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Complex Angle Variables for Constrained Integrable Hamiltonian Systems

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Abstract

We propose Dirac formalism for constraint Hamiltonian systems as an useful tool for the algebro-geometrical and dynamical characterizations of a class of integrable systems, the so called hyperelliptically separable systems. As a model example, we apply it to the classical geodesic flow on an ellipsoid.

Consider an n-dimensional Hamiltonian system on the phase space $x_1, \ldots, x_n$ with Hamilton function $H(x)$ and Poisson bracket $\{\,,\}$. Let $q_1, \ldots, q_r$, $p_1, \ldots, p_r$ be (local) Darboux coordinates on a symplectic leave $\mathcal{M}$ of the bracket, so that the symplectic form to $\mathcal{M}$ is $\Omega = \sum\limits_{i=1}^{r} dq_i \wedge dp_i$. Suppose that the restriction of the system on $\mathcal{M}$ is an algebraically completely integrable system in the following sense [4]: there is a family of algebraic curves of genus g such that the complex invariant manifolds associated to the system are open subsets of customary ($g = r$) or generalized ($g < r$) Jacobian varieties of such algebraic curves.

Following [7, 3], $\mathcal{M}$ can be regarded as a fiber bundle $\mathcal{M} \to \mathcal{U}$ with the base $\mathcal{U}$ parametrizing the corresponding curves and the fibers being (generalized) Jacobians of the curves. $\mathcal{U}$ is a subvariety in the moduli space of curves of genus g.

For simplicity, let us suppose $g = r$, the other case can be considered along similar lines (see [7]). In particular, there exists a canonical transformation $(p, q) \to (\lambda, \mu)$ to *separating variables* such that $\Omega = \sum\limits_{i=1}^{g} d\lambda_i \wedge d\mu_i$ and on each invariant manifold the pairs of conjugated variables satisfy algebraic relations

$$F(\lambda_i, \mu_i; c) = 0, \qquad i = 1, \ldots, g,$$

defining a family of algebraic curves Γ_c of genus g. $c = (c_1, \ldots, c_g)$ are, among the coefficients of F, those which are independent first integrals of the system in involution. Moreover, c form a basis of coordinates on the base $\mathcal{U}$. Solving $F(\lambda, \mu; c) = 0$ in terms of μ, we obtain the generating function

$$G(\lambda, c) = \sum_i \int_{\lambda_0}^{\lambda_i} \mu(\lambda, c_1, \ldots, c_g)\, d\lambda$$

of another canonical transformation $(\lambda,\mu)\to(c,\phi)$ described explicitly by the Abel–Jacobi mapping $\Gamma^{(g)}\to\mathrm{Jac}(\Gamma)$

$$\frac{\partial G}{\partial c_i}\equiv\sum_i\int_{\lambda_0}^{\lambda_i}\frac{\partial\mu(\lambda,c)}{\partial c_i}\,d\lambda=\phi_i,\qquad i=1,\ldots,g.$$

$\phi_1,\ldots,\phi_g$ are coordinates on the universal covering of $\mathrm{Jac}(\Gamma)$ and are also the *complex angle variables.*

There are many Liouville integrable systems, as well as integrable PDE, which are not algebraically completely integrable, but enjoy the following property [1]: *their complex invariant manifolds are open subsets of r-dimensional nonlinear strata of customary or generalized Jacobian varieties associated to algebraic curves of genus g.* Notice that their generic solutions are not meromorphic functions of complex time.

A trivial example is the one-dimensional system with the Hamiltonian

$$H=\frac{1}{2}p^2+R_r(q)=h,\qquad h=\text{const},$$

where $R_r(q)$ is a polynomial of degree $r>4$ with simple roots. Integration of the system leads to the inversion of a single Abelian integral

$$t-t_0=\int_{q_0}^{q}\frac{dq}{\sqrt{2(h-R_r(q))}},$$

associated to the genus $g\geq 2$ hyperelliptic curve $\Gamma=\{w^2=h-R_r(q)\}$, which is the generic complex invariant manifold of the system and which cannot be completed into an Abelian variety. Moreover, the generic solution $q(t)$ is a single-valued function only on an *infinitely* sheeted covering of the complex plane t with an infinite number of algebraic branch points whose projections on t form a dense set (except rare cases of reducibility of the Abelian integral to elliptic ones).

