Sufficient conditions for close-to-starlikeness and close-to-convexity of order β

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Abstract. The object of the present paper is to obtain certain sufficient conditions for close-to-starlikeness and close-to-convexity of order β.

Introduction

The function, for which the equation $f(z)=w$ has p roots in D for every complex number w, is said to be p-valent (or multivalent) function, where D is a domain in the extended complex plane C. Let H be the class of analytic functions in $U=\{z\in C:|z|<1\}$, and A_p be the subclass of H consisting of functions of the form

$$f(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+\cdots, p\in N, z\in U \tag{1}$$

with $A_1=A$.

A function $f(z)\in A_p$ is consist as starlike of order $\alpha(0\le\alpha<p)$ in U (see [1]), that is, $f(z)\in S_p^*(\alpha)$, if and only if

$$Re(\frac{zf'(z)}{f(z)})>\alpha, 0\le\alpha<p, z\in U \tag{2}$$

with $S_1^*(0):=S^*$.

A function $f(z)\in A_p$ is consist as p-valently strongly close-to-star of order $\beta(0<\beta\le 1)$ in U with respect to $g(z)$, that is, $f(z)\in CS_p^*(\beta)$, if and only if

$$|\arg(\frac{f(z)}{g(z)})|<\frac{\beta\pi}{2}, z\in U \tag{3}$$

for some real $\beta(0<\beta\le 1)$ and for some starlike function $g(z)\in A_p$ (see [2]). For $g(z)=z^p$ in condition (3), we have that $f(z)\in A_p$ is p-valently strongly close-to-star of order $\beta(0<\beta\le 1)$ in U if

$$|\arg(\frac{f(z)}{g(z)})|<\frac{\beta\pi}{2}, z\in U. \tag{4}$$

A function $f(z)\in A_p$ is consist as p-valently strongly close-to-convex of order $\beta(0<\beta\le 1)$ in U with respect to $g(z)$, that is, $f(z)\in CC_p(\beta)$, if and only if

$$|\arg(\frac{zf'(z)}{g(z)})|<\frac{\beta\pi}{2}, z\in U \tag{5}$$

for some real $\beta(0<\beta\le 1)$ and for some starlike function $g(z)\in A_p$ (see [3]). For $g(z)=z^p$ in condition (5), we have that $f(z)\in A_p$ is p-valently strongly close-to-convex of order $\beta(0<\beta\le 1)$ in U if

$$|\arg(\frac{zf'(z)}{g(z)})|<\frac{\beta\pi}{2}, z\in U. \quad (6)$$

In proving our main theorem, we need the following lemma due to Owa, Nunokawa, Saitoh and Fukui.

Lemma 1.1. (see [4])Let Let $p(z)$ be analytic, $p(z)\neq 0$ in U and $p(0)=1$. Suppose that there exists a point $z_0 \in U$ such that

$$|\arg p(z)|<\frac{\pi\beta}{2}\ for\ |z|<|z_0| \quad (7)$$

and

$$|\arg p(z_0)|=\frac{\pi\beta}{2} \quad (8)$$

where $\beta>0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)}=ik\beta \quad (9)$$

where

$$k\geq\frac{1}{2}(\gamma+\frac{1}{\gamma})\geq 1 when \arg p(z_0)=\frac{\pi\beta}{2} \quad (10)$$

and

$$k\leq-\frac{1}{2}(\gamma+\frac{1}{\gamma})\leq -1 when \arg p(z_0)=-\frac{\pi\beta}{2} \quad (11)$$

where

$$p(z_0)^{1/\beta}=\pm i\gamma and \gamma>0. \quad (12)$$

Main Result

Theorem 2.1. If $f(z)\in A_p$ satisfies the condition,

$$|1+\frac{zf''(z)}{f'(z)}-\alpha|<\alpha, z\in U \quad (13)$$

where

$$\alpha=\frac{p^2+\beta^2}{2p} \quad (14)$$

and $0<\beta\leq 1$, then $f(z)\in CC_p(\beta)$.

