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Change of the Time for the Toda Lattice

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Abstract

For the Toda lattice we consider properties of the canonical transformations of the extended phase space, which preserve integrability. At the special values of integrals of motion the integral trajectories, separated variables and the action variables are invariant under change of the time. On the other hand, mapping of the time induces shift of the generating function of the Bäcklund transformation.

On the $2n$-dimensional symplectic manifold $\mathcal{M}$ (phase space) with coordinates $\{p_j, q_j\}_{j=1}^n$ let us consider hamiltonian system with some Hamilton function $H(p,q)$. By adding to $\mathcal{M}$ the time $q_{n+1} = t$ and the Hamilton function $p_{n+1} = -H$ one gets $2n+2$-dimensional extended phase space $\mathcal{M}_E$ of the given hamiltonian system. By definition canonical transformations of the extended phase space $\mathcal{M}_E$ preserve the Hamilton–Jacobi equation and differential form

$$\alpha = \sum_{j=1}^{n} p_j\, dq_j - H dt.$$

So, any canonical transformation of the time looks like

$$d\widetilde{t} = v^{-1}(p,q)\, dt, \qquad \widetilde{H} = v(p,q) H.$$

All the canonical transformation of the initial phase space $\mathcal{M}$ map any integrable system into the other integrable system. However, we have not the regular way to obtain canonical transformation of the extended phase space $\mathcal{M}_E$, which maps a given integrable system into the other integrable system.

The periodical Toda lattice is described by the following Hamilton function

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + a_i\, e^{q_i - q_{i+1}}. \tag{1}$$

Here $\{p_i, q_i\}$ are canonical variables and the periodicity conventions $i+n=i$ are always assumed. The $n\times n$ Lax matrices for the Toda lattice are

$$\begin{aligned} L^{(n)}(\mu) &= \sum_{i=1}^{n} p_j\, E_{i,i} + \sum_{i=1}^{n-1}\left(e^{q_i - q_{i+1}} E_{i+1,i} + E_{i,i+1}\right) + \mu\, e^{q_n - q_1} E_{1,n} + \mu^{-1} E_{n,1}, \\ A^{(n)}(\mu) &= \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} E_{i+1,i} + \mu\, e^{q_n - q_1} E_{1,n}, \end{aligned} \tag{2}$$

where $E_{i,k}$ stands for the $n \times n$ matrix with unity on the intersection of the i-th row and the k-th column as the only nonzero entry.

Now let us introduce another 2×2 Lax representation for the same Toda lattice

$$T^{(1\ldots n)}(\lambda) = T_1(\lambda) \cdots T_n(\lambda) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}(\lambda), \tag{3}$$

such that $\{H, T_j\} = T_j A_j - A_{j-1} T_j$, $\{H, T\} = [T, A_n]$, where

$$T_j = \begin{pmatrix} \lambda + p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \qquad A_j = \begin{pmatrix} \lambda & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \tag{4}$$

Sometimes, we shall omit the superscripts n of the matrix $L(\mu)$ (2) and the $1 \ldots n$ of the monodromy matrix $T(\lambda)$ (3).

According to [5], canonical transformations of $\mathcal{M}_E$

$$d\widetilde{t} = e^{q_j - q_{j+1}} dt, \qquad \widetilde{H} = e^{q_{j+1} - q_j}(H + b), \qquad b \in \mathbb{R} \tag{5}$$

preserves integrability. This change of the time gives rise to the following transformation of the Lax matrices

$$\widetilde{L} = L - \widetilde{H}\, E_{j,j+1}, \qquad \widetilde{A} = v^{-1}(q) A, \tag{6}$$

and

$$\begin{aligned} \widetilde{T}^{(1\ldots n)} &= T^{(1\ldots n)} + T^{(1\ldots j-1)} \begin{pmatrix} H + b & 0 \\ 0 & 0 \end{pmatrix} T^{(j+2\ldots n)}, \\ \widetilde{A}_n &= v^{-1}(q) A_n. \end{aligned} \tag{7}$$

The corresponding transformation of the spectral curves $\det(L(\mu) + \lambda I) = 0$ or $\det(T(\lambda) + \mu I) = 0$ looks like

$$\begin{aligned} C: &\quad -\mu - \frac{1}{\mu} = \lambda^n + \lambda^{n-1} p + \lambda^{n-2} \left(\frac{p^2}{2} - H \right) + \cdots, \\ \widetilde{C}: &\quad -\mu - \frac{1 - \widetilde{H}}{\mu} = \lambda^n + \lambda^{n-1} p + \lambda^{n-2} \left(\frac{p^2}{2} + b \right) + \cdots. \end{aligned} \tag{8}$$

