

Superanalogs of the Calogero Operators and Jack Polynomials

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Abstract

A depending on a complex parameter k superanalog \mathcal{SL} of Calogero operator is constructed; it is related with the root system of the Lie superalgebra $\mathfrak{gl}(n|m)$. For $m = 0$ we obtain the usual Calogero operator; for $m = 1$ we obtain, up to a change of indeterminates and parameter k the operator constructed by Veselov, Chalykh and Feigin [2, 3]. For $k = 1, \frac{1}{2}$ the operator \mathcal{SL} is the radial part of the 2nd order Laplace operator for the symmetric superspaces corresponding to pairs $(GL(V) \times GL(V), GL(V))$ and $(GL(V), OSp(V))$, respectively. We will show that for the generic m and n the superanalogs of the Jack polynomials constructed by Kerov, Okunkov and Olshanskii [5] are eigenfunctions of \mathcal{SL} ; for $k = 1, \frac{1}{2}$ they coincide with the spherical functions corresponding to the above mentioned symmetric superspaces. We also study the inner product induced by Berezin's integral on these superspaces.

The Hamiltonian of the quantum Calogero problem is of the form

$$\mathcal{L} = \sum_{i=1}^n \left(\frac{\partial}{\partial t_i} \right)^2 - \frac{1}{2} k(k-1) \sum_{i < j} \frac{\omega^2}{\sinh^2 \frac{\omega}{2} (t_i - t_j)}. \quad (1)$$

In this form it is a particular case (corresponding to the root system R of $\mathfrak{gl}(n)$) of the operator constructed in the famous paper by Olshanetsky and Perelomov [8]

$$\mathcal{L} = \Delta - \sum_{\alpha \in R^+} k_\alpha (k_\alpha - 1) \frac{(\alpha, \alpha)}{\left(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha} \right)^2}. \quad (2)$$

Veselov, Feigin and Chalykh [2] suggested the following generalization of operator (1)

$$\begin{aligned} \mathcal{L}' = & \sum_{i=1}^n \left(\frac{\partial}{\partial t_i} \right)^2 + \left(\frac{\partial}{\partial t_{n+1}} \right)^2 - \frac{1}{2} k(k-1) \sum_{i < j} \frac{\omega^2}{\sinh^2 \frac{\omega}{2} (t_i - t_j)} \\ & + \frac{1}{2} (k-1) \sum_{i=1}^n \frac{\omega^2}{\sinh^2 \frac{\omega}{2} (t_i - \sqrt{-k} t_{n+1})}. \end{aligned} \quad (3)$$

It is known ([6]) that eigenfunctions of operator (1) can be expressed in terms of Jack polynomials $P_\lambda(x_1, \dots, x_n; k)$, where λ is a partition of n . (For definition and properties of Jack polynomials see [7, 11].) It is known ([7]) that for $k = 1, \frac{1}{2}, 2$ (our k is inverse of α , the parameter of Jack polynomials Macdonald uses in [7]) Jack polynomials are interpreted as spherical functions on symmetric spaces corresponding to pairs $(GL \times GL, GL)$, (GL, SO) and (GL, Sp) , respectively. In these cases the corresponding operators are radial parts of the corresponding second order Laplace operators.

Superroots of $\mathfrak{gl}(n|m)$:

Let $I = I_{\bar{0}} \amalg I_{\bar{1}}$ be the union of the “even” indices $I_{\bar{0}} = \{1, \dots, n\}$ and “odd” indices $I_{\bar{1}} = \{\bar{1}, \dots, \bar{m}\}$. Let $\dim V = (n|m)$ and $e_1, \dots, e_n, e_{\bar{1}}, \dots, e_{\bar{m}}$ be a basis of V such that the parity of each vector is equal to that of its index. Let $\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\bar{1}}, \dots, \varepsilon_{\bar{m}}$ be the left dual basis of V^* . Then the set of roots can be described as follows: $R = R_{11} \amalg R_{22} \amalg R_{12} \amalg R_{21}$, where

$$\begin{aligned} R_{11} &= \{\varepsilon_i - \varepsilon_j \mid i, j \in I_{\bar{0}}\}, & R_{22} &= \{\varepsilon_i - \varepsilon_j \mid i, j \in I_{\bar{1}}\}, \\ R_{12} &= \{\varepsilon_i - \varepsilon_j \mid i \in I_{\bar{0}}, j \in I_{\bar{1}}\}, & R_{21} &= \{\varepsilon_i - \varepsilon_j \mid i \in I_{\bar{1}}, j \in I_{\bar{0}}\}. \end{aligned} \quad (4)$$

