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Hard Loss of Stability in Painlevé-2 Equation

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Abstract

A special asymptotic solution of the Painlevé-2 equation with small parameter is studied. This solution has a critical point t_* corresponding to a bifurcation phenomenon. When $t < t_*$ the constructed solution varies slowly and when $t > t_*$ the solution oscillates very fast. We investigate the transitional layer in detail and obtain a smooth asymptotic solution, using a sequence of scaling and matching procedures.

1 Introduction

In this work a special asymptotic solution for the equation Painlevé-2

$$\varepsilon^2 u'' + 2u^3 + tu = 1 \tag{1}$$

is constructed as $\varepsilon \to 0$.

The behaviour of wanted solution differs in different intervals of the parameter t. The qualitative behaviour of numerical solution is indicated in the figure (at $\varepsilon^2 = 0.1$). Calculations for another small values of ε give pictures like this. Let us explain these numeric results using asymptotic theory for small ε.

In the area I the special solution is approximated by an asymptotic solution in which the leading term is a least root of a cubic equation $2u^3 + tu = 1$. Corrections are algebraic functions of t. Such algebraic asymptotics becomes invalid near the point t_*, where two of the roots of the cubic equation coalesce. In the left neighborhood of the point t_* this asymptotics goes into an asymptotics of a special solution of the Painlevé-1 equation with respect to new scaling variable $\tau = (t - t_*)\varepsilon^{-4/5}$. This is the area II in the figure. This special solution of the Painlevé-1 equation has poles at $\tau = \tau_k$, $k = 0, 1, \dots$. In the neighborhoods of these poles one more scaling is done. New variable is $\theta = (\tau - \tau_k)\varepsilon^{-1/5}$ (area III). In this area the asymptotics is defined by separatrix solution of a nonlinear autonomous equation. This combined asymptotic structure becomes invalid as $\tau \to \infty$, because the poles of the solution Painlevé-1 equations close to each other. As $\tau \to \infty$ a fast oscillating asymptotics is valid. It is the area IV in the figure.

The qualitative behaviour of solutions of second-order ordinary differential equations with respect to an additional parameter was explained, for example, in the book [1]. In [1]

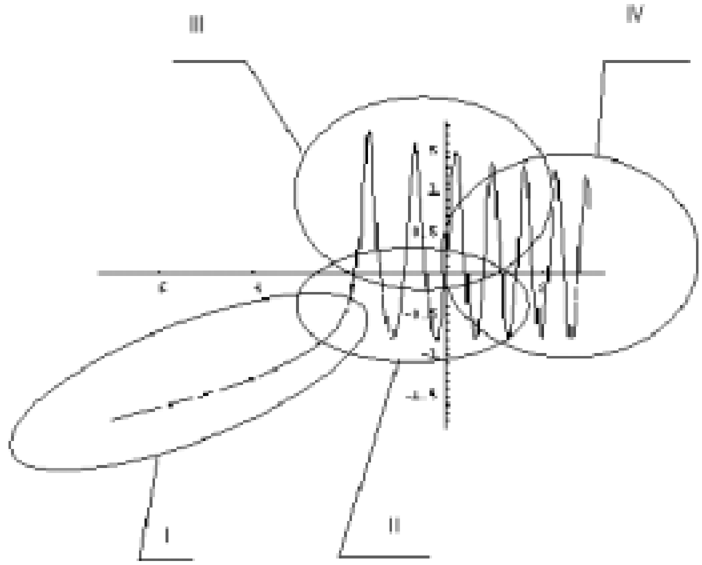

Figure 1. The hard loss of stability.

the various types of bifurcations for equilibrium positions of conservative second-order ordinary differential equations were described also.

The equation (1) is nonautonomous, however one can separate the dependence on slow t and fast variables (τ, θ) in the asymptotic solution of this equation. We may speak about the phase plane and phase trajectory of this equation with respect to the fast variable t/ε. The equilibrium positions on the phase plane depend on t. When $t < t_*$ there are three equilibrium positions. At the critical point t_* the bifurcation "saddle-center" occurs. It means that one of stable and unstable equilibrium positions coalesce. When $t > t_*$ only one equilibrium position exists. Such bifurcation leads to instability [2].

The bifurcations of slowly varying equilibrium positions of second order equation with the algebraic nonlinearity and slowly varying parameters were considered in [3] only in a preliminary fashion. When this text was written, new work [4] became known. In that work, a change of an energy and a phase jump has been studied for a solution in a very narrow layer near a saddle-center bifurcation point in general case. However results of the work [4] are inapplicable to the Painlevé-2 equation, because this equation has degenerate behaviour with respect to equations considered in [4].

Amongst another works, in which the asymptotic solutions of the nonlinear equations with varying coefficients were studied we should note the work [5]. In that work a changing of an adiabatic invariant was studied in a problem when a solution passed through separatrix in the nondegenerate case. One more work where the passage through the separatrix in nondegenerate case was studied is [6].

More complicated bifurcation is the pitchfork in the equation Painlevé-2. Asymptotics with respect to a small parameter of the solutions for the equation Painlevé-2 with zero in the right hand side of the equation (1) was investigated in works [7, 8]. In this case a solution in an interior layer near a bifurcation value of the parameter t_* is determined by the equation Painlevé-2, but already without a small parameter; and the problem, generally speaking, does not become simpler.

The asymptotics of the solutions for the Painlevé equations with a leading term as an elliptic function with the modulated parameters were studied, for example, in works [9]–[20]. We must mention some of the works about the scaling limits or double asymptotics for the Painlevé-2 equation. The general approach to the scaling limits of the Painlevé equations based on the Bäcklund transformations was studied in [21]. The scaling limit passage from the equation Painlevé-2 to Painlevé-1 was studied on the level of classical solutions in the work [22]. In work [23] different approaches to the double asymptotics were developed. The qualitative analysis for the relation of the algebraic and fast oscillating asymptotic solutions of the equation (1) was done in the work [24] also. However, the asymptotic solutions constructed by this way are non-uniform with respect to two variable t and ε.

The major difference of the presented work in comparsion with the others cited is constructing uniform asymptotic solution with respect to two parameters t and ε as $\varepsilon \to 0$ into an interval of t, where the main term of asymptotics (elliptic function) is degenerated. This uniform asymptotic solution is valid on a segment containing the saddle-center bifurcation point. In this work we specify a different types of asymptotic approximations of the being studied solution, their valid intervals and orders of neglected terms which result when these asymptotics are substituted into a being solved equation.

The contents of the various sections are as follows. Section 2 states the main problem. The results are formulated in Section 3. An asymptotic expansion to be valid before the bifurcation point t_* is written in Section 4. Section 5 is devoted to inner asymptotic expansions which are matched with each other and with asymptotics from Section 4. A fast oscillating asymptotic expansion of Kuzmak-type, a degeneration of the oscillations and the matching of the fast oscillations with inner expansions from Section 5 are described in Section 6. Open problems are discussed in Section 7.

2 Naive statement of the problem

Let's consider a cubic equation:

$$2u^3 + tu = 1, \tag{2}$$

which is obtained as a rejection of the term with the small parameter in the equation (1). There exist the point t_* and the value u_* such, that if $t = t_*$, then u_* is double root of the equation (2). The values u_* and t_* are easy to obtain by solving of the equations:

$$2u_*^3 + t_* u_* = 1, \qquad 6u_*^2 + t_* = 0.$$

There are $t_* = -3\,2^{-1/3}$, $u_* = -4^{-1/3}$.

The discriminant of the equation (2) has the form:

$$D = \left(\frac{t}{6}\right)^3 + \left(\frac{1}{4}\right)^2.$$

The discriminant $D < 0$ when $t < t_*$ and hence the cubic equation (2) has three real roots $u_1(t) < u_2(t) < u_3(t)$. If $t > t_*$ then $D > 0$ and the cubic equation (2) has one real root and two complex conjugate roots. At $t = t_*$ the roots $u_1(t)$ and $u_2(t)$ coalesce $u_1(t_*) = u_2(t_*) = u_*$.

When $t < t_*$ it is possible to construct a real formal solution of the equation (1):

$$u(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{2k} \overset{2k}{u}(t) \tag{3}$$

by taking any of roots $u_j(t)$ as a leading term $\overset{0}{u}(t)$. These three formal solutions are slowly varying equilibrium positions for the equation (1).

Consider an equation

$$\varepsilon^2 v'' + \left(6u_j^2(t) + t\right) v = 0, \qquad j = 1,2,3, \tag{4}$$

which describe small perturbations of the leading term $u_j(t)$ of the asymptotic expansions (3). If $j = 2$, then $6u_2^2(t) + t < 0$, so that the equation (4) has one exponentially growing solution and one exponentially decreasing solution. Hence corresponding asymptotic expansion is unstable with respect to small perturbation of the leading term. Otherwise if $j = 1, 3$ then the coefficients in the equation (4) are positive and the formal solution (3) with $u_j(t)$ as the leading term of the asymptotics is stable.

When $t = t_*$ the roots $u_1(t)$ and $u_2(t)$ coalesce: $u_1(t_*) = u_2(t_*) = u_*$ and when $t > t_*$, there exists only one equilibrium $u_3(t)$. Therefore t_* is the bifurcation point for the asymptotic solution (3) in case the leading term $\overset{0}{u}(t) \equiv u_1(t)$.

Our proposal is to construct a smooth asymptotic solution of the equation (1) with the leading term $u_1(t)$ when $t < t_*$ on a segment $[t_* - a, t_* + a]$, $a = \text{const} > 0$.

3 The main results

Here we describe a smooth asymptotic solution constructed in this work. Following by V P Maslov [25] we will use the words "asymptotic solution with respect to mod $(O(\varepsilon^\alpha))$", namely, a function is said to be an asymptotic solution of mod $(O(\varepsilon^a))$ of the equation (1) if after its substitution into this equation the latter is satisfied up to the terms of the order $O(\varepsilon^a)$.

When $t_* - a \le t < t_*$, $(a = \text{const} > 0)$ and $(t_* - t)\varepsilon^{-4/5} \gg 1$ the asymptotic solution with respect to mod $\left(O\left(\varepsilon^6\right) + O\left(\varepsilon^6\left(t - t_*\right)^{-13/2}\right)\right)$ has the form

$$u(t,\varepsilon) = u_1(t) + \varepsilon^2 \frac{-2tu_1(t)}{(6u_1^2(t) + t)^4} + \varepsilon^4 \overset{2}{u}(t). \tag{5}$$

The last term of the formal asymptotic solution as $t \to t_* - 0$ can be written as:

$$\overset{2}{u}(t) = O\left((t - t_*)^{-9/2}\right).$$

When $|t - t_*| \ll 1$ the asymptotic solution is defined by two different types of the formal asymptotic expansions. The first one has the form

$$u(t,\varepsilon) = u_* + \varepsilon^{2/5} \overset{0}{v}(\tau) + \varepsilon^{4/5} \overset{1}{v}(\tau). \tag{6}$$

Here the variable τ is defined by the formula $\tau = (t - t_*)\varepsilon^{-4/5}$, the function $\overset{0}{v}(\tau)$ is defined as the solution of the equation Painlevé-1:

$$\frac{d^2\overset{0}{v}(\tau)}{d\tau^2} + 6u_*\overset{0}{v}{}^2 + u_*\tau = 0,$$

with the pure algebraic asymptotic behavior as $\tau \to -\infty$:

$$\overset{0}{v}(t) = -\sqrt{-\frac{\tau}{6}} + O\left(\tau^{-2}\right).$$

The formula (6) is asymptotic solution with respect to mod $\left(O\left(\varepsilon^{8/5}\tau^2\right) + O\left(\varepsilon^{8/5}\right)\right)$ as $1 \ll -\tau \ll \varepsilon^{-4/5}$.