Here $p = \sum p_j$ is a total momentum, H and $\widetilde{H}$ are the corresponding Hamilton functions. The Poisson brackets relations for the $n \times n$ Lax matrices can be expressed in the r-matrix form

$$\left\{ \overset{1}{L}(\mu), \overset{2}{L}(\nu) \right\} = \left[r_{12}(\mu, \nu), \overset{1}{L}(\mu) \right] + \left[r_{21}(\mu, \nu) \overset{2}{L}(\nu) \right].$$

Here Π is the permutation operator in $\mathbb{C}^n \times \mathbb{C}^n$ and we used the standard notations $\overset{1}{L}(\mu) = L(\mu) \otimes I$, $\overset{2}{L}(\nu) = I \otimes L(\nu)$, $r_{21}(\mu, \nu) = -\Pi r_{12}(\nu, \mu) \Pi$. Change of the time (5) maps the constant r-matrix for the Toda lattice

$$r_{12}(\mu, \nu) = r_{12}^{\mathrm{const}}(\mu, \nu) = \frac{1}{\mu - \nu} \left(\nu \sum_{m \geq i} + \mu \sum_{m < i} \right) E_{im} \otimes E_{mi}$$

into the following dynamical r-matrix

$$r_{12}(\mu,\nu)=r_{12}^{\rm const}(\mu,\nu)+r_{12}^{\rm dyn}(\mu,\nu),\qquad r_{12}^{\rm dyn}(\mu,\nu)=\widetilde{A}(\nu,q)\otimes E_{j,j+1}.$$

The 2×2 monodromy matrix $T(\lambda)$ (3) satisfies the following Sklyanin r-matrix relations

$$\left\{\overset{1}{T}(\lambda),\overset{2}{T}(\nu)\right\}=\left[R(\lambda-\nu),\overset{1}{T}(u)\overset{2}{T}(\nu)\right],\qquad R(\lambda-\nu)=\frac{\Pi}{\lambda-\nu}. \tag{9}$$

Change of the time (5) transforms these quadratic relations into the following poly-linear relations

$$\left\{\overset{1}{\widetilde{T}}(\lambda),\overset{2}{\widetilde{T}}(\nu)\right\}=\left[R(\lambda-\nu),\overset{1}{\widetilde{T}}(\lambda)\overset{2}{\widetilde{T}}(\nu)\right]+\left[r_{12}^{\rm dyn}(\lambda,\nu),\overset{1}{\widetilde{T}}(\lambda)\right]+\left[r_{21}^{\rm dyn}(\lambda,\nu),\overset{2}{\widetilde{T}}(\nu)\right].$$

The corresponding dynamical r-matrix is given by

$$r_{12}^{\rm dyn}(\lambda,\nu)=A_n(\lambda,q)\otimes\left(T_1\cdots T_{j-1}\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot T_{j+1}\otimes T_n\right).$$

Here all the matrices T_k depend on the spectral parameter ν and $A_n(\lambda,q)$ is the second Lax matrix (3).

Now let us look at the Darboux coordinates and the action-angle variables in framework of the traditional consideration of the Toda lattice [1, 3]. Let us study the Toda system and the dual system simultaneously and, for the simplicity, consider change of the time (5) at $j=1$ such that

$$\widetilde{H}=\exp(q_2-q_1)(H+b)$$

and the corresponding 2×2 matrix looks like

$$\widetilde{T}^{(1\ldots n)}=T^{(1\ldots n)}+\begin{pmatrix}H&0\\0&0\end{pmatrix}T^{(3\ldots n)}. \tag{10}$$

The separation variables $\{\lambda_1\,\lambda_2,\ldots,\lambda_{n-1}\}$ for the both system are eigenvalues of $T(\lambda)$ and zeroes of the polynomial $\mathcal{C}(\lambda)$

$$\mathcal{C}(\lambda)=\gamma\prod_{i=1}^{n-1}(\lambda-\lambda_i),\qquad\text{and}\quad \mu_i=\mathcal{D}(\lambda_i),\quad i=1,\ldots,n-1.$$

