asset $S(t)$ which satisfies stochastic differential equation

$$dS_t = S_t[\mu dt + \sigma dB_H(t) + (e^{J(t)} - 1)dQ_t] \quad (2)$$

where the expected rate μ and the volatility rate $\sigma > 0$ are constants, $\{B_H(t), t \geq 0\}$ is the fractional Brownian motion on the complete probability space (Ω, \mathcal{F}, P) , Q_t is the random jump times of the underlying asset price at the time $[0, t]$, which follows the Poisson process with intensity λ . $e^{J(t)} - 1$ is the relative jump height of the underlying asset price, and $J(t) \sim N(-\sigma_J^2/2, \sigma_J^2)$. We assume that $\{B_H(t), t \geq 0\}$, $\{J(t), t \geq 0\}$ and $\{Q_t, t \geq 0\}$ are independent.

Lemma 1[5] The solution of the stochastic differential equation (2) is

$$S_t = S_0 \exp\{\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_H(t) + \sum_{i=1}^Q J(i)\}.$$

Definition 2[6] The expectation return rate $\beta(u)$ of the $\{S_t, t \geq 0\}$ on $[t, T]$ is defined by

$$\exp\left\{\int_t^T \beta(u) du\right\} = \frac{E[S_T]}{S_t}.$$

Lemma 3[5] The expectation return rate $\beta(u)$ of the $\{S_t, t \geq 0\}$ on $[t, T]$ satisfies $\beta(u) = \mu$, $u \in [t, T]$.

Reload Option Pricing Formula

Definition 4[7] Consider that the option is reloaded only once before maturity time T , the value of reload option at the reloaded time $T_1 (0 < T_1 < T)$ is

$$V_{T_1} = \max(S_{T_1} - K, 0),$$

and the value of reload option at maturity time T is

$$V_T = \begin{cases} \frac{K}{S_{T_1}} \max\{S_T - S_{T_1}, 0\}, & S_{T_1} > K, \\ \max\{S_T - K, 0\}, & S_{T_1} < K, \end{cases} \quad \text{or} \quad V_T = \begin{cases} K \max\{\frac{S_T}{S_{T_1}} - 1, 0\}, & S_{T_1} > K, \\ \max\{S_T - K, 0\}, & S_{T_1} < K, \end{cases}$$

where K is strike price, S_{T_1} is stock price at reload time T_1 , S_T is stock price at time T .

Definition 5 The actuarial price of reload option is defined by

$$V_0 = E\left\{\left\{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} - K \exp\{-rT_1\}\right\} I_A\right\} + KE\left\{\left\{S_T \exp\left\{-\int_0^T \beta(u) du\right\}\right\} \middle/ \left\{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} - \exp\{-rT\}\right\} I_{AB}\right\} \\ + E\left\{\left\{S_T \exp\left\{-\int_0^T \beta(u) du\right\} - K \exp\left\{-\int_0^T r(u) du\right\}\right\} I_{CD}\right\},$$

where

$$A = \{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} > K \exp\{-rT_1\}\}, \quad B = \left\{\left\{S_T \exp\left\{-\int_0^T \beta(u) du\right\}\right\} \middle/ \left\{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} - \exp\{-rT\}\right\}\right\}, \\ C = \{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} < K \exp\{-rT_1\}\}, \quad D = \{S_T \exp\left\{-\int_0^T \beta(u) du\right\} > K \exp\{-rT\}\}.$$