The function $\overset{0}{v}(\tau)$ has poles of second order at some points τ_k, $k = 1, 2, \dots$ (see, e.g. [26]):

$$\overset{0}{v}(\tau) = -\frac{1}{u_*(\tau - \tau_k)^2} + O\left(\tau_k(\tau - \tau_k)^2\right).$$

Near the poles the last term of the asymptotic solution can be written as

$$\overset{1}{v}(\tau) = O\left((\tau - \tau_k)^{-4}\right), \qquad \text{as} \quad \tau \to \tau_k.$$

The expansion (6) is suitable at $\varepsilon^{-1/5}|\tau - \tau_k| \gg 1$. The formula (6) is asymptotic solution with respect to mod $\left(O\left(\varepsilon^{8/5}\right) + O\left(\varepsilon^{8/5}\tau^2\right) + O\left(\varepsilon^{8/5}\tau_k\left(\tau - \tau_k\right)^{-8}\right)\right)$.

As $\tau \to \infty$ the main term of the asymptotics (6) is

$$\overset{0}{v}(\tau, \varepsilon) = \sqrt{\tau}\wp(s, g_2, g_3) + O\left(\tau^{-\gamma}\right), \qquad \gamma = \text{const} > 0. \tag{7}$$

Here

$$s = \frac{4}{5}\tau^{5/4} + \overset{0}{\sigma}(\chi), \qquad \text{where} \quad \chi = \varepsilon^{2/5}\frac{5}{7}\tau^{7/4},$$

the phase shift $\overset{0}{\sigma}$ is defined in Section 5.4.1 by a formula (36). The parameter of the Weierstrass elliptic function $g_2 = -2u_*$. The second parameter g_3 is defined by a solution of an equation (see, e.g. [11]):

$$\text{Re} \int_\Gamma d\lambda\, \omega = 0,$$

where Γ is any circle on an algebraic curve $\omega^2 = \lambda^3 + \lambda/2 - g_3/4$. The last term of the asymptotics (6) has the form:

$$\overset{1}{v}(\tau) = O(\tau) + O\left(\frac{\tau}{(\tau - \tau_k)^4}\right) \qquad \text{as} \quad \tau \to \infty \quad \text{and} \quad \tau \neq \tau_k.$$

The formula (6) is asymptotic solution with respect to mod$\Big(O\left(\varepsilon^{8/5}\right) + O\left(\varepsilon^{8/5}\tau^{3/2}\right) + O\left(\varepsilon^{8/5}\tau_k^{3/2}(\tau - \tau_k)^{-8}\right)\Big)$. The expansion (6) is suitable as $\tau \ll \varepsilon^{-4/5}$ and $\varepsilon^{-1/5}|\tau - \tau_k|\tau_k^{-1/4} \gg 1$.

The second one, which is valid in the neighborhoods $|\tau-\tau_k||\tau_k|^{1/5}\ll 1$ of poles τ_k of the function $\overset{0}{v}(\tau)$, reads as

$$u(t,\varepsilon)=u_*+\overset{0}{w}(\theta_k)+\varepsilon^{4/5}\overset{1}{w}(\theta_k), \tag{8}$$

where $\theta_k=(\tau-\tau_k)\varepsilon^{-1/5}+\varepsilon^{1/5}\overset{1}{\theta}_k$. The phase shift $\overset{1}{\theta}_k$ is defined by a formula (22). The function $\overset{0}{w}(\theta_k)$ is defined by the formula:

$$\overset{0}{w}(\theta_k)=-\frac{16u_*}{4+16u_*^2\theta_k^2}.$$

The last term of the formal asymptotics (8) at $|\theta|\to\infty$ can be written as

$$\overset{1}{w}(\theta_k)=O\left(\theta_k^2|\tau_k|\right).$$

The formula (8) is asymptotic solution with respect to mod $\left(O\left(\varepsilon^{8/5}\right)+O\left(\varepsilon^{8/5}\theta_k^4\tau_k^2\right)\right)$.

When $(t-t_*)\varepsilon^{-2/3}\gg 1$ and $t<t_*+a$ the asymptotic solution with respect to mod $\left(O\left(\varepsilon^2\right)+O\left(\varepsilon^2(t-t_*)^{-3}\right)\right)$ has the fast oscillating behavior:

$$u(t,\varepsilon)=\overset{0}{U}(t_1,t)+\varepsilon\overset{1}{U}(t_1,t). \tag{9}$$

Here the last term of the asymptotics (9) at $t\to t_*+0$ can be written as

$$\overset{1}{U}(t_1,t)=O\left((t-t_*)^{-3/2}\right).$$

The leading term of the asymptotic solution satisfies the Cauchy problem:

$$(S')^2\left(\partial_{t_1}\overset{0}{U}\right)^2=-\overset{0}{U}{}^4-t\overset{0}{U}{}^2+2\overset{0}{U}+E(t),\qquad \overset{0}{U}|_{t_1=0}=u_*.$$

Here $t_1=S(t)/\varepsilon+\phi(t)$. The function $E(t)$ is defined by the equation

$$I_0\equiv 2\int_{\beta(t)}^{\alpha(t)}\sqrt{-x^4-tx^2+2x+E(t)}\,dx=2\pi,$$

where $\alpha(t)$ and $\beta(t)$ ($\alpha(t)>\beta(t)$) are two real roots of the equation $-x^4-tx^2+2x+E(t)=0$, other roots of this equation are complex.

The phase function $S(t)$ is the solution of the Cauchy problem:

$$T=S'\sqrt{2}\int_{\beta(t)}^{\alpha(t)}\frac{dx}{\sqrt{-x^4-tx^2+2x+E(t)}},\qquad S|_{t=t_*}=0.$$

Where T is the constant defined by the formula

$$T=\frac{\sqrt{2}C_*(k)}{2|u_*|^{1/2}}\left(\frac{3}{6-2k^2}\right)^{1/4},$$

where $k \approx 0.463$ is the unique solution of the equation

$$\int_0^\infty dy \frac{-ky + k^2 + 1}{[(y-k)^2+1]^{5/2}} y^{5/2} = 0,$$

and

$$C_*(k) = \int_0^\infty \frac{dy}{\sqrt{y\,[(y-k)^2+1]}}.$$

The phase shift $\phi(t)$ is defined by an equation (see [27]):

$$\frac{\partial_E I_0}{\partial_E S'} \phi' = a = \text{const}.$$

Remark 1. In this work the constant a remains out side of our analysis. Its value may be defined by using the monodromy-preserve method for the Painlevé-2 equation [28].

Remark 2. The domains of validity of the asymptotic solution (5) and the asymptotic solution (6) intersect, so that these expansions match. The solution of the Painlevé-1 equation which defines the asymptotics (6) has infinite sequence of the poles τ_k, $k = 1, 2, \dots$. Near all of these poles we match the asymptotic solutions (6) and (8). As the number of the pole $k \to \infty$ the domain of validity of this complicated combine asymptotics (6) and (8) intersects with the domain of validity for the fast oscillating asymptotic solution (9). Its allows to match the sandwiched asymptotics with the fast oscillating asymptotic solution.

4 The outer algebraic asymptotics

The algebraic asymptotic solution (5) of the equation (1) is constructed here. This asymptotic solution is suitable when $t < t_*$ and asymptotic behavior of this solution is investigated as $t \to t_* - 0$.

We construct the asymptotic solution of the equation (1) as:

$$u(t,\varepsilon) = \overset{0}{u}(t) + \varepsilon^2 \overset{1}{u}(t) + \varepsilon^4 \overset{2}{u}(t) + \cdots. \tag{10}$$

Let's formulate the result of this section. The asymptotic solution (5) with respect to mod $\left(O\left(\varepsilon^6 (t-t_*)^{-13/2}\right)\right)$, where $\overset{0}{u}(t) \equiv u_1(t)$ is least of the solutions of the equation (2), is suitable when $(t_* - t)\varepsilon^{-4/5} \gg 1$ and $t > t_* - a$, where $a = \text{const} > 0$.

4.1 Constructing the algebraic asymptotic solution

Let's obtain the coefficients of the asymptotics (10). Substituting the ansatz (10) into the equation (1) and equating coefficients at identical powers of ε we find the sequence of the formulas for $\overset{k}{u}(t)$, $k = 0, 1, 2, \dots$.

$$2\overset{0}{u}{}^3(t) + t\overset{0}{u}(t) = 1, \qquad \left(6\overset{0}{u}{}^2(t) + t\right)\overset{1}{u}(t) = -\overset{0}{u}{}''(t),$$

$$\left(6\overset{0}{u}{}^2(t) + t\right)\overset{2}{u}(t) = -6\overset{0}{u}(t)\overset{1}{u}{}^2(t) - \overset{1}{u}{}''(t).$$

The cubic equation for $\overset{0}{u}(t)$ when $t < t_*$ has three real roots $u_1(t) < u_2(t) < u_3(t)$. As the leading term of asymptotic expansion (10) we choose $u_1(t)$. The second derivative of $\overset{0}{u}(t)$ has the form:

$$\overset{0}{u}'' = \left(\frac{\overset{0}{u}}{6\overset{0}{u}^2 + t}\right)' = \frac{2t\overset{0}{u}^2(t)}{\left(6\overset{0}{u}^2(t) + t\right)^3}.$$

This allows to obtain the formula for $\overset{1}{u}(t)$.

It is easy to get the expressions for the following terms of the asymptotic solution (10). In an explicit form they are not adduced here, however, it is important to note, that the power of the denominator $\left(6\overset{0}{u}^2(t) + t\right)$ in the coefficients of the asymptotics grows with each next step. The n-th term of the asymptotic expansion as $\left(6\overset{0}{u}^2(t) + t\right) \to 0$ has the form

$$\overset{n}{u}(t) = O\left(\left(6\overset{0}{u}^2(t) + t\right)^{-5n+1}\right). \tag{11}$$

Let's write the asymptotic behavior of the asymptotic expansion (10) as $t \to t_*$. For this purpose we shall calculate the asymptotics of the expression $\left(6\overset{0}{u}^2(t) + t\right)$:

$$\left(6\overset{0}{u}^2(t) + t\right)\Big|_{t\to t_*} = -2u_*\sqrt{6}\sqrt{t_* - t} + \frac{2}{3}(t_* - t) - \frac{5}{9\sqrt{6}u_*}(t - t_*)^{3/2} + O\left((t_* - t)^2\right).$$

Using this formula and $\overset{0}{u}(t)$, $\overset{1}{u}(t)$ we obtain:

$$u(t,\varepsilon) = u_* - \frac{1}{\sqrt{6}}\sqrt{t_* - t} + \frac{1}{18u_*}(t_* - t) + \varepsilon^2\left[-\frac{1}{3\,2^{10/3}}(t_* - t)^{-2} - O\left((t_* - t)^{-3/2}\right)\right]$$
$$+ O\left(\varepsilon^4(t_* - t)^{-9/2}\right) + O\left((t_* - t)^{3/2}\right).$$

4.2 The domain of validity of the algebraic asymptotic solution

The domain of validity for this expansion as $t \to t_* - 0$ is determined from the relation $\varepsilon^2\overset{n+1}{u}(t)/\overset{n}{u}(t) \ll 1$. It follows from the formula (11), that the expansion (10) is suitable when $(t_* - t)\varepsilon^{-4/5} \gg 1$.