On V^* , define the depending on parameter k inner product by setting

$$(v_1^*, v_2^*)_k = \sum_{i=1}^n v_1^*(e_i)v_2^*(e_i) - k \sum_{j=1}^m v_1^*(e_{\bar{j}})v_2^*(e_{\bar{j}}) \quad (5)$$

and set $\rho_{(k)} = k\rho_1 + \frac{1}{k}\rho_2 - \rho_{12}$, where

$$\rho_1 = \frac{1}{2} \sum_{\alpha \in R_{11}^+} \alpha; \quad \rho_2 = \frac{1}{2} \sum_{\beta \in R_{22}^+} \beta; \quad \rho_{12} = \frac{1}{2} \sum_{\gamma \in R_{12}} \gamma.$$

Set $\partial_i(e^{v^*}) = v^*(e_i)e^{v^*}$, $\partial_{\bar{j}}(e^{v^*}) = v^*(e_{\bar{j}})e^{v^*}$. Define the superanalog of the Calogero operator to be

$$\begin{aligned} \mathcal{SL} &= \sum_{i=1}^n \partial_i^2 - k \sum_{j=1}^m \partial_{\bar{j}}^2 - k(k-1) \sum_{\alpha \in R_{11}^+} \frac{(\alpha, \alpha)_k}{\left(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\right)^2} \\ &+ \frac{1}{k} \left(\frac{1}{k} - 1\right) \sum_{\beta \in R_{22}^+} \frac{(\beta, \beta)_k}{\left(e^{\frac{1}{2}\beta} - e^{-\frac{1}{2}\beta}\right)^2} - 2 \sum_{\gamma \in R_{12}} \frac{(\gamma, \gamma)_k}{\left(e^{\frac{1}{2}\gamma} - e^{-\frac{1}{2}\gamma}\right)^2}. \end{aligned} \quad (6)$$

In order to describe the eigenfunctions of \mathcal{SL} , it is convenient to present it in terms of operator \mathcal{M} described below. Set

$$\delta^{(k)} = \prod_{\alpha \in R_{11}^+} \left(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\right)^k \prod_{\beta \in R_{22}^+} \left(e^{\frac{1}{2}\beta} - e^{-\frac{1}{2}\beta}\right)^{1/k} \prod_{\gamma \in R_{12}} \left(e^{\frac{1}{2}\gamma} - e^{-\frac{1}{2}\gamma}\right)^{-1}. \quad (7)$$

Set

$$\mathcal{M} = \left(\delta^{(k)}\right)^{-1} \left(\mathcal{L} - (\rho_{(k)}, \rho_{(k)})_k\right) \delta^{(k)}.$$

Lemma. *The explicit form of \mathcal{M} is*

$$\mathcal{M} = \sum_{i=1}^n \partial_i^2 - k \sum_{\bar{j}=1}^m \partial_{\bar{j}}^2 + k \sum_{\alpha \in R_{11}^+} \frac{e^\alpha + 1}{e^\alpha - 1} \partial_\alpha - \sum_{\beta \in R_{22}^+} \frac{e^\beta + 1}{e^\beta - 1} \partial_\beta - 2 \sum_{\gamma \in R_{12}} \frac{e^\gamma + 1}{e^\gamma - 1} \partial_{\gamma,k}, \quad (8)$$

where

$$\begin{aligned} \partial_\alpha &= \partial_i - \partial_j & \text{for } \alpha = \varepsilon_i - \varepsilon_j, \\ \partial_\beta &= \partial_{\bar{i}} - \partial_{\bar{j}} & \text{for } \beta = \varepsilon_{\bar{i}} - \varepsilon_{\bar{j}}, \\ \partial_{\gamma,k} &= \partial_i - \partial_{\bar{j}} & \text{for } \gamma = \varepsilon_i - k\varepsilon_{\bar{j}}. \end{aligned} \quad (9)$$

In terms of new indeterminates $x_i = e^{\varepsilon_i}$ and $y_{\bar{j}} = e^{\varepsilon_{\bar{j}}}$ the operator \mathcal{M} takes the form

$$\begin{aligned} \mathcal{M} &= \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} \right)^2 - k \sum_{\bar{j}=1}^m \left(y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}} \right)^2 + k \sum_{1 \leq i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) \\ &\quad - \sum_{1 \leq i < j \leq n} \frac{y_i + y_j}{y_i - y_j} \left(y_i \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial y_j} \right) - \sum_{1 \leq i \leq n, 1 \leq j \leq m} \frac{x_i + y_j}{x_i - y_j} \left(x_i \frac{\partial}{\partial x_i} - y_j \frac{\partial}{\partial y_j} \right). \end{aligned} \quad (10)$$