Theorem 6 The actuarial price of reload option is given by

$$V_0 = \sum_{m=0}^{\infty} \frac{e^{-\lambda T_1} (\lambda T_1)^m}{m!} \{S_0 \Phi(-d^{(m)} + \sigma_1^{(m)}) - K \exp\{-rT_1\} \Phi(-d^{(m)})\} \\ + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^m [\lambda(T - T_1)]^n}{m!n!} \{K \Phi(\rho_1^{(m,n)} \sigma_2^{(n)} - d^{(m)}, \sigma_2^{(n)} - d^{(m)}, \rho_1^{(m,n)}) - K \exp\{-rT\} \Phi(-d^{(m)}, -d^{(n)}, \rho_1^{(m,n)})\} \\ + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^m [\lambda(T - T_1)]^n}{m!n!} \{S_0 \Phi(d^{(m)} - \rho_2^{(m,n)} \sigma^{(m,n)}, \sigma^{(m,n)} - c^{(m,n)}, -\rho_2^{(m,n)}) - K \exp\{-rT\} \Phi(d^{(m)}, -c^{(m,n)}, -\rho_2^{(m,n)})\},$$

where

$$d^{(m)} = \{\ln(K/S_0) - rT_1 + \frac{1}{2}\sigma^2 T_1^{2H} + m\sigma_J^2/2\} / \sigma_1^{(m)}, \quad a^{(n)} = \{-rT + \frac{1}{2}\sigma^2(T^{2H} - T_1^{2H}) + n\sigma_J^2\} / \sigma_2^{(n)}, \\ c^{(m,n)} = \{\ln(K/S_0) - rT_1 + \frac{1}{2}\sigma^2 T_1^{2H} + (m+n)\sigma_J^2/2\} / \sigma^{(m,n)}, \quad \sigma_1^{(m)} = \sqrt{\sigma^2 T_1^{2H} + m\sigma_J^2}, \\ \sigma_2^{(n)} = \sqrt{\sigma^2(T^{2H} - T_1^{2H}) + n\sigma_J^2}, \quad \sigma^{(m,n)} = \sqrt{\sigma^2 T^{2H} + (m+n)\sigma_J^2}, \\ \rho_1^{(m,n)} = \{\sigma^2(T^{2H} - T_1^{2H} - (T - T_1)^{2H})\} / \{2\sigma_1^{(m)} \sigma_2^{(n)}\}, \quad \rho_2^{(m,n)} = \{\sigma^2(T^{2H} + T_1^{2H} - (T - T_1)^{2H}) + 2m\sigma_J^2\} / \{2\sigma_1^{(m)} \sigma^{(m,n)}\}, \\ \varphi(u) = 1/\sqrt{2\pi} \exp\{-u^2/2\}, \quad \varphi(u, v, \rho) = 1/\{2\pi\sqrt{1-\rho^2}\} \exp\{-\{u^2 - 2\rho uv + v^2\}/\{2(1-\rho^2)\}\}, \\ \Phi(x) = \int_{-\infty}^x \varphi(u) du, \quad \Phi(x, y, \rho) = \int_{-\infty}^x \int_{-\infty}^y \varphi(u, v, \rho) dudv.$$

Proof: Let

$$V_1 = E\left\{\left\{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} - K \exp\{-rT_1\}\right\} I_A\right\}, \quad V_2 = E\left\{\left\{S_T \exp\left\{-\int_0^T \beta(u) du\right\}\right\} \middle/ \left\{S_{T_1} \exp\left\{-\int_0^{T_1} \beta(u) du\right\} - \exp\{-rT\}\right\} I_{AB}\right\},$$

$$V_3 = E\{ \{S_T \exp\{-\int_0^T \beta(u)du\} - K \exp\{-rT\}\} I_{CD} \}.$$

Assume there are the jump $J_1(i), i=1,2,\dots,m$ at $[0, T_1]$, and the jump $J_2(i), i=1,2,\dots,n$ at $[T_1, T]$.