Evaluate the residual which is obtained when one substitutes the asymptotic solution (5) into the equation (1)

$$F(t,\varepsilon) = -\varepsilon^6\left(\overset{2}{u}'' + 2\overset{0}{u}\overset{1}{u}\overset{2}{u} + 6\overset{1}{u}^3\right) - \varepsilon^8\left(6\overset{0}{u}\overset{2}{u}^2 + \overset{1}{u}^2\overset{2}{u}\right) - \varepsilon^{12}\overset{2}{u}^3.$$

Using the asymptotic behaviour of the $\overset{k}{u}$, $k = 0,1,2$ as $t \to t_* - 0$ one can obtain

$$F(t,\varepsilon) = O\left(\varepsilon^6(t - t_*)^{-13/2}\right).$$

5 The inner asymptotics

In this section the asymptotic expansions of solution of (1) which are suitable in the small neighborhood of a point t_* are constructed. By following terminology of the matching method [29], they are called "the inner asymptotic expansions".

5.1 First inner expansion

It follows from the consideration of the validity of the outer expansion, made in the previous section, that it is natural to make the following scaling of variables:

$$(u-u_*)=\varepsilon^{2/5}v, \qquad (t-t_*)=\varepsilon^{4/5}\tau.$$

As a result we write the equation (1) as

$$\frac{d^2v}{d\tau^2}+6u_*v^2+u_*\tau=-\varepsilon^{2/5}\left(\tau v+2v^3\right). \tag{12}$$

In the limit as $\varepsilon\to 0$ we obtain the equation Painlevé-1. This asymptotic reduction is known as one of the scaling limits for the Painlevé-2 equation [30] (see also [3, 19]).

A solution of this equation has the asymptotic expansion as $\tau\to-\infty$:

$$v(\tau,\varepsilon)=\left(-\sqrt{-\tau/6}+\frac{1}{48u_*\tau^2}+\frac{49}{768\sqrt{6}u_*^2(-\tau)^{9/2}}+\cdots\right)$$
$$+\varepsilon^{2/5}\left(-\frac{\tau}{18u_*}+\frac{1}{144\sqrt{6}(-\tau)^{3/2}}+\cdots\right)+O\left(\varepsilon^{4/5}\tau^{3/2}\right).$$

The asymptotic solution of the equation (12) we build as:

$$v(\tau,\varepsilon)=\overset{0}{v}(\tau)+\sum_{n=1}^{\infty}\varepsilon^{2n/5}\overset{n}{v}(\tau), \tag{13}$$

where the function $\overset{0}{v}(\tau)$ is the solution of the Painlevé-1 equation.

Here it is shown, that the asymptotic solution (13) is suitable in the neighborhood of infinity (when $-\tau\ll\varepsilon^{-4/5}$) and in the neighborhood of the poles for the function $\overset{0}{v}(\tau)$: $(\tau-\tau_k)\varepsilon^{-1/5}\gg 1$.

5.1.1 Asymptotic behaviour as $\tau\to-\infty$

The coefficients of the asymptotics are calculated from the matching condition for the asymptotic expansion (10) as $t\to t_*$ and the expansion (13) as $\tau\to-\infty$. In particular, $\overset{0}{v}(\tau)$ has the algebraic asymptotics:

$$\overset{0}{v}(\tau)|_{\tau\to-\infty}=-\sqrt{-\tau/6}+\frac{1}{48u_*\tau^2}+\frac{49}{768\sqrt{6}u_*^2(-\tau)^{9/2}}+O\left(\tau^{-7}\right). \tag{14}$$

In the book [26] it is shown, that there exists the solution of the Painlevé-1 equation with the asymptotics (14). The data of a monodromy for the solution of the Painlevé-1 equation with the asymptotics (14) are calculated in the work [9].

The first correction in the asymptotics (13) satisfies the equation

$$\frac{d^2\overset{1}{v}}{d\tau^2} + 12u_*\overset{0}{v}\overset{1}{v} = -\tau\overset{0}{v} - 2\overset{0}{v}{}^3. \tag{15}$$

The asymptotics of the solution for this equation as $\tau \to -\infty$ has the form

$$\overset{1}{v}(\tau) = -\frac{\tau}{18u_*} + \frac{1}{144\sqrt{6}(-\tau)^{3/2}} + O\left(\tau^{-4}\right).$$

Asymptotics of the higher corrections is constructed by ordinary way. The n-th correction as $\tau \to -\infty$ has an order:

$$\overset{n}{v}(\tau) = O\left((-\tau)^{(n+1)/2}\right).$$

5.1.2 Validity of the asymptotic solution as $\tau \to -\infty$

The requirement of validity for the asymptotics is $\varepsilon^{2/5}\overset{1}{v}/\overset{0}{v} \ll 1$. It reduces to the condition $(-\tau) \ll \varepsilon^{-4/5}$.

The residual of the asymptotic solution (6) has the form:

$$F(\tau,\varepsilon) = -\varepsilon^{8/5}\left(6u_*\overset{1}{v}{}^2 + \tau\overset{1}{v} + 6\overset{1}{v}\overset{0}{v}{}^2\right) - \varepsilon^{10/5}\overset{0}{v}\overset{1}{v}{}^2 - \varepsilon^{12/5}\overset{1}{v}{}^3.$$

Using the asymptotic behaviour of $\overset{k}{v}$, $k = 0, 1$ as $(-\tau) \ll \varepsilon^{-4/5}$ one can obtain

$$F(\tau,\varepsilon) = O\left(\varepsilon^{8/5}\left(\tau^2\right)\right).$$

5.1.3 Asymptotic behaviour near the poles

The function $\overset{0}{v}(\tau)$ has the poles when $\tau \in (-\infty,\infty)$. Let's denote these poles by τ_k. In the neighborhood of the pole $\tau \to \tau_k \pm 0$ the function $\overset{0}{v}(\tau)$ is defined by the converging power series (see e.g. [26])

$$\overset{0}{v}(\tau) = -\frac{1}{u_*(\tau-\tau_k)^2} + \frac{\tau_k u_*}{10}(\tau-\tau_k)^2 + \frac{u_*}{6}(\tau-\tau_k)^3 + c_k(\tau-\tau_k)^4 + O\left((\tau-\tau_k)^5\right). \tag{16}$$

The constants τ_k and c_k are the parameters of this solution. In the review [11] it is noted, that the problem on the connection between the asymptotics of this solution at infinity and the constants τ_k and c_k is not investigated yet. The points of the poles τ_k and appropriate constants c_k can be obtained with the help of the numerical calculation using the given asymptotics at infinity (14).

The asymptotics of $\overset{1}{v}$ as $\tau \to \tau_k \pm 0$ may be written as a sum of a certain solution of a nonhomogeneous linearized Painlevé-1 equation $\overset{1}{v}_c(\tau)$

$$\overset{1}{v}_c(\tau) = -\frac{1}{(\tau-\tau_k)^4} + \frac{\tau_k}{120u_*} - \frac{1}{24u_*}(\tau-\tau_k) + \frac{9c_k}{10u_*^2}(\tau-\tau_k)^2 + O\left((\tau-\tau_k)^5\right),$$

and two solutions of a homogeneous linearized equation $v_1(\tau)$, $v_2(\tau)$:

$$v_1(\tau) = \frac{1}{(\tau-\tau_k)^3} + \frac{\tau_k u_*^2}{10}(\tau-\tau_k) + \frac{u_*^2}{5}(\tau-\tau_k)^2 + 2c_k u_*(\tau-\tau_k)^3 + O\left((\tau-\tau_k)^5\right),$$

$$v_2(\tau) = (\tau-\tau_k)^4 + O\left((\tau-\tau_k)^8\right).$$

Thus:

$$\overset{1}{v} = \overset{1}{v}_c(\tau) + \overset{1}{a}{}^{\pm}_k v_1(\tau) + \overset{1}{b}{}^{\pm}_k v_2(\tau). \tag{17}$$

Here $\overset{1}{a}{}^{\pm}_k$ and $\overset{1}{b}{}^{\pm}_k$ are constants.

Higher corrections have the same form:

$$\overset{n}{v}(\tau) = \overset{n}{v}_c(\tau) + \overset{n}{a}{}^{\pm}_k v_1(\tau) + \overset{n}{b}{}^{\pm}_k v_2(\tau),$$

where

$$\overset{n}{v}_c(\tau) = O\left((\tau-\tau_k)^{-2(n+1)}\right), \qquad \text{as} \quad \tau \to \tau_k.$$

5.1.4 Validity of the asymptotic solution as $\tau \to \tau_k$

By using the asymptotics (16) and (17) it is easy to see, that the asymptotic expansion (13) is suitable at

$$\varepsilon^{-1/5}|\tau-\tau_k| \gg 1.$$

The residual of the asymptotic solution as $\tau \to \tau_k$ when $\varepsilon^{-1/5}|\tau-\tau_k| \gg 1$ is

$$F(\tau,\varepsilon) = \varepsilon^{8/5} O\left(\frac{\tau_k}{(\tau-\tau_k)^4}\right) + \varepsilon^{8/5} O\left((\tau-\tau_k)^{-8}\right).$$

5.2 Second inner expansion

For the construction of the uniform asymptotics in the neighborhood of the pole of the function $\overset{0}{v}$ it is necessary to make one more scaling of the independent variable and the function (see [3]):

$$(\tau-\tau_k) = \varepsilon^{1/5}\theta, \qquad \varepsilon^{-2/5} v = w.$$

For function w we obtain the equation:

$$\frac{d^2 w}{d\theta^2} + 6u_* w^2 + 2w^3 = -\varepsilon^{4/5}\tau_k(u_* + w) - \varepsilon\theta(u_* + w). \tag{18}$$

The solution of this equation has following asymptotic expansion as $\theta \to -\infty$:

$$
\begin{aligned}
w = &-\frac{1}{u_*\theta^2} + \frac{1}{4u_*^3\theta^4} + O\left(\theta^{-6}\right) + \varepsilon^{1/5}\left(\frac{\overset{1}{a}{}_k^-}{\theta^3} + O\left(\theta^{-4}\right)\right)\\
&+ \varepsilon^{2/5}\left(\left(\overset{1}{a}{}_k^-\right)^2\frac{1}{u_*\theta^4} + O\left(\theta^{-6}\right)\right) + \varepsilon^{3/5}\left(\overset{2}{a}{}_k^-\frac{1}{\theta^3} + O\left(\theta^{-4}\right)\right)\\
&+ \varepsilon^{4/5}\left(-\frac{120\tau_k}{u_*} + \frac{\tau_k u_*}{10}\theta^2 + O\left(\theta^{-1}\right)\right)\\
&+ \varepsilon\left(\frac{u_*\theta^3}{6} + \frac{\theta}{24u_*} + \overset{1}{a}{}_k^-\tau_k u_*^2\theta + O(1)\right)\\
&+ \varepsilon^{6/5}\left(\frac{9c_k}{10u_*^2}\theta^2 + c_k\theta^4 + \frac{\overset{1}{a}{}_k^- u_*^2}{5}\theta^2 + O\left(\theta^1\right)\right) + O\left(\varepsilon^{7/5}\theta^5\right)\\
&+ \varepsilon^{8/5}\left(\overset{1}{b}{}_k^-\theta^4 - \frac{\tau_k^2u_*^3}{300}\theta^6 + O\left(\theta^2\right)\right) + O\left(\varepsilon^{9/5}\right).
\end{aligned}
\tag{19}
$$

This long asymptotic formula shows, that the constant $\overset{1}{b}{}_k^-$ appears only in the correction of an order $\varepsilon^{8/5}$. If we want to construct the first correction of the asymptotics for the first inner expansion after the pole τ_k, we must construct the correction in order $\varepsilon^{8/5}$ for the second inner expansion.