Other examples are finite-dimensional reductions of the shallow water (Camassa–Holm) equation and the Dym-type equation [5] and generalizations of the integrable case of the Henon–Heiles system on $\mathbf{R}^2$ with potentials of degree ≥ 4, in particular with the Hamiltonian

$$H=\frac{1}{2}\left(p_x^2+p_y^2\right)+V^{(5)}(x,y),\qquad V^{(5)}=y^5+y^3x^2+\frac{3}{16}yx^4.$$

For the above system, separation of variables in parabolic coordinates gives rise to the quadratures

$$\frac{d\lambda_1}{2\sqrt{R(\lambda_1)}}+\frac{d\lambda_2}{2\sqrt{R(\lambda_2)}}=d\phi_1,\qquad \frac{\lambda_1\,d\lambda_1}{2\sqrt{R(\lambda_1)}}+\frac{\lambda_2\,d\lambda_2}{2\sqrt{R(\lambda_2)}}=d\phi_2,$$

$$R(\lambda)=\lambda\left(c\lambda-d-\lambda^6\right),\qquad d\phi_1=0,\qquad d\phi_2=dt,$$

containing 2 holomorphic differentials on the genus 3 curve $\Gamma=\{w^2=R(\lambda)\}$. These describe a mapping of the symmetric product $\Gamma^{(2)}$ to the (3-dimensional) Jacobian of Γ. Its image is a 2-dimensional *nonlinear* subvariety (stratum) of $\mathrm{Jac}(\Gamma)$, which is a translation of the theta-divisor of the Jacobian.

Our main observation here (see also [2]) is that such systems can be obtained from restrictions of algebraic integrable ones to subvarieties of the phase space using the Hamiltonian formalism with constraints developed by Dirac [6].

A complete presentation of this formalism to generalized algebraically integrable systems will be considered elsewhere [2]. Here we present the main theorem in the case of systems with invariant manifolds on strata of Jacobi varieties and a model example associated to strata of generalized Jacobi varieties.

Let us constrain the algebraically integrable system onto the symplectic subvariety $\mathcal{N} \subset \mathcal{M}$ defined by 2d constraints

$$\begin{aligned} &\lambda_{g-d+1} = C_{g-d+1}, \ \ldots, \ \lambda_g = C_d, \qquad C_{g-d+1}, \ldots, C_d = \text{const}, \\ &\mu_{g-d+1} = E_{g-d+1}, \ldots, \mu_g = E_g, \qquad E_i = \text{const}. \end{aligned}$$

Theorem. *The constraint variety $\mathcal{N}$ intersects the family of Jacobians along $(g-d)$-dimensional nonlinear subvarieties (strata), the images of the mapping $\Gamma^{(g-d)} \to Jac(\Gamma)$. The latter are complexified invariant manifolds of the system restricted on $\mathcal{N}$.*

Now the angle variables $\phi_1, \ldots, \phi_g$ play the role of redundant coordinates on the strata [1].

A model example is the geodesic flow on an ellipsoid, which can be obtained constraining the free motion in $\mathbf{R^3}$. Consider a family of confocal quadrics in $\mathbf{R^3}$ $\left(\mathbf{C}^3\right) = (X_1, X_2, X_3)$

$$Q(s) = \left\{ \frac{X_1^2}{D_1 - s} + \frac{X_2^2}{D_2 - s} + \frac{X_3^2}{D_3 - s} = 1 \right\}, \qquad s \in \mathbf{R}, \quad 0 < D_1 < D_2 < D_3.$$

and associated ellipsoidal coordinates λ_1, λ_2, λ_3 such that

$$X_i^2 = \frac{(D_i - \lambda_1)(D_i - \lambda_2)(D_i - \lambda_3)}{(D_i - D_j)(D_i - D_k)}, \qquad (i, j, k) = (1, 2, 3).$$