Following Kerov, Okunkov and Olshanskii [5], determine superanalogs of Jack polynomials. Let

$$S^p = x_1^p + \dots + x_n^p \quad \text{and} \quad S^\mu = S^{\mu_1} \dots S^{\mu_l} \quad \text{for any partition } \mu \text{ of } n$$

and let $P_\lambda(x; k) = \sum \chi_\mu^\lambda S^\mu$ be the decomposition of the classical Jack polynomials into sums of powers. Set further

$$S^{p,k} = \sum_{1 \leq i \leq n} x_i^p - \frac{1}{k} \sum_{1 \leq j \leq m} y_j^p \quad \text{and} \quad S^{\mu,k} = S^{\mu_1,k} \dots S^{\mu_l,k} \quad \text{for any partition } \mu \text{ of } n.$$

Then the superanalogs of Jack polynomials are of the form

$$P_\lambda(x, y; k) = \sum \chi_\mu^\lambda(k) S^{\mu,k}. \quad (11)$$

Theorem 1. *The polynomials $P_\lambda(x, y; k)$ defined by eq. (9) are eigenfunctions of operator (10).*

Spherical functions:

In this paper we adopt an algebraic approach to the theory of spherical functions.

Let \mathfrak{g} be a finite dimensional Lie superalgebra, $U(\mathfrak{g})$ its enveloping algebra, $\mathfrak{b} \subset \mathfrak{g}$ a subalgebra. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be an irreducible representation and V^* the dual module. If $v \in V$ is a nonzero \mathfrak{b} -invariant vector, then there exists a nonzero vector $v^* \in V^*$ which is also \mathfrak{b} -invariant. The matrix coefficient

$$\theta^\pi(v^*, v) \in U(\mathfrak{g})^*$$

will be called the *spherical function associated with the triple* (π, v^*, v) .

Let $l \in U(\mathfrak{g})^*$ be a left and right \mathfrak{b} -invariant functional, i.e.,

$$l(xuy) = l(u) \quad \text{for any } x, y \in \mathfrak{b} \text{ and } u \in U(\mathfrak{g}).$$

If $z \in Z(\mathfrak{g})$, then $L^*(z)l$, where L^* is left coregular representation of \mathfrak{g} , is also a left and right \mathfrak{b} -invariant functional.

Let $\mathfrak{g} = \mathfrak{gl}(V) \oplus \mathfrak{gl}(V)$ and let $\mathfrak{b}_1 \simeq \mathfrak{g}(V)$ be the first summand of \mathfrak{g} , whereas $\mathfrak{b} \simeq \mathfrak{g}(V)$ is the diagonal subalgebra, i.e., $\mathfrak{b} = \{(x, x) \mid x \in \mathfrak{g}(V)\}$. Let \mathfrak{h} be the Cartan subalgebra of $\mathfrak{gl}(V)$, let λ be a partition of $l \in \mathbb{N}$ and V^λ an irreducible $\mathfrak{gl}(V)$ -module in $V^{\otimes l}$, corresponding to λ , see [9].

The \mathfrak{g} -module $W^\lambda = V^\lambda \otimes (V^\lambda)^*$ is irreducible and contains a unique, up to a constant factor, invariant vector v_λ . The dual module $(W^\lambda)^*$ contains a similar vector v_λ^* . Let $\varphi_\lambda = \theta^\pi(v_\lambda^*, v_\lambda)$ be the corresponding spherical function.

Let $\dim V = (n|m)$, and let $I_0 = \{1, \dots, n\}$ and $I_1 = \{\bar{1}, \dots, \bar{m}\}$; let $\{e_{ij} \mid i, j \in I = I_0 \amalg I_1\}$ be the basis of $\mathfrak{gl}(V)$ consisting of matrix units. Set

$$C_2 = \sum_{i \in I_0} e_{ii}^2 - \sum_{j \in I_1} e_{jj}^2 + \sum_{i, j \in I_0; i \neq j} e_{ij}e_{ji} - \sum_{i, j \in I_1; i \neq j} e_{ij}e_{ji} - \sum_{i \in I_0; j \in I_1} e_{ij}e_{ji} - \sum_{i \in I_1; j \in I_0} e_{ij}e_{ji}.$$

As is easy to verify, C_2 is a central element in the enveloping algebra of $\mathfrak{gl}(V)$ and \mathfrak{g} , if $\mathfrak{gl}(V)$ is embedded as the first summand.

Theorem 2.