It is convenient to include a time shift depended on ε into the main term, and construct the asymptotic expansion depended on a new time variable:

$$
\theta_k = \theta + \varepsilon^{1/5}\overset{1}{\theta}_k + \varepsilon^{3/5}\overset{2}{\theta}_k,
$$

where $\overset{n}{\theta}_k = \mathrm{const}$.

We search the asymptotic expansion for the solution of this equation as a segment of an asymptotic series

$$
w(\theta_k,\varepsilon) = \overset{0}{w}(\theta_k) + \varepsilon^{4/5}\overset{1}{w}(\theta_k) + \varepsilon\overset{2}{w}(\theta_k) + \varepsilon^{6/5}\overset{3}{w}(\theta_k) + \varepsilon^{8/5}\overset{4}{w}(\theta_k). \tag{20}
$$

In this case the equation for the $w(\theta_k,\varepsilon)$ looks like:

$$
\begin{aligned}
\frac{d^2w}{d\theta_k^2} + 6u_*w^2 + 2w^3 = &-\varepsilon^{4/5}\tau_k(u_*+w) - \varepsilon\theta_k(u_*+w)\\
&+ \varepsilon^{6/5}\overset{1}{\theta}_k(u_*+w) + \varepsilon^{8/5}\overset{2}{\theta}_k(u_*+w) + \cdots.
\end{aligned}
$$

It is shown here, that the asymptotic solution (8) is the formal asymptotic solution of the equation (18) with respect to mod $\left(O\left(\varepsilon^{8/5}\tau_k^2\theta^4\right) + O\left(\varepsilon^{9/5}\tau_k\theta^5\right) + O\left(\varepsilon^2\theta^6\right)\right)$ when $|\theta\tau_k^{1/5}| \ll \varepsilon^{1/5}$.

The solution of the equation for the leading term of the asymptotics (20) is defined by the asymptotics as $\tau \to \tau_k$ of the asymptotic expansion (13), which is outer with respect to (20). This solution has the form

$$\overset{0}{w}(\theta_k) = -\frac{16u_*}{4 + 16u_*^2\theta_k^2}. \tag{21}$$

The constants $\overset{n}{\theta}_k$ are defined by asymptotics of the function $w(\theta, \varepsilon)$. Using the formula (19) we obtain:

$$\overset{n}{\theta}_k = \frac{u_*}{2}\overset{n}{a}{}_k^-, \qquad n = 1, 2, \dots. \tag{22}$$

The corrections in the expansion (20) satisfy the linearized equations

$$\frac{d^2\overset{1}{w}}{d\theta_k^2} + \left(12u_*\overset{0}{w} + 6\overset{0}{w}{}^2\right)\overset{1}{w} = -\tau_k\left(u_* + \overset{0}{w}\right),$$

$$\frac{d^2\overset{2}{w}}{d\theta_k^2} + \left(12u_*\overset{0}{w} + 6\overset{0}{w}{}^2\right)\overset{2}{w} = \theta_k\left(u_* + \overset{0}{w}\right),$$

$$\frac{d^2\overset{3}{w}}{d\theta^2} + \left(12u_*\overset{0}{w} + 6\overset{0}{w}{}^2\right)\overset{3}{w} = \overset{1}{\theta}_k\left(u_* + \overset{0}{w}\right),$$

$$\frac{d^2\overset{4}{w}}{d\theta^2} + \left(12u_*\overset{0}{w} + 6\overset{0}{w}{}^2\right)\overset{3}{w} = -6\overset{1}{w}{}^2\left(\overset{0}{w} + u_*\right) + \overset{2}{\theta}_k\left(u_* + \overset{0}{w}\right).$$

The expression for $\overset{0}{w}$ can be used to obtain two linearly independent solutions of the homogeneous equation for the corrections:

$$w_1 = \frac{8\theta_k}{\left(1 + 4u_*^2\theta_k^2\right)^2},$$

$$w_2 = \left[-\frac{1}{8} + 2u_*^2\theta_k^2 - u_*\theta_k^4 + \frac{2}{5}\theta_k^6 + \frac{2u_*^2}{7}\theta_k^8\right]\frac{1}{\left(1 + 4u_*^2\theta_k^2\right)^2}.$$

By using these solutions of the homogeneous equation it is easy to get the solutions of the nonhomogeneous equations for the corrections. The asymptotics of the corrections as $\theta \to \infty$ has the form:

$$\overset{1}{w} = \frac{\tau_k u_*}{10}\theta_k^2 + \frac{\tau_k}{120u_*} + \frac{\tau_k}{120}\theta_k^{-2} - \frac{1}{160u_*^2}\theta_k^{-4} + O\left(\theta_k^{-6}\right),$$

$$\overset{2}{w} = \frac{u_*}{6}\theta_k^3 + \frac{1}{24u_*}\theta_k + O\left(\theta_k^{-5}\right),$$

$$\overset{3}{w} = \frac{1}{56u_*^2} c_k \theta_k^4 + O\left(\theta_k^2\right),$$

$$\overset{4}{w} = -\frac{u_*^3 \tau_0^2}{300} \theta_k^6 + \left(\frac{1}{56u_*^2} \overset{k}{b}{}_k^- - \frac{11u_* \tau_k^2}{2100} \right) \theta_k^4 + O\left(\theta_k^2\right).$$

It is important to note that the leading term of the asymptotics of $\overset{3}{w}$ as $\theta_k \to \pm\infty$ is the same. This term defines the constants c_k and hence the solution of the Painlevé-1 equation before and after the pole. We have the same value of the constant c_k as $\theta_k \to -\infty$ and $\theta_k \to \infty$ and then we have the same asymptotic solution of Painlevé-1 equation before and after the pole τ_k.

An opposite result takes place for the coefficient $\overset{4}{w}$ in order θ_k^4 as $\theta_k \to \infty$. This coefficient is changed. It is equal $\overset{1}{b}{}_k^- / \left(56u_*^2\right)$ as $\theta_k \to -\infty$ and $\overset{1}{b}{}_k^- / \left(56u_*^2\right) - 11u_* \tau_k^2/2100$ as $\theta_k \to \infty$.

5.2.1 Validity of second internal asymptotic solution

Using the asymptotics for the corrections and the leading term, we obtain, that the expansion (20) is suitable when $|\theta\tau_k^{1/5}| \ll \varepsilon^{-1/5}$. On the other hand, the expansion (13) is suitable at $|\theta| \gg 1$. Hence, the domains of the applicability for the expansions (13) and (20) are intersected at realization of the condition $|\tau_k| \ll \varepsilon^{-1}$. If we take into account also the requirement of fitness of the asymptotic expansion (20), then get the restriction $|\tau_k| \ll \varepsilon^{-4/5}$. From this inequality it follows, that in this section the formal asymptotic expansions to be suitable when $|t - t_*| \ll 1$ are constructed.

We calculate the residual of second internal asymptotic expansion using the asymptotic behaviour of $\overset{k}{w}$, $k = 0, 1, 2, 3, 4$:

$$F(\theta, \tau_k, \varepsilon) = \varepsilon^{9/5} O\left(\tau_k \theta^5\right), \qquad (23)$$

when $|\theta||\tau_k|^{1/5} \ll \varepsilon^{-1/5}$.

5.3 Dynamics in the internal layer

Using the asymptotic expansion of the second inner expansion as $\theta \to \infty$ and the first inner expansion as $\tau \to \tau_k + 0$ we find that the first inner expansion after the pole has the form (13) where

$$\overset{n}{v}(\tau) = \overset{n}{v}_c(\tau) + \overset{n}{a}{}_k^+ v_1(\tau) + \overset{n}{b}{}_k^+ v_2(\tau).$$

Here

$$\overset{n}{a}{}_k^+ = \overset{n}{a}{}_k^-, \qquad \overset{n}{b}{}_k^+ = \overset{n}{b}{}_k^- + \overset{n}{\Delta}_k, \qquad n = 1, 2.$$

The shift $\overset{n}{\Delta}_k$ may be calculated from the asymptotics of the second inner expansion as $\theta \to \infty$, for $n = 1$ we have obtained:

$$\overset{1}{\Delta}_k = -\frac{22u_*^3 \tau_k^2}{75}.$$

Thus a behaviour of the asymptotic solution in the internal layer is combined by the first and the second inner asymptotic expansions.

5.4 The asymptotics of the inner expansions as $\tau \to \infty$

In the above sections we demonstrate the asymptotic behaviour of the asymptotic solution at $\tau \to -\infty$ and near the poles of the solution for the Painlevé-1 equation. Below we study the asymptotic behaviour of the solution as $\tau \to \infty$.

The regular asymptotic expansion on ε is constructed in the previous section concerning the first inner asymptotic expansion. This asymptotics is not valid as large τ. To use the first inner asymptotic expansion as $\tau \to \infty$ we must modulate the parameters of the solution of the Painlevé-1 equation and cancel secular terms in the first and the second corrections in the asymptotic expansion (13) using singular perturbation theory. Instead of modulating the parameters of the solution of the Painlevé-1 equation it is more convenient to study the modulation equation for parameters of the main term of the asymptotics for the solution of the Painlevé-1 equation as $\tau \to \infty$. This study will be developed in this subsection.