(i) Calculate V_1 . Because

$$S_{T_1} = S_0 \exp\{\mu T_1 - \frac{1}{2}\sigma^2 T_1^{2H} - m\sigma_J^2/2 + \sigma_1^{(m)}\xi\}, \quad \xi = \{\sigma B_H(T_1) + \sum_{i=1}^m J_1(i) + m\sigma_J^2/2\} / \{\sigma_1^{(m)}\} \sim N(0,1),$$

$$A = \{S_{T_1} \exp\{-\int_0^{T_1} \beta(u)du\} > K \exp\{-rT_1\}\} = \{\xi > d^{(m)}\}.$$

Hence

$$\begin{aligned} V_1 &= E\{ \{S_{T_1} \exp\{-\int_0^{T_1} \beta(u)du\} - K \exp\{-rT_1\}\} I_A \} = E\{E\{ \{S_{T_1} \exp\{-\int_0^{T_1} \beta(u)du\} - K \exp\{-rT_1\}\} I_A | \mathcal{Q}_{T_1} \} \} \\ &= \sum_{m=0}^{+\infty} \frac{e^{-\lambda T_1} (\lambda T_1)^m}{m!} E\{ \{S_{T_1} \exp\{-\int_0^{T_1} \beta(u)du\} - K \exp\{-rT_1\}\} I_A | \mathcal{Q}_{T_1} = m \} \triangleq \sum_{m=0}^{+\infty} \frac{e^{-\lambda T_1} (\lambda T_1)^m}{m!} \{V_{11} - V_{12}\}, \end{aligned}$$

where

$$\begin{aligned} V_{11} &= E\{S_{T_1} \exp\{-\int_0^{T_1} \beta(u)du\} I_A | \mathcal{Q}_{T_1} = m\} = E\{S_0 \exp\{-\frac{1}{2}\sigma^2 T_1^{2H} - m\sigma_J^2/2 + \sigma_1^{(m)}\xi\} I_{\{\xi > d^{(m)}\}} \} \\ &= E\{S_0 \int_{d^{(m)}}^{+\infty} \exp\{-(\sigma_1^{(m)})^2/2 + \sigma_1^{(m)}u\} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{u^2}{2}\} du \} = S_0 \int_{d^{(m)}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{(u - \sigma_1^{(m)})^2}{2}\} du = S_0 \Phi(-d^{(m)} + \sigma_1^{(m)}), \\ V_{12} &= E\{K \exp\{-rT_1\} I_A | \mathcal{Q}_{T_1} = m\} = K \exp\{-rT_1\} E\{I_{\{\xi > d^{(m)}\}} | \mathcal{Q}_{T_1} = m\} = K \exp\{-rT_1\} P\{\xi > d^{(m)}\} = K \exp\{-rT_1\} \Phi(-d^{(m)}). \end{aligned}$$

(ii) Calculate V_2 . Because

$$S_T / S_{T_1} = \exp\{\mu(T - T_1) - \frac{1}{2}\sigma^2(T^{2H} - T_1^{2H}) - n\sigma_J^2/2 + \sigma_2^{(n)}\eta\}, \quad \eta = \{\sigma(B_H(T) - B_H(T_1)) + \sum_{i=1}^n J_2(i) + n\sigma_J^2/2\} / \{\sigma_2^{(n)}\} \sim N(0,1),$$

$$B = \{\frac{S_T}{S_{T_1}} \exp\{-\mu(T - T_1)\} > \exp\{-rT\}\} = \{\eta > a^{(n)}\}, \quad \rho_{\xi, \eta} = E(\xi\eta) = \rho_1^{(m,n)}.$$

Hence

$$\begin{aligned} V_2 &= E\{ \{\frac{S_T}{S_{T_1}} \exp\{-\int_{T_1}^T \beta(u)du\} - \exp\{-rT\}\} I_{AB} \} = E\{E\{ \{\frac{S_T}{S_{T_1}} \exp\{-\int_{T_1}^T \beta(u)du\} - \exp\{-rT\}\} I_{AB} | \mathcal{Q}_{T_1}, \mathcal{Q}_{T-T_1} \} \} \\ &= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{e^{-\lambda T} (\lambda T_1)^m [\lambda(T - T_1)]^n}{m!n!} E\{ \{\frac{S_T}{S_{T_1}} \exp\{-\mu(T - T_1)\} - \exp\{-rT\}\} I_{\{\xi > d^{(m)}, \eta > a^{(n)}\}} | \mathcal{Q}_{T_1} = m, \mathcal{Q}_{T-T_1} = n \} \\ &= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{e^{-\lambda T} (\lambda T_1)^m [\lambda(T - T_1)]^n}{m!n!} \{V_{21} - V_{22}\}, \end{aligned}$$