- i) Every left and right invariant functional $l \in U(\mathfrak{g})^*$ is uniquely determined by its restriction onto $S(\mathfrak{h}) \subset S(\mathfrak{b}_1)$.
- ii) Let $S(\mathfrak{h})^{inv}$ be the set of restrictions of left and right invariant functional $l \in U(\mathfrak{g})^*$ onto $S(\mathfrak{h}) \subset S(\mathfrak{b}_1)$. Then for every $z \in Z(\mathfrak{g})$ there exists a uniquely determined operator Ω_z on $S(\mathfrak{h})^{inv}$ (the radial part of z). It is determined from the formula

$$(\Omega_z l')(u) = (L^*(z)l)(u) \quad \text{for any } l' \in S(\mathfrak{h})^{inv} \text{ and any its extension } l \in S(\mathfrak{g}).$$
- iii) The above defined operator Ω_{C_2} corresponding to C_2 coincides with the operator \mathcal{M} determined by formula (10) for $k = 1$.
- iv) The functions φ_λ , as functionals on $S(\mathfrak{h})$, coincide, up to a constant factor, with Jack polynomials $P_\lambda(x, y; 1)$, where $x_i = e^{\varepsilon_i}$ for $i \in I_0$ and $y_j = e^{\varepsilon_j}$ for $j \in I_1$.

Let $\mathfrak{g} = \mathfrak{gl}(V)$, where $\dim V = (n|m)$ and $m = 2r$ is even. Let $\mathfrak{b} = \mathfrak{osp}(V)$ be the orthosymplectic Lie subsuperalgebra in $\mathfrak{gl}(V)$ which preserves the tensor

$$\sum_{i \in I_0} e_i^* \otimes e_i^* + \sum_{j \in I_1} \left(e_j^* \otimes e_{j+r}^* + e_{j+r}^* \otimes e_j^* \right). \quad (12)$$

Let ψ be an involutive automorphism of \mathfrak{g} that singles out $\mathfrak{osp}(V)$:

$$\mathfrak{osp}(V) = \{x \in \mathfrak{gl}(V) \mid \psi(x) = -x\}.$$

Let V^λ be a \mathfrak{g} -module. By [10], V^λ contains a \mathfrak{b} -invariant vector \tilde{v}_λ if and only if $\lambda = 2\mu$ and all its rows are of even length. The vector $\tilde{v}_\lambda^* \in (V^\lambda)^*$ is similarly defined. Let $\tilde{\varphi}_\lambda = (\tilde{v}_\lambda^*, v_\lambda)$ be the corresponding matrix coefficient. Set $\mathfrak{h}^+ = \{x \in \mathfrak{h} \mid \psi(x) = x\}$, where $\mathfrak{h} \subset \mathfrak{g}$ is Cartan subalgebra.

Theorem 3.

- i) Every left and right invariant functional on $U(\mathfrak{g})$ is uniquely determined by its restriction onto $S(\mathfrak{h}^+)$.
- ii) Let $S(\mathfrak{h})^{inv}$ be the set of restrictions of left and right invariant functionals. Then for every $z \in Z(\mathfrak{g})$ there exists a uniquely determined operator Ω_z on $S(\mathfrak{h})^{inv}$ (the radial part of z). It is determined from the formula

$$(\Omega_z l')(u) = (L^*(z)l)(u) \quad \text{for any } l' \in S(\mathfrak{h})^{inv} \text{ and any its extension } l \in S(\mathfrak{g}).$$
- iii) The operator Ω_{C_2} corresponding to C_2 coincides with the operator \mathcal{M} determined by formula (10) for $m = r$ and $k = \frac{1}{2}$.
- iv) The functions $\tilde{\varphi}_\lambda$, as functionals on $S(\mathfrak{h})$, coincide, up to a constant factor, with Jack polynomials $P_\lambda(x, y; \frac{1}{2})$, where $\lambda = 2\mu$, $x_i = e^{2\varepsilon_i}$ for $1 \leq i \leq n$ and $y_j = e^{2\varepsilon_j}$ for $1 \leq j \leq r$.

Invariant functional F (the Berezin integral):

For every \mathfrak{g} -module W , define in $U(\mathfrak{g})^*$ the subspace $C(W)$ consisting of the linear hull of the matrix coefficients of W . Denote by $\mathfrak{A}_{n,m}$ the subalgebra of $U(\mathfrak{g})^*$ generated by the matrix coefficients of the identity representation V of $\mathfrak{g} = \mathfrak{gl}(V)$ and its dual.

Theorem 4.

- i) On $\mathfrak{A}_{n,m}$, there exists a unique up to a constant factor nontrivial left and right invariant (with respect to the left and right coregular representations) linear functional F .
- ii) On $\mathfrak{A}_{n,m}$, define the inner product $\langle l_1, l_2 \rangle = F(l_1^t l_2)$, where $l \mapsto l^t$ is the principal automorphism of $U(\mathfrak{g})^*$. Then $\langle l_1, l_2 \rangle = 0$ for any $l_1 \in C(V^\lambda)$, $l_2 \in C(V^\mu)$ and $\lambda \neq \mu$.
- iii) If $\dim V_0^\lambda \neq \dim V_1^\lambda$, then $\langle l_1, l_2 \rangle = 0$ for any $l_1, l_2 \in C(V^\lambda)$.

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