5.4.1 Asymptotic behaviour of first inner expansion

The elliptic asymptotics of the solution for the Painlevé-1 equation as $\tau \to \infty$ was obtained by P Boutroux [12]. Here we are interested in a connection formula for the solution of the Painlevé-1 equation. Namely we have the asymptotic behaviour of the solution as $\tau \to -\infty$ and we need the asymptotic behaviour of the same solution as $\tau \to \infty$. The Painlevé-1 equation is integrable by the monodromy-preserving method [28]. If the monodromy data are known, then the solution of the Painlevé-1 equation is uniquely defined. In the correspondence with [9], in our case the monodromy data are constants s_2 and s_3 and these constants are equal to zero. The asymptotics of the function $\overset{0}{v}(\tau)$ outside of the poles has the form (see e.g. [11])

$$\overset{0}{v} = \sqrt{\tau}\,\overset{0}{\rho}(\sigma) + O\left(\tau^{-\gamma}\right), \tag{24}$$

where $\gamma > 0$ is some constant, the function $\overset{0}{\rho}(\sigma)$ is determined by the Weierstrass elliptic function

$$\overset{0}{\rho}(\sigma) = -\wp(\sigma, g_2, g_3)/u_*. \tag{25}$$

The phase function $\sigma = \frac{4}{5}\tau^{5/4}$. It is important to note, that in the formula (24) the shift of the phase function σ is equal to zero. A parameter is $g_2 = -2u_*$ and a parameter g_3 is defined as a solution of the equation:

$$\mathrm{Re}\int_\gamma \omega\, d\lambda = 0,$$

where γ is any circle on an algebraic curve: $\omega^2 = \lambda^3 + \lambda/2 - g_3/4$.

We are interested in the asymptotic solution of the perturbed Painlevé-1 equation (12). Therefore the asymptotics of the works [19, 11] for unperturbed Painlevé-1 equation are the main term of our asymptotics on ε. Substitute into (12):

$$v(\tau, \varepsilon) = \sqrt{\tau}\rho(\sigma, \varepsilon), \tag{26}$$

where $\sigma = 4\tau^{5/4}/5$. As a result we obtain an equation:

$$\rho'' + 6u_*\rho^2 + u_* = -\left(\frac{5\sigma}{4}\right)^{-1}\rho' + \left(\frac{5\sigma}{4}\right)^{-2}\rho - \varepsilon^{2/5}\left(\frac{5\sigma}{4}\right)^{2/5}\left(\rho + 2\rho^3\right). \tag{27}$$

Let us to construct the asymptotics of the perturbed equation as a segment of the asymptotic series:

$$\rho(\sigma,\varepsilon) = \overset{0}{\rho}(s) + \varepsilon^{2/5}\left(\frac{5\sigma}{4}\right)^{2/5}\overset{1}{\rho}(s) + \varepsilon^{4/5}\left(\frac{5\sigma}{4}\right)^{4/5}\overset{2}{\rho}(s), \tag{28}$$

where

$$s = \sigma + \overset{0}{\sigma}(\chi), \qquad \chi = \varepsilon^{2/5}\frac{5}{7}\left(\frac{5\sigma}{4}\right)^{7/5},$$

$\overset{0}{\sigma}$ is modulated phase shift and χ is once more slow variable.

Substitute the formula (28) into the equation (27). Then let's equate the coefficients with identical powers of ε. As a result we obtain the sequence of the equations:

$$\overset{0}{\rho}{}'' + 6u_*\overset{0}{\rho}{}^2 + u_* = \left[-\left(\frac{5}{4}\sigma\right)^{-1}\overset{0}{\rho}{}' + \frac{1}{4}\left(\frac{5}{4}\sigma\right)^{-2}\overset{0}{\rho}\right], \tag{29}$$

$$\begin{aligned}\overset{1}{\rho}{}'' + 12u_*\overset{0}{\rho}\overset{1}{\rho} = &-\left(\overset{0}{\rho} + 2\overset{0}{\rho}{}^3\right) - \overset{0}{\sigma}{}'\overset{0}{\rho}{}'' \\ &+\left[-\frac{1}{2}\left(\frac{5}{4}\sigma\right)^{-1}\overset{0}{\sigma}{}'\overset{0}{\rho}{}' - 2\left(\frac{5}{4}\sigma\right)^{-1}\overset{1}{\rho}{}' + \frac{7}{8}\left(\frac{5}{4}\sigma\right)^{-2}\overset{1}{\rho}\right],\end{aligned} \tag{30}$$

$$\begin{aligned}\overset{2}{\rho}{}'' + 12u_*\overset{0}{\rho}\overset{2}{\rho} = &-6u_*\overset{1}{\rho}{}^2 - \overset{1}{\rho}\left(1 + 6\overset{0}{\rho}{}^2\right) - \left(\overset{0}{\sigma}{}'\right)^2\overset{0}{\rho}{}'' - \overset{0}{\sigma}{}''\overset{0}{\rho}{}' - \overset{0}{\sigma}{}'\overset{1}{\rho}{}'' \\ &+\left(\frac{5}{4}\sigma\right)^{-1}L_1\left(\overset{0}{\rho},\overset{1}{\rho},\overset{2}{\rho}\right) + \left(\frac{5}{4}\sigma\right)^{-2}L_2\left(\overset{0}{\rho},\overset{1}{\rho},\overset{2}{\rho}\right).\end{aligned} \tag{31}$$

Here $L_{1,2}$ are linear operators.

Solutions of these equations may be represented as asymptotic expansions as $\sigma \to \infty$. We will assume that the corrections of these expansions are small as $\sigma \to \infty$. Therefore we will neglect the correction terms of this asymptotics and consider only the main terms of these asymptotics with respect to σ. The equations for the main terms $\overset{0}{\rho}_0$, $\overset{1}{\rho}_0$, $\overset{2}{\rho}_0$ have the form:

$$\overset{0}{\rho}{}_0'' + 6u_*\overset{0}{\rho}{}_0^2 + u_* = 0, \tag{32}$$

$$\overset{1}{\rho}{}_0'' + 12u_*\overset{0}{\rho}_0\overset{1}{\rho}_0 = -\left(\overset{0}{\rho}_0 + 2\overset{0}{\rho}{}_0^3\right) - \overset{0}{\sigma}{}'\overset{0}{\rho}{}_0'', \tag{33}$$

$$\overset{2}{\rho}_0'' + 12u_*\overset{0}{\rho}_0\overset{2}{\rho}_0 = -6u_*\overset{1}{\rho}{}_0^2 - \overset{1}{\rho}_0\left(1 + 6\overset{0}{\rho}{}_0^2\right) - \left(\overset{0}{\sigma}'\right)^2\overset{0}{\rho}_0'' - \overset{0}{\sigma}''\overset{0}{\rho}_0' - \overset{0}{\sigma}'\overset{1}{\rho}_0'' \tag{34}$$

The requirement of validity for the first inner asymptotics at $\tau \to \infty$ is $\varepsilon^{2n/5}\sigma^{2n/5}\overset{n}{\rho}/\overset{0}{\rho} \ll 1$, where $n = 1, 2$. We find the modulated equation for the $\overset{0}{\sigma}$ such that the first inner expansion is valid as large τ.

The solution of the equation for the main term is the function $\overset{0}{\rho}(s)$ which is defined by (25). The equations for the corrections is the Lame equations with external force. To write the solutions of these equations we will use two linear independent solutions of the Lame equation. Denote one of these solutions by

$$p_1(s) = \partial_s\overset{0}{\rho}_0.$$

We denote the second solution as $p_2(s)$. The solutions $p_1(s)$ and $p_2(s)$ are such that a wronskian:

$$W(p_1, p_2) = 1.$$

The second solution is aperiodic:

$$p_2(s + \Omega) = Cp_1(s) + p_2(s), \qquad \text{where} \quad C = \text{const} \neq 0.$$

The solution of the equation (33) can be written as:

$$\begin{aligned}\overset{1}{\rho}_0(s) = \overset{1}{A}_k p_1(s) + \overset{1}{B}_k p_2(s) + p_1(s)\int_{s_0}^{s} dz\left(-\overset{0}{\rho}_0(z) - 2\overset{0}{\rho}{}_0^3(z)\right)p_2(z) \\ - p_2(s)\int_{s_0}^{s} dz\left(-\overset{0}{\rho}_0(z) - 2\overset{0}{\rho}{}_0^3(z)\right)p_1(z) - \overset{0}{\sigma}'sp_1(s).\end{aligned} \tag{35}$$

Here $s_0 = s_k + \Omega/2$, where s_k is pole of the function $\overset{0}{\rho}_0(s)$ and Ω is a real period of the ρ. The $\overset{1}{A}_k$ and $\overset{1}{B}_k$ are constants.

The first correction $\overset{1}{\rho}$ is bounded as $s \in \mathbb{R}$ if

$$\overset{1}{B}_k = \frac{1}{C}\overset{0}{\sigma}' + \frac{1}{C}R.P.\int_0^{\Omega} dz\left(\overset{0}{\rho}_0(z) + 2\overset{0}{\rho}{}_0^3(z)\right)p_1(z).$$

In this formula the integral must be regularized. Namely:

$$R.P.\int_0^{\Omega} dz\left(\overset{0}{\rho}_0(z) + 2\overset{0}{\rho}{}_0^3(z)\right)p_1(z) = \text{res}_{r=0}\left[\frac{1}{r}\int_r^{\Omega-r} dz\left(\overset{0}{\rho}_0(z) + 2\overset{0}{\rho}{}_0^3(z)\right)p_1(z)\right].$$

Using a perturbation theory for the second order equation developed in works [32]–[27], we can show, that the solution for the $\overset{2}{\rho}$ has an order $O(\sigma)$, $\sigma \neq \sigma_k$, $k \in \mathbb{Z}$, if the function $\overset{0}{\sigma}$ is a solution of the differential equation: $\overset{0}{\sigma}'' = 0$. This equation and the equation for the $\overset{1}{B}_k$ allow to write the Cauchy problem for the $\overset{0}{\sigma}$ in the form:

$$\begin{aligned}&\overset{0}{\sigma}' = C\overset{1}{B}_k + R.P.\int_0^{\Omega} dz\left(\overset{0}{\rho}_0(z) + 2\overset{0}{\rho}{}_0^3(z)\right)p_2(z), \\ &\chi \in (\chi_k, \chi_{k+1}), \qquad \overset{0}{\sigma}|_{\chi=0} = 0,\end{aligned} \tag{36}$$

where

$$\chi_k = \varepsilon^{2/5}\frac{5}{7}\left(\frac{5\sigma_k}{4}\right)^{7/5}.$$

The constants $\overset{1}{A}_k$ and $\overset{1}{B}_k$ are defined by matching conditions of the asymptotics (26), (28) and (13) as $\tau_k \to \tau_k + 0$:

$$\overset{1}{A}_k = \frac{\overset{1}{a}{}_k^{+}}{\tau_k} - R.P.\int_0^{\Omega/2} dz\left(\overset{0}{\rho}_0(z) + 2\overset{0}{\rho}{}_0^3(z)\right)p_2(z),$$

$$\overset{1}{B}_k = \frac{u_*\overset{1}{b}{}_k^{+}}{14\tau_k} - R.P.\int_0^{\Omega/2} dz\left(\overset{0}{\rho}_0(z) + 2\overset{0}{\rho}{}_0^3(z)\right)p_1(z).$$

As a result we obtain the formula (7) for the main term of the asymptotics (28).