From $\det T(\lambda)=1$ and $\det\widetilde{T}(\lambda)=(1-\widetilde{H})$ one immediately gets

$$\mathcal{A}(\lambda_i)=\mu_i^{-1},\qquad\text{and}\quad \mu_i+\mu_i^{-1}=P(\lambda_i)=\operatorname{tr}T(\lambda_i),$$
$$\widetilde{\mathcal{A}}(\lambda_i)=(1-\widetilde{H})\mu_i^{-1},\qquad\text{and}\quad \mu_i+(1-\widetilde{H})\mu_i^{-1}=\widetilde{P}(\lambda_i)=\operatorname{tr}\widetilde{T}(\lambda_i).$$

By using the standard form of the hyperelliptic curves C and $\widetilde{C}$ (8) and by applying Arnold's method, action variables have the form

$$\begin{aligned}s_i&=\oint_{\alpha_i}\frac{1}{2}\left(P(\lambda)+\sqrt{P(\lambda)^2-4}\right)d\lambda,\\ \widetilde{s}_i&=\oint_{\widetilde{\alpha}_i}\frac{1}{2}\left(\widetilde{P}(\lambda)+\sqrt{\widetilde{P}(\lambda)^2-4(1-\widetilde{H})}\right)d\lambda,\end{aligned} \tag{11}$$

where α_i and $\widetilde{\alpha}$ are α-cycles of the Jacobi variety of the algebraic curves (8), respectively.

Finally let us consider the known Bäcklund transformation B_ν for the Toda lattice [1, 2]. As is well known, transformation B_ν is canonical transformation $(p,q)\mapsto(P,Q)$ of the initial phase space $\mathcal{M}$, which can be described via the generating function

$$F_\nu(q\,|\,Q)=\sum_{i=1}^{n}\left(e^{q_i-Q_i}-e^{Q_i-q_{i+1}}-\nu(q_i-Q_i)\right). \tag{12}$$

such that

$$p_i=\frac{\partial F}{\partial q_i},\qquad P_i=-\frac{\partial F}{\partial Q_i}. \tag{13}$$

To prove that B_ν preserves integrals of motion $I_k(p,q)=I_k(P,Q)$, one verifies that B_ν preserves the spectrum of the Lax matrix $L(\mu)$ (2)

$$M(\mu,q,Q)L(\mu,p,q)=L(\mu,P,Q)M(\mu,q,Q), \tag{14}$$

where

$$M(\mu,q,Q)=\sum_{i=1}^{n-1}e^{Q_i-q_{i+1}}E_{i+1,i}+\mu e^{Q_n-q_1}E_{1,n}. \tag{15}$$

Canonical transformation (5) of the extended phase space $\mathcal{M}_E$ associated with arbitrary root β_j induces the following shift of the generating function

$$\widetilde{F}_\nu(q\,|\,Q)=F_\nu(q,Q)+\widetilde{H}e^{Q_j-q_{j+1}}=F_\lambda(q,Q)+\Delta F. \tag{16}$$

Here and in (13) we reinterpret the Hamiltonian $\widetilde{H}$ not as function on the phase space $\mathcal{M}$, but rather as constant of motion or the element of the extended phase space $\mathcal{M}_E$. In this case the equality (14) and matrix $M(\mu,q,Q)$ (15) are invariant with respect to the change of the time.

For the corresponding 2×2 Lax matrices T and $\widetilde{T}$ the intertwining relations are equal to

$$M_i(\lambda,\nu)T_i(p,q)=T_i(P,Q)M_{i+1}(\lambda,\nu),$$

where

$$M_i(\lambda,\nu)=\begin{pmatrix}1 & e^{Q_{i-1}}\\ -e^{-q_i} & \nu-\lambda-e^{Q_{i-1}-q_i}\end{pmatrix},$$

As above, these relations and these matrices $M_i(\lambda,\mu)$ are invariant with respect to the change of the time.

Recall, the correspondence between the kernel of the corresponding quantum Baxter $\mathbb{Q}$-operator and the function $F_\lambda(q\,|\,Q)$ is given by the semiclassical relation [2, 4]. Change of the time (5) gives rise to factorization of the $\mathbb{Q}$-operator at the semiclassical limit

$$\widetilde{\mathbb{Q}}\sim\exp(-i\widetilde{F}/\hbar)=\mathbb{Q}\cdot\exp(-i\Delta F/\hbar).$$

So, it will be interesting to investigate the spectrum of the new quantum $\widetilde{\mathbb{Q}}$-operator by using known spectrum of the dual system.

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