where

$$\begin{aligned} V_{21} &= E\{\frac{S_T}{S_{T_1}} \exp\{-\mu(T - T_1)\} I_{\{\xi > d^{(m)}, \eta > a^{(n)}\}} | \mathcal{Q}_{T_1} = m, \mathcal{Q}_{T-T_1} = n\} = E\{\exp\{-\frac{1}{2}\sigma^2(T^{2H} - T_1^{2H}) - n\sigma_J^2/2 + \sigma_2^{(n)}\eta\} I_{\{\xi > d^{(m)}, \eta > a^{(n)}\}} \} \\ &= \exp\{-\frac{1}{2}(\sigma_2^{(n)})^2\} \int_{d^{(m)}}^{+\infty} \int_{a^{(n)}}^{+\infty} \exp\{\sigma_2^{(n)}v\} \varphi(u, v, \rho_1^{(m,n)}) du dv = \int_{d^{(m)}}^{+\infty} \int_{a^{(n)}}^{+\infty} \varphi(u - \rho_1^{(m,n)}\sigma_2^{(n)}, v - \sigma_2^{(n)}, \rho_1^{(m,n)}) du dv \\ &= \int_{d^{(m)} - \rho_1^{(m,n)}\sigma_2^{(n)}}^{+\infty} \int_{a^{(n)} - \sigma_2^{(n)}}^{+\infty} \varphi(u, v, \rho_1^{(m,n)}) du dv = \Phi(\rho_1^{(m,n)}\sigma_2^{(n)} - d^{(m)}, \sigma_2^{(n)} - a^{(n)}, \rho_1^{(m,n)}), \\ V_{22} &= E\{\exp\{-rT\} I_{\{\xi > d^{(m)}, \eta > a^{(n)}\}} | \mathcal{Q}_{T_1} = m, \mathcal{Q}_{T-T_1} = n\} = \exp\{-rT\} E\{I_{\{-\xi < -d^{(m)}, -\eta < -a^{(n)}\}}\} = \exp\{-rT\} \Phi(-d^{(m)}, -a^{(n)}, \rho_1^{(m,n)}). \end{aligned}$$

(iii) Calculate V_3 . Because

$$C = \{S_{T_1} \exp\{-\int_0^{T_1} \beta(u)du\} < K \exp\{-rT_1\}\} = \{\xi < d^{(m)}\}, \quad S_T = S_0 \exp\{\mu T - \frac{1}{2}\sigma^2 T^{2H} - (m+n)\sigma_J^2/2 + \sigma^{(m,n)}\gamma\},$$

$$\gamma = \{\sigma B_H(T) + \sum_{i=1}^m J_1(i) + \sum_{i=1}^n J_2(i) + (m+n)\sigma_J^2/2\} / \sigma^{(m,n)} \sim N(0,1),$$

$$D = \{S_T \exp\{-\int_0^T \beta(u)du\} > K \exp\{-rT\}\} = \{\gamma > c^{(m,n)}\}, \quad \rho_{\xi, \gamma} = E(\xi\gamma) = \rho_2^{(m,n)},$$

then

$$V_3 = E\{ \{S_T \exp\{-\int_0^T \beta(u)du\} - K \exp\{-rT\}\} I_{CD} \}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T_1)^m [\lambda(T-T_1)]^n}{m!n!} E\{S_T \exp\{-\mu T\} - K \exp\{-rT\} I_{\{\xi < d^{(m)}, \gamma > c^{(m,n)}\}} \mid Q_{T_1} = m, Q_{T-T_1} = n\} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T_1)^m [\lambda(T-T_1)]^n}{m!n!} \{V_{31} - V_{32}\},
\end{aligned}$$