5.4.2 Validity of the first inner expansion as $\tau \to \infty$

Let us denote by

$$\overset{n}{v}(\tau) = \tau^{(n+1)/2}\overset{n}{\rho}(\sigma), \qquad n = 0, 1, 2.$$

Then the condition of the validity $\varepsilon^{2/5}\overset{n}{v}/\overset{0}{v} \ll 1$, $n = 1, 2$ fulfills as $\varepsilon^{2/5}\tau/\sqrt{\tau} \ll 1$ or as the same:

$$\tau \ll \varepsilon^{-4/5}.$$

Near the pole τ_n we obtain the asymptotics:

$$\overset{0}{v} = O\left(\frac{\sqrt{\tau}}{(\tau-\tau_k)^2}\right), \qquad \text{and} \qquad \overset{1}{v} = O\left(\frac{\tau}{(\tau-\tau_k)^4}\right).$$

Hence the first inner asymptotic expansion is suitable near the poles τ_k as

$$\varepsilon^{-1/5}(\tau-\tau_k)\tau_k^{-1/4} \gg 1.$$

The residual of the first inner expansion is

$$F(\tau,\varepsilon) = -\varepsilon^2\left(12u_*\overset{1}{v}\overset{2}{v} - 6\overset{0}{v}\overset{1}{v}{}^2 + 6\overset{0}{v}{}^2\overset{2}{v} + \tau\overset{2}{v}\right) - \varepsilon^{12/5}\left(2\overset{1}{v}{}^3 + 12\overset{0}{v}\overset{1}{v}\overset{2}{v} + 6u_*\overset{2}{v}{}^2\right)$$
$$+\,\varepsilon^{14/5}\left(6\overset{1}{v}{}^2\overset{2}{v} + 6\overset{0}{v}\overset{2}{v}{}^2\right) + \varepsilon^{16/5}6\overset{1}{v}\overset{2}{v}{}^2 + \varepsilon^{18/5}2\overset{2}{v}{}^3.$$

Using the results of the above section outside of the poles of $\overset{0}{v}$ one can obtain:

$$F = O\left(\varepsilon^2\tau^{5/2}\right) + O\left(\frac{\varepsilon^2\tau^{5/2}}{(\tau-\tau_k)^{10}}\right).$$

5.4.3 Validity of the second inner expansion as $\tau_k \to \infty$

In the second inner expansion the asymptotics as $\tau \to \infty$ corresponds to the asymptotics at $\tau_k \to \infty$. It is easy to see, that in this case the first correction grows. This growth limits the value τ_k, at which the asymptotics (20) is correct: $|\tau_k| \ll \varepsilon^{-4/5}$. One can use the formula (23) to obtain the residual of the second inner asymptotics as $\tau_k \to \infty$.

6 Fast oscillating asymptotics

6.1 The Kuzmak's approximation

In this section we apply formulas obtained in [31]–[34], [27] to the fast oscillating formal asymptotic solution of the Painlevé-2 equation. These formulas are usable when $t > t_*$.

The fast oscillating asymptotics is constructed as

$$u(t,\varepsilon) = \overset{0}{U}(t_1,t) + \varepsilon\overset{1}{U}(t_1,t) + \varepsilon^2\overset{2}{U}(t_1,t) + \cdots . \tag{37}$$

As the argument t_1 we use expression

$$t_1 = S(t)/\varepsilon + \phi(t),$$

where $S(t)$ and $\phi(t)$ are unknown functions.

The equations for the leading term and the corrections of the asymptotics (37) look like:

$$(S')^2\partial_{t_1}^2\overset{0}{U} + 2\overset{0}{U}{}^3 + \overset{0}{U}t = 1, \tag{38}$$

$$(S')^2\partial_{t_1}^2\overset{1}{U} + \left(6\overset{0}{U}{}^2 + t\right)\overset{1}{U} = -2S'\partial_{tt_1}^2\overset{0}{U} - S''\partial_{t_1}\overset{0}{U} - 2S'\phi'\partial_{t_1}^2\overset{0}{U}, \tag{39}$$

$$\begin{aligned}(S')^2\partial_{t_1}^2\overset{2}{U} + \left(6\overset{0}{U}{}^2 + t\right)\overset{2}{U} = -6\overset{0}{U}\overset{1}{U}{}^2 - 2S'\partial_{tt_1}^2\overset{1}{U} - S''\partial_{t_1}\overset{1}{U} - 2S'\phi'\partial_{t_1}^2\overset{1}{U}\\ - \partial_t^2\overset{0}{U} - (\phi')^2\partial_{t_1}\overset{0}{U} - \phi''\partial_{t_1}\overset{0}{U} - 2\phi'\partial_t\partial_{t_1}\overset{0}{U}.\end{aligned} \tag{40}$$

Integrating once with respect to t_1 the equation for $\overset{0}{U}$ we obtain:

$$(S')^2\left(\partial_{t_1}\overset{0}{U}\right)^2 = -\overset{0}{U}{}^4 - t\overset{0}{U}{}^2 + 2\overset{0}{U} + E(t), \tag{41}$$

where $E(t)$ is the “constant of integration”.

We study the equation when the right-hand side has two real roots $\beta(t) < \alpha(t)$. We define the initial data as

$$\overset{0}{U}|_{t_1=0} = \beta(t).$$

Solution of the equation (41) is an elliptic function. The left hand side of the formula (41) is positive for real functions S' and $\overset{0}{U}$. The term of highest order in the right hand side is $-\overset{0}{U}{}^4$. Therefore the real solution of the equation (41) has no poles when $t_1 \in \mathbb{R}$ and $|E(t)| < \infty$.

To construct an uniform asymptotics (37) we choose the unknown functions $S(t)$, $\phi(t)$ and the "constant of integration" $E(t)$ by a special way. They must satisfy to anti-resonant conditions for equations (38)–(40). The first condition is the boundedness of the right hand side of (39) as $t_1 \in \mathbb{R}$. It is satisfied if a period of the oscillates of the function $\overset{0}{U}(t_1, t)$ on t_1 is a constant (see for example [31]):

$$T = \sqrt{2}S' \int_{\beta(t)}^{\alpha(t)} \frac{dx}{\sqrt{-x^4 - tx^2 + 2x + E(t)}}. \tag{42}$$

The next condition is a boundedness of the first correction $\overset{1}{U}(t_1, t)$ when $t_1 \in \mathbb{R}$. It gives equation defining a main term of an action I_0 (see, [31]):

$$I_0 = S' \int_0^T \left[\partial_{t_1} \overset{0}{U}(t_1, t)\right]^2 dt_1 = \text{const}.$$

Using the explicit expression for the derivative with respect to t_1 we present this formula in a some other form:

$$I_0 = 2 \int_{\beta(t)}^{\alpha(t)} \sqrt{-x^4 - tx^2 + 2x + E(t)}\, dx = \text{const}, \tag{43}$$

where $\alpha(t)$ and $\beta(t)$ are the solutions of the equation $-x^4 - tx^2 + 2x + E(t) = 0$. The equation (43) means that the action is a constant. At the other hand-side the equation (43) defines the energy $E(t)$ as a function with respect to slow time t.

At last the necessary condition of boundedness of the second correction $\overset{2}{U}$ with respect to t_1 is the equation (see, [27]):

$$\frac{\partial_E I_0}{\partial_E S'} \phi' = a = \text{const}. \tag{44}$$

The equations (42)–(44) define the parameters of the main term of the asymptotics as functions with respect to t. To find these functions one should define corresponding constants.

Notice that to define the value of the action I_0 we construct the asymptotic solution of the equation (1) when $t > t_*$. Thus the polynomial of the fourth power on $\overset{0}{U}$ in the right hand side of the equation (41) can have no more than two various real roots $\alpha(t)$ and $\beta(t)$. Hence this polynomial can be submitted as:

$$F(x, t) = (\alpha(t) - x)(x - \beta(t)) \left((x - m(t))^2 + n^2(t)\right).$$

The degeneration of the elliptic integral at $t = t_*$ corresponds to the case $m(t_*) = \beta(t_*) = u_*$ and $n(t_*) = 0$. For this case it is easy to calculate the constant in the right hand side of the equation (43), which is equal to 2π (i.e. $I_0 = 2\pi$) and the value of the parameter $E(t_*) = E_* = \frac{4}{3}\left(\frac{1}{2}\right)^{2/3}$.

6.2 Degeneration of the fast oscillating asymptotics

In this subsections we calculate the asymptotic behaviour of the phase functions $S(t)$ as $t \to t_* + 0$.

The oscillating solution is degenerated as $t \to t_* + 0$. Let's construct the asymptotics of this solution in the neighborhood of the degeneration point. For this purpose we calculate the asymptotics of the phase function $S(t)$ and the function $E(t)$. Let's write the equation (43) as:

$$\int_\beta^\alpha \sqrt{(\alpha - x)(x-\beta)\left[(x-m)^2 + n^2\right]}\, dx = \pi, \tag{45}$$

where α, β, m, n are real functions when $t \geq t_*$. These functions satisfy the Viéta equations:

$$\begin{aligned} &\alpha + \beta + 2m = 0, \\ &m^2 + n^2 + \alpha\beta + 2m(\alpha+\beta) = t, \\ &(\alpha+\beta)\left(m^2+n^2\right) + 2m\alpha\beta = 2, \\ &\alpha\beta\left(m^2+n^2\right) = -E. \end{aligned} \tag{46}$$

The equation (42) and three equation from (46) define the dependency α, β, m, n on the parameter t. The last equation in (46) defines the function $E(t)$. Let's make changes of variables: $E = E_* + g_1$, $t = t_* + \eta$, $4m = m_* + m_1$. After simple transformations of the equations (46) we obtain:

$$\begin{aligned} &2m_*\left[6m_1^2 - 2n^2 + \eta\right] + \left[2m_1^2 - 2n^2 + \eta\right] 2m_1 = 0, \\ &m_*^2\left(12m_1^2 - 4n^2 + \eta\right) + 2m_* m_1\left(6m_1^2 - 2n^2 + \eta\right) \\ &\qquad + \left(3m_1^2 - n^2 + \eta\right)\left(m_1^2 + n^2\right) = -g_1. \end{aligned} \tag{47}$$

Construct the solution of this system as $t \to t_* + 0$ as:

$$m_1 = \mu\sqrt{\eta} + O(\eta), \qquad n = \nu_1\sqrt{\eta} + O(\eta), \qquad g_1 = \gamma_1\eta + O\left(\eta^{3/2}\right).$$

Let's substitute these expressions in (47), equate the coefficients at the identical powers of η. As a result we obtain:

$$6\mu_1^2 - 2\nu_1^2 = -1, \qquad \gamma_1 = m_*^2.$$

To define the constants μ_1 and ν_1 it is necessary to construct the asymptotics as $\eta \to +0$ of the left hand side of the equation (45). The asymptotics of the outside the integral coefficient in the equation (45) has the form

$$(\alpha-\beta)^3 = 64|m|^3\left[1 - \frac{3}{2}\frac{\mu_1\sqrt{-\eta}}{m_*} + \frac{3}{2}\frac{\nu_1^2 - \mu_1^2 - 1}{4m_*^2}\eta + O\left(\eta^{3/2}\right)\right]. \tag{48}$$

The integral in the equation (45) is presented as

$$I(k,\delta) = \int_0^1 dz\, \sqrt{(1-z)z}\sqrt{(z-k\delta)^2 + \delta^2},$$

where

$$z=\frac{x-\beta}{\alpha-\beta},\qquad \frac{m-\beta}{\alpha-\beta}=k\delta,\qquad \delta^2=\frac{n^2}{(\alpha-\beta)^2},\tag{49}$$

The value of the constant k will be defined from an asymptotics below.