Proof. Let us put

$$F(z)=\frac{f'(z)}{pz^{p-1}}. \quad (15)$$

By logarithmic differentiation of (15), we have

$$\frac{zF'(z)}{F(z)}=1+\frac{zf''(z)}{f'(z)}-p \quad (16)$$

or

$$\frac{zF'(z)}{F(z)}+p=1+\frac{zf''(z)}{f'(z)} \quad (17)$$

Suppose there exist a point $z_0\in U$ such that

$$|\arg F(z)|<\frac{\pi}{2}\beta\ for\ |z|<|z_0| \quad (18)$$

and

$$|\arg F(z_0)|=\frac{\pi}{2}\beta, \quad (19)$$

then from Lemma 1.1, we have

$$\frac{z_0F'(z_0)}{F(z_0)} = i\beta k \tag{20}$$

where

$$k \geq \frac{1}{2}(\gamma + \frac{1}{\gamma}) \geq 1 when \arg p(z_0) = \frac{\pi\beta}{2} \tag{21}$$

and

$$k \leq -\frac{1}{2}(\gamma + \frac{1}{\gamma}) \leq -1 when \arg p(z_0) = -\frac{\pi\beta}{2} \tag{22}$$

where

$$p(z_0)^{1/\beta} = \pm i\gamma and \gamma > 0. \tag{23}$$

At first, let us suppose $p(z_0)^{1/\beta} = i\gamma$, then we have

$$\frac{z_0F'(z_0)}{F(z_0)} + p = i\beta k + p \tag{24}$$

where

$$k \geq \frac{1}{2}(\gamma + \frac{1}{\gamma}) \geq 1. \tag{25}$$

From this, we have

$$\mathrm{Re}(\frac{z_0F'(z_0)}{F(z_0)} + p)^{-1} = \frac{p}{p^2 + \beta^2k^2} \leq \frac{p}{p^2 + \beta^2} \tag{26}$$

Since $|w - \alpha| < \alpha \Leftrightarrow \mathrm{Re}(1/w) > \frac{1}{2\alpha}$, this contradicts the assumption of this theorem.

For the case $p(z_0)^{1/\beta} = -i\gamma$, applying the same method as the above, we have the condition (26). Therefore we complete the proof.

Theorem 2.2. If $f(z) \in A_p$ satisfies the condition,

$$\left|\frac{zf'(z)}{f(z)} - \alpha\right| < \alpha, z \in U \tag{27}$$

where

$$\alpha = \frac{p^2 + \beta^2}{2p} \tag{28}$$

and $0 < \beta \leq 1$, then $f(z) \in SS_p^*(\beta)$.

Proof. Let us put

$$F(z) = \frac{f(z)}{z^p}. \tag{29}$$

By logarithmic differentiation of (15), we have

$$\frac{zF'(z)}{F(z)} = \frac{zf'(z)}{f(z)} - p \tag{30}$$

or

$$\frac{zF'(z)}{F(z)} + p = \frac{zf'(z)}{f(z)} \tag{31}$$

Suppose there exist a point $z_0 \in U$ such that

$$|\arg F(z)| < \frac{\pi}{2}\beta for |z| < |z_0| \tag{32}$$

and

$$|\arg F(z_0)| = \frac{\pi}{2}\beta, \tag{33}$$

then from Lemma 1.1, we have

$$\frac{z_0 F'(z_0)}{F(z_0)} = i\beta k \tag{34}$$

where

$$k \geq \frac{1}{2}(\gamma + \frac{1}{\gamma}) \geq 1 when \arg p(z_0) = \frac{\pi\beta}{2} \tag{35}$$

and

$$k \leq -\frac{1}{2}(\gamma + \frac{1}{\gamma}) \leq -1 when \arg p(z_0) = -\frac{\pi\beta}{2} \tag{36}$$

where

$$p(z_0)^{1/\beta} = \pm i\gamma and \gamma > 0. \tag{37}$$

At first, let us suppose $p(z_0)^{1/\beta} = i\gamma$, then we have

$$\frac{z_0 F'(z_0)}{F(z_0)} + p = i\beta k + p \tag{38}$$

where

$$k \geq \frac{1}{2}(\gamma + \frac{1}{\gamma}) \geq 1. \tag{39}$$

From this, we have

$$\mathrm{Re}(\frac{z_0 F'(z_0)}{F(z_0)} + p)^{-1} = \frac{p}{p^2 + \beta^2 k^2} \leq \frac{p}{p^2 + \beta^2} \tag{40}$$

Since $|w - \alpha| < \alpha \Leftrightarrow \mathrm{Re}(1/w) > \frac{1}{2\alpha}$, this contradicts the assumption of this theorem.

For the case $p(z_0)^{1/\beta} = -i\gamma$, applying the same method as the above, we have the condition (26). Therefore we complete the proof.

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