where

$$\begin{aligned}
V_{31} &= E\{S_T \exp\{-\mu T\} I_{\{\xi < d^{(m)}, \gamma > c^{(m,n)}\}} \mid Q_{T_1} = m, Q_{T-T_1} = n\} = E\{S_0 \exp\{-\frac{1}{2}(\sigma^{(m,n)})^2 + \sigma^{(m,n)}\gamma\} I_{\{\xi < d^{(m)}, \gamma > c^{(m,n)}\}}\} \\
&= S_0 \exp\{-\frac{1}{2}(\sigma^{(m,n)})^2\} \int_{-\infty}^{d^{(m)}} \int_{c^{(m,n)}}^{+\infty} \exp\{\sigma^{(m,n)}v\} \varphi(u, v, \rho_2^{(m,n)}) dudv = S_0 \int_{-\infty}^{d^{(m)}} \int_{c^{(m,n)}}^{+\infty} \varphi(u - \rho_2^{(m,n)}\sigma^{(m,n)}, v - \sigma^{(m,n)}, \rho_2^{(m,n)}) dudv \\
&= S_0 \int_{-\infty}^{d^{(m)} - \rho_2^{(m,n)}\sigma^{(m,n)}} \int_{c^{(m,n)} - \sigma^{(m,n)}}^{+\infty} \varphi(u, v, \rho_2^{(m,n)}) dudv = S_0 \Phi(d^{(m)} - \rho_2^{(m,n)}\sigma^{(m,n)}, \sigma^{(m,n)} - c^{(m,n)}, -\rho_2^{(m,n)}), \\
V_{32} &= E\{K \exp\{-rT\} I_{\{\xi < d^{(m)}, \gamma > c^{(m,n)}\}} \mid Q_{T_1} = m, Q_{T-T_1} = n\} = K \exp\{-rT\} P\{\xi < d^{(m)}, \gamma > c^{(m,n)}\} = K \exp\{-rT\} \Phi(d^{(m)}, -c^{(m,n)}, -\rho_2^{(m,n)}).
\end{aligned}$$

Remark 7 When the jump intensity $\lambda=0$, we get the actuarial price of reload option in the fractional Brownian motion environment

$$\begin{aligned}
V_0 &= S_0 \Phi(-d + \sigma_1) - K \exp\{-rT_1\} \Phi(-d) + K \Phi(\rho_1 \sigma_2 - d, \sigma_2 - a, \rho_1) - K \exp\{-rT\} \Phi(-d, -a, \rho_1) \\
&\quad + S_0 \Phi(d - \rho_2 \sigma_3, \sigma_3 - c, -\rho_2) - K \exp\{-rT\} \Phi(d, -c, -\rho_2),
\end{aligned}$$

where

$$\begin{aligned}
d &= \{\ln(K/S_0) - rT_1 + \frac{1}{2}\sigma^2 T_1^{2H}\} / \sigma_1, \quad a = \{-rT + \frac{1}{2}\sigma^2 (T^{2H} - T_1^{2H})\} / \sigma_2, \quad c = \{\ln(K/S_0) - rT_1 + \frac{1}{2}\sigma^2 T_1^{2H}\} / \sigma_3, \\
\sigma_1 &= \sigma T_1^H, \quad \sigma_2 = \sigma \sqrt{(T^{2H} - T_1^{2H})}, \quad \sigma_3 = \sigma T^H, \\
\rho_1 &= \{T^{2H} - T_1^{2H} - (T - T_1)^{2H}\} / \{2T_1^H \sqrt{(T^{2H} - T_1^{2H})}\}, \quad \rho_2 = \{T^{2H} + T_1^{2H} - (T - T_1)^{2H}\} / \{2T_1^H T^H\}.
\end{aligned}$$

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