The asymptotics of an integral $I(k,\delta)$ as $\delta\to 0$ has the form

$$I(k,\delta)=\frac{\pi}{16}-k\delta\frac{\pi}{8}+\delta^2\frac{\pi}{4}+c(k)\delta^{5/2}+O\left(\delta^3\right),\tag{50}$$

where

$$c(k)=-\frac{8}{5}\int_0^\infty dy\frac{-ky+k^2+1}{[(y-k)^2+1]^{5/2}}y^{5/2}.$$

First three terms in this formula are calculated by standard way. Let's show as we can obtain the function $c(k)$. For this purpose the following trick ([35]) is applicable. Let's calculate third derivative with respect to δ of the function $I(k,\delta)$:

$$\frac{\partial^3 I}{\partial\delta^3}=-3\int_0^1 dz\,\sqrt{(1-z)z}\frac{-kz+k^2\delta+\delta}{[(z-k\delta)^2+\delta]^{5/2}}.$$

On the right hand side we replace z by δy and we present the integral as

$$\frac{\partial^3 I}{\partial\delta^3}=-3\delta^{-1/2}\int_0^\infty dy\,y^{5/2}\frac{-ky+k^2+1}{[(y-k)^2+1]^{5/2}}+O(1).\tag{51}$$

Solving the ordinary differential equation (51) in the neighborhood of $\delta=0$, we get:

$$I(k,\delta)=c_0+\delta c_1+\delta^2 c_2+\delta^{5/2}\frac{8}{15}c_3(k)+O\left(\delta^3\right),$$

where

$$C_3(k)=-3\int_0^\infty du\frac{-ky+k^2+1}{[(y-k)^2+1]^{5/2}}y^{5/2}.$$

After that it is easy to obtain the asymptotics (50).

To define the value of k we substitute the asymptotics (48) and (50) in (45) and equate to zero the coefficients at identical powers of η. In the result we get at $\eta^{5/4}$ the equation

$$c(k)=0.$$

This is the transcendental equation for the definition of the parameter k. The numerical solution gives $k\sim 0.463$. Using the formula (49), we get:

$$\mu_1=\frac{k}{3}|\nu|,\qquad \nu_1=\sqrt{\frac{3}{2\left(3-k^2\right)}}.$$

To construct the asymptotics of $S(t)$ as $t \to t_* + 0$ we use the equation connecting the period of fast oscillations with its phase ([31]):

$$T = \sqrt{2} S' \int_\beta^\alpha \frac{dx}{\sqrt{(\alpha - x)(x - \beta)\left[(x - m)^2 + n^2\right]}}. \tag{52}$$

Present the integral in the right hand side as

$$J = \frac{1}{\alpha - \beta} \int_0^1 \frac{dz}{\sqrt{(1 - z) z \left[(z - k\delta)^2 + \delta^2\right]}}.$$

After the same replacements, as at the construction of the asymptotics $\frac{\partial^3 I}{\partial \delta^3}$, as $\delta \to 0$ we get:

$$J = \frac{\delta^{-1/2}}{\alpha - \beta} \int_0^\infty \frac{dy}{\sqrt{y\left[(y + k)^2 + 1\right]}} + O(1).$$

We substitute this expression into the equation (52), use the asymptotics δ and $(\alpha - \beta)$ as $\eta \to +0$ and in the result we get:

$$S' = (t - t_*)^{1/4} S_*(k) + O\left((t - t_*)^{1/2}\right),$$

where

$$S_*(k) = \frac{T}{\sqrt{2}} \frac{2|m_*|^{1/2}}{C_*(k)} \left(\frac{3}{6 - 2k^2}\right)^{1/4}, \qquad C_*(k) = \int_0^\infty \frac{dy}{\sqrt{y\left[(y - k)^2 + 1\right]}}.$$

The period of the oscillations for the function $\overset{0}{U}(t_1, t)$ with respect to the variable t_1 in the Krylov–Bogolubov's method is an arbitrary constant. Let's choose it such, that $S_*(k) = 1$:

$$T = \frac{S_*(k)\sqrt{2} C_*(k)}{2|u_*|^{1/2}} \left(\frac{3}{6 - 2k^2}\right)^{1/4}. \tag{53}$$

In the result the phase of the oscillations as $t \to t_*$ has a form

$$S(t) = \frac{4}{5}(t - t_*)^{5/4} + O\left((t - t_*)^{3/2}\right) + S_0, \tag{54}$$

where S_0 is some constant. Its value will be defined below at the matching of the asymptotics (37) and inner asymptotics (13), (20) as $t \to t_* + 0$.

6.3 The domain of validity of the fast oscillating asymptotics

In this subsection we establish the domain of validity of the fast oscillating asymptotics and compute the residual of this asymptotic solution.

The validity of the asymptotics is defined by the formula $\varepsilon \overset{1}{U} \ll \overset{0}{U}$. Let us check this requirement. For this we must obtain the order of the first correction as $t \to t_* + 0$. Evaluate the order of the right hand side of the equation for the first correction:

$$F_1(t_1, t, \varepsilon) = -2S' \partial^2_{tt_1} \overset{0}{U} - S'' \partial_{t_1} \overset{0}{U}.$$

From the equation for $\overset{0}{U}$ one can evaluate second term in $F_1(t_1,t,\varepsilon)$ as $t\to t_*+0$:

$$S''\partial_{t_1}\overset{0}{U}=O\left((t-t_*)^{-1}\right).$$

One must reduce formula for the derivative of $\overset{0}{U}$ with respect to t to evaluate of the first term in the formula for $F_1(t_1,t,\varepsilon)$.

The function $\overset{0}{U}$ is the inverse function with respect to the elliptic integral

$$t_1+t_0=S'\int_{\beta(t)}^{\overset{0}{U}}\frac{dy}{\sqrt{-y^4-ty^2+2y+E(t)}}.$$

Both limits of the integration are functions with respect to t, it is not convenient for us. Make the substitution $y=(\alpha-\beta)z+\beta$. Then we obtain

$$t_1+t_0=\frac{S'}{(\alpha-\beta)}\int_0^{\frac{\overset{0}{U}-\beta}{\alpha-\beta}}\frac{dz}{\sqrt{z(1-z)}\sqrt{(z-\gamma)^2+\delta^2}}.$$

Now we differentiate this formula with respect to t, as a result we obtain the formula for the $\partial_t\left[\frac{\overset{0}{U}-\beta}{\alpha-\beta}\right]$:

$$\partial_t\left[\frac{\overset{0}{U}-\beta}{\alpha-\beta}\right]=\frac{1}{(\alpha-\beta)S'}\sqrt{\left(\alpha-\overset{0}{U}\right)\left(\overset{0}{U}-\beta\right)\left(\left(\overset{0}{U}-m\right)^2+n^2\right)}$$
$$\times\left[-\partial_t\left(\frac{S'}{(\alpha-\beta)}\right)\int_0^{\frac{\overset{0}{U}-\beta}{\alpha-\beta}}\frac{dz}{\sqrt{z(1-z)\left((z-\gamma)^2+\delta^2\right)}}\right.$$
$$\left.+\frac{S'}{(\alpha-\beta)}\int_0^{\frac{\overset{0}{U}-\beta}{\alpha-\beta}}\frac{dz\,\delta\delta'-(z-\gamma)\gamma'}{\sqrt{z(1-z)}\left((z-\gamma)^2+\delta^2\right)^{3/2}}\right],$$

where $\gamma=\frac{m}{\alpha-b}$.

In the same way we can obtain the formula:

$$\partial_t\left[\frac{\overset{0}{U}-\alpha}{\alpha-\beta}\right]=\frac{1}{(\alpha-\beta)S'}\sqrt{\left(\alpha-\overset{0}{U}\right)\left(\overset{0}{U}-\beta\right)\left(\left(\overset{0}{U}-m\right)^2+n^2\right)}$$
$$\times\left[-\partial_t\left(\frac{S'}{(\alpha-\beta)}\right)\int_{-1}^{\frac{\overset{0}{U}-\alpha}{\alpha-\beta}}\frac{dz}{\sqrt{z(1-z)\left((z-\Gamma)^2+\delta^2\right)}}\right.$$
$$\left.+\frac{S'}{(\alpha-\beta)}\int_0^{\frac{\overset{0}{U}-\alpha}{\alpha-\beta}}\frac{dz\,\delta\delta'-(z-\Gamma)\Gamma'}{\sqrt{z(1-z)}\left((z-\Gamma)^2+\delta^2\right)^{3/2}}\right],$$

where $\Gamma=\frac{\alpha-m}{\alpha-\beta}$.

These formulas will be useful when we will reduce the formula for the second derivative of $\overset{0}{U}$ with respect to t.

The first derivative of $\overset{0}{U}$ with respect to t has the form:

$$\partial_t \overset{0}{U} = (\alpha-\beta)\left[\frac{\alpha'-\beta'}{(\alpha-\beta)^2} + \partial_t\left(\frac{-\beta}{\alpha-\beta}\right)\right]$$

$$+\frac{1}{(\alpha-\beta)S'}\sqrt{\left(\alpha-\overset{0}{U}\right)\left(\overset{0}{U}-\beta\right)\left(\left(\overset{0}{U}-m\right)^2+n^2\right)}$$

$$\times\left[-\partial_t\left(\frac{S'}{(\alpha-\beta)}\right)\int_0^{\frac{\overset{0}{U}-\beta}{\alpha-\beta}} \frac{dz}{\sqrt{z(1-z)}\left((z-\gamma)^2+\delta^2\right)}\right.$$

$$\left.+\frac{S'}{(\alpha-\beta)}\int_0^{\frac{\overset{0}{U}-\beta}{\alpha-\beta}} \frac{dz\,(z-\gamma)\gamma'+\delta\delta'}{\sqrt{z(1-z)}\left((z-\gamma)^2+\delta^2\right)^{3/2}}\right].$$

Now we can evaluate the second derivative $\partial^2_{tt_1}\overset{0}{U}$.

$$\partial^2_{tt_1}\overset{0}{U} = \partial_t\left[\frac{1}{S'}\sqrt{\left(\alpha-\overset{0}{U}\right)\left(\overset{0}{U}-\beta\right)\left(\left(\overset{0}{U}-m\right)^2+n^2\right)}\right].$$

Using the formula for $\partial_t\overset{0}{U}$ one can obtain as $t\to t_*+0$:

$$\partial^2_{tt_1}\overset{0}{U} = O\left((t-t_*)^{-5/4}\right).$$

This formula allows to evaluate the right hand side in the equation (39) as $t\to t_*+0$:

$$F_1(t_1,t,\varepsilon) = O\left((t-t_*)^{-5/4}\right).$$

The first correction is periodical function with respect to t_1. One can derive the solution of the equation for the first correction using two linear independent solution of the equation

$$(S')^2\partial^2_{t_1}V + \left(6\overset{0}{U}{}^2+t\right)V = 0.$$

Here our goal is to write these solutions in the terms of $\overset{0}{U}$ because then we evaluate the order of derivatives of the first correction of asymptotic solution (37) using the formula for $\partial_t\overset{0}{U}$.