A free motion of a particle in $\mathbf{R^3}$ is described by the Hamiltonian

$$\begin{aligned} &H = 2 \sum_{i=1}^{3} \frac{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}{\Phi(\lambda_i)} \dot{\lambda}_i^2 = \frac{1}{2} \sum_{i=1}^{3} \frac{\Phi(\lambda_i)}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \mu_i^2, \\ &\Phi(\lambda) = (\lambda - D_1)(\lambda - D_2)(\lambda - D_3), \end{aligned}$$

where μ_1, μ_2, μ_3 are momenta canonically conjugated to λ. The canonical variables satisfy algebraic relations

$$\mu_i^2 = \frac{c_0(\lambda_i - c_1)(\lambda_i - c_2)}{\Phi(\lambda_i)}, \qquad i = 1, 2, 3,$$

defining genus 2 hyperelliptic curve $\Gamma = \left\{ w^2 = \Phi(\lambda)(\lambda - c_1)(\lambda - c_2) \right\}$. The constants of motion c_1, c_2 have a transparent geometric interpretation: in the configuration space $\mathbf{R^3}$ the straight line trajectory is tangent to the quadrics $Q(c_1)$, $Q(c_2)$ of the above confocal family.

The Hamilton equations for λ, μ and the generating function

$$G(\lambda, c) = \sum_{i}^{3} \int_{\lambda_0}^{\lambda_i} \frac{\sqrt{c_0(\lambda - c_1)(\lambda - c_2)}}{\sqrt{\Phi(\lambda)}} \, d\lambda$$

result in the following transformation $(\lambda,\mu)\to(\phi,c)$ written in a differential form

$$\sum_{i=1}^{3}\frac{d\lambda_i}{2\sqrt{R(\lambda_i)}}=d\phi_1,\qquad \sum_{i=1}^{3}\frac{\lambda_i\,d\lambda_i}{2\sqrt{R(\lambda_i)}}=d\phi_2,\qquad \sum_{i=1}^{3}\frac{\lambda_i^2\,d\lambda_i}{2\sqrt{R(\lambda_i)}}=d\phi_3,$$

$$R(\lambda)=-c_0\Phi(\lambda)(\lambda-c_1)(\lambda-c_2),\qquad d\phi_1=d\phi_2=0,\qquad d\phi_3=dt,$$

which contain two holomorphic differentials and one meromorphic differential of the second kind on the genus 2 curve Γ.

This defines the Abel–Jacobi mapping of the symmetric product $\Gamma^{(3)}$ to the (3-dimensional) *generalized* Jacobian variety.

As a result of inversion of the mapping, we get theta-functional expressions

$$X_i=\frac{\phi_3\theta[\eta_i](\phi_1,\phi_2)-\partial_V\theta[\eta_i](\phi_1,\phi_2)]}{\theta(\phi_1,\phi_2)},\qquad i=1,2,3, \tag{1}$$

where $\theta[\eta_i](\phi)$ are theta-functions associated to Γ with certain half-integer theta-characteristics η_i and ∂_V is the derivative w.r.t. $V=-2\phi_2$. As expected, X_i are linear in ϕ_3 and therefore in t.

Now let us restrict the system onto the symplectic subvariety $\mathcal{N}=\{\lambda_3=0,\ \mu_3=0\}\subset T\mathbf{R}^3$. $\mathcal{N}$ coincides with the cotangent bundle of the triaxial ellipsoid $Q(0)$ on which the constant of motion $c_2=0$. The subvariety $\mathcal{N}$ intersects the family of generalized Jacobians along 2-dimensional nonlinear strata W_2 which, in the angle coordinates ϕ, have the form

$$W_2=\{\phi_3\theta(\phi_1,\phi_2)-\partial_V\theta(\phi_1,\phi_2)=0\}.$$

Then, from the dynamical point of view, we restrict the free motion in the space $\mathbf{R^3}$ to the geodesic flow on $Q(0)\subset\mathbf{R}^3$; from the algebraic geometrical point of view, by imposing the constraint, we force the linear motion on the generalized Jacobians to take place on their nonlinear strata W_2, where the angle variables ϕ_1, ϕ_2, ϕ_3 play the role of *redundant* coordinates.

According to definition of W_2, ϕ_1 becomes a transcendental function of ϕ_2, ϕ_3. Substituting it into (1), we get the explicit solution for the geodesic problem in terms of the natural parameter $t=\phi_3$.

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