The first one is

$$U_1(t_1,t,\varepsilon) \equiv \partial_{t_1}\overset{0}{U} = \pm\frac{1}{S'}\sqrt{-\overset{0}{U}{}^4 - t\overset{0}{U}{}^2 + 2\overset{0}{U} + E(t)}.$$

Here the sign before the root is $+$, when $\partial_{t_1}\overset{0}{U} > 0$ and vice versa.

The second solution of the homogeneous linearized equation for the first correction is

$$U_2(t_1,t,\varepsilon)=\pm\frac{1}{S'}\sqrt{-\overset{0}{U}{}^4-t\overset{0}{U}{}^2+2\overset{0}{U}+E(t)}\int_{t_0}^{t_1}\frac{d\sigma}{-\overset{0}{U}{}^4-t\overset{0}{U}{}^2+2\overset{0}{U}+E(t)}.$$

Integral in this formula must be regularized because integrand has second order poles at points $\overset{0}{U}=\alpha$ and $\overset{0}{U}=\beta$. One of the possibile way of the regularization is done in [36]. Here we will follow [36].

Near the singular points $\sigma=t_1^j$ $j\in\mathbb{Z}$ one must represent the integral as

$$S'\int_{t_1^j-\lambda}^{t_1^j+\lambda}\frac{d\sigma}{U_1^2(\sigma,t,\varepsilon)}=-S'\int_{t_1^j-\lambda}^{t_1^j+\lambda}\frac{1}{\partial_\sigma U_1(\sigma,t,\varepsilon)}d_\sigma\left(\frac{1}{U_1(\sigma,t,\varepsilon)}\right)$$

$$=-\frac{S'}{U_1(\sigma,t,\varepsilon)\partial_\sigma U_1(\sigma,t,\varepsilon)}\Bigg|_{t_1^j-\lambda}^{t_1^j+\lambda}-S'\int_{t_1^j-\lambda}^{t_1^j+\lambda}\frac{d\sigma\,\partial_\sigma^2U_1(\sigma,t,\varepsilon)}{U_1(\sigma,t,\varepsilon)\left(\partial_\sigma U_1(\sigma,t,\varepsilon)\right)^2}.$$

The parameter λ may be for example $\lambda=T/4$, where T is the period of oscillations of the function $\overset{0}{U}$ with respect to t_1.

Change the second derivative of U_1 in the last integrand as

$$\partial_{t_1}^2U_1=-\left(\frac{1}{S'}\right)^2\left(6\overset{0}{U}{}^2+t\right)U_1,$$

and the first derivative of U_1 as

$$\partial_{t_1}U_1=S'\partial_{t_1}^2\overset{0}{U}=\frac{1}{S'}\left(1-t\overset{0}{U}-2\overset{0}{U}{}^3\right).$$

As a result we obtain formula for regularization of the integral:

$$S'\int_{t_1^j-\lambda}^{t_1^j+\lambda}\frac{d\sigma}{U_1^2(\sigma,t,\varepsilon)}=-\frac{(S')^3}{U_1(\sigma,t,\varepsilon)\left(1-t\overset{0}{U}(\sigma,t,\varepsilon)-2\overset{0}{U}{}^3(\sigma,\tau,\varepsilon)\right)}\Bigg|_{\sigma=t_1^j-\lambda}^{\sigma=t_1^j+\lambda}$$

$$+S'\int_{t_1^j-\lambda}^{t_1^j+\lambda}\frac{d\sigma\left(6\overset{0}{U}{}^2(\sigma,t,\varepsilon)+t\right)}{\left(1-t\overset{0}{U}(\sigma,t,\varepsilon)-2\overset{0}{U}{}^3(\sigma,t,\varepsilon)\right)^2}.$$

Using the functions U_1 and U_2 one can solve the equation (39) for the first correction of the asymptotic solution (37) and obtain the solution the terms of $\overset{0}{U}$.

One can see the first correction $\overset{1}{U}$ has the order of the right hand side of the equation for the first correction multiplyed on S'. It means that

$$\overset{1}{U}=O\left((t-t_*)^{-3/2}\right)$$

as $t\to t_*+0$.

This formula allows to obtain the restriction for the validity of the formal asymptotic solution (37):

$$(t-t_*)\varepsilon^{-2/3} \gg 1.$$

Evaluate the residual of the asymptotic solution (37). For this we must evaluate the function

$$F(t_1,t,\varepsilon) = -\varepsilon^2\partial_t^2\overset{0}{U} - \varepsilon^2\partial_t\left(\frac{1}{S'}\partial_{t_1}\overset{1}{U}\right) - \varepsilon^2\frac{1}{S'}\partial_{t_1}\partial_t\overset{1}{U} - \varepsilon^3\partial_t^2\overset{1}{U}.$$

For all $t\in(t_*,t_*+a]$ the order of $F(t_1,t,\varepsilon)=O\left(\varepsilon^2\right)$. But the order of F grows as $t\to t_*+0$, because the derivatives with respect to t have singularity at $t=t_*$. In the right hand side of this formula there is only functions on the $\overset{0}{U}$, then we can differentiate the right hand side for deriving of the second derivation of $\overset{0}{U}$ with respect to t. The formula for $\partial_t^2\overset{0}{U}$ will be very large if we will write it in these definitions but now one can evaluate the order of $\partial_t^2\overset{0}{U}$ as $t\to t_*+0$ using the formulas

$$S' = O\left((t-t_*)^{1/4}\right), \qquad O(\gamma)=O(\delta)=O\left((t-t_*)^{1/2}\right), \qquad \text{as} \quad t\to t_*+0.$$

As a result one obtain:

$$\partial_t^2\overset{0}{U}(t_1,t,\varepsilon) = O\left((t-t_*)^{-10/4}\right).$$

Using the same formulas one can evaluate the order of the $F(t_1,t,\varepsilon)$ as $t\to t_*+0$:

$$F(t_1,t,\varepsilon) = O\left(\varepsilon^2(t-t_*)^{-11/4}\right) + O\left(\varepsilon^3(t-t_*)^{-17/4}\right).$$

6.4 The matching of the fast oscillating asymptotic solution and the inner asymptotics

The matching of this asymptotics with the inner asymptotics (13) and (20) is carried out. From the matching condition for the phase function we obtain the initial condition $S(t)|_{t=t_*}=0$.

Now we turn to the evaluation of the asymptotics for the function $\overset{0}{U}$ as $t\to t_*+0$. The function $\overset{0}{U}$ may be written in the implicit form:

$$t_1 = -S'\int_{\overset{0}{U}}^{\alpha(t)} \frac{dx}{\sqrt{-x^4-tx^2+2x+E(t)}}. \tag{55}$$

We remind the parameter t_1 is equal to $\varepsilon^{-1}S(t)+\phi(t)$ and the additional term S_0 is undefined in the function $S(t)$. This term we define in this subsection.

The formula (55) allows us to obtain the main term of asymptotics for the $\overset{0}{U}$ at $t\to t_*+0$. Denote

$$\overset{0}{U} = u_* + W(t_1,t)$$

and

$$x = u_* + y.$$

Using asymptotics of $E(t)$ and $\alpha(t)$ as $t \to t_* + 0$ one can get

$$t_1 = -S' \int_W^{u_*+\alpha(t)} \frac{dy}{\sqrt{-y^4 - 4u_* y^3 + O(t-t_*)}}$$
$$= S' \left(\int_W^{4u_*} \frac{dy}{\sqrt{-y^4 - 4u_* y^3}} + O\left((t-t_*)^{1/2}\right) + O\left(\frac{(t-t_*)}{W^{5/2}}\right)\right).$$

This formula allows to write the asymptotic expansion of $\overset{0}{U}$ as $t \to t_* + 0$ in the form:

$$\overset{0}{U}(t_1, t) = W_0(t_1/S') + O\left((t-t_*)^{1/2}\right) + O\left(\frac{(t-t_*)}{W^{5/2}}\right). \tag{56}$$

The main term of the asymptotics is defined by formula:

$$W_0(t_1/S') = -\frac{4u_*}{1 + 4u_*^2 (t_1/S')^2}.$$

This asymptotics is applicable as $W_0^{5/2} \gg (t - t_*)$. The function $\overset{0}{U}(t_1, t)$ is periodic with respect to t_1. It means the asymptotic (56) is applicable on some segments of the interval $\varepsilon^{4/5} \ll (t - t_*) \ll 1$. The argument of the function W_0 in the neighborhood of some point t_k as $t \to t_* + 0$ is:

$$\left(\frac{t_1}{\varepsilon S'}\right)\bigg|_{t=t_k} \sim \frac{S(t_k) + S'(t_k)(t-t_k) + O\left(S''(t_k)(t-t_k)^2\right)}{S'(t_k) + O\left(S''(t_k)(t-t_k)\right)}$$
$$\sim \frac{t-t_k}{\varepsilon} + S_k + O\left(\frac{(t-t_k)^2}{\varepsilon t_k}\right),$$

where $S_k = \varepsilon^{-1} S(t_k)/S'(t_k)$.

It is easy to see the argument of the function W_0 may be represented as

$$\frac{t_1}{\varepsilon S'} \sim \theta + S_k,$$

where S_k is some constants depending on S_0 and number k. One can see the main term of the asymptotics (56) coincides up to shift S_k with the main term of the second inner asymptotic expansion which is the function $\overset{0}{w}(\theta_k)$. It is easy to see for full definition of the function $\overset{0}{U}(t_1, t)$ one must find the phase shift S_0.

We defined the additional constant S_0 by matching the functions $\overset{0}{U}(t_1, t)$ and the first inner asymptotic expansion.

The formula (56) is suitable when $|W_0(\theta)|^{5/2} \gg |t - t_*|$. When $W_0(\theta)$ is small, we consider other asymptotic formula for the function $\overset{0}{U}(t_1, t)$:

$$\overset{0}{U}(t_1, t) = u_* + \sqrt{t - t_*}\,\mathcal{P}\left(\frac{S(t)}{\varepsilon} + \phi(t), t\right). \tag{57}$$

Substitute this formula to the second-order equation for the function $\overset{0}{U}(t_1, t)$ (38). Expand the function $\mathcal{P}\left(S(t)/\varepsilon + \phi(t), t\right)$ with respect to the small parameter $(t - t_*)$. In a result the equation for the main term of the asymptotic expansion is

$$p'' + 6u_*p^2 + u_* = 0.$$

This equation coincides with the equation for the asymptotics of the first correction of the first inner asymptotic expansion. The boundary conditions for the function $p(S(t)/\varepsilon + \phi(t))$ is obtained from the condition of the matching (57) with the asymptotics of the expansion (56) as $|\theta| \to \infty$. The additional constant S_0 in the formula (54) is finally defined at the matching of the asymptotic expansions (56), (57) with asymptotics of the inner asymptotic expansions. This get: $S_0 = 0$.

7 Open problems

In this work the bifurcation of the slowly varying equilibrium of the Painlevé-2 equation was studied by matching method on the formal asymptotic approach. However it is necessary to note two important problems which remind out of side of our analysis.

1. The phase shift of the oscillating asymptotic solution is undefined in our approach. Its definition demands much more thin calculations for the corrections of the asymptotic formulas.
2. A problem of a justification of the remainder term for the constructed asymptotic solution remains open as well.

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