The Maupertuis Principle and Canonical Transformations of the Extended Phase Space

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Abstract

We discuss some special classes of canonical transformations of the extended phase space, which relate integrable systems with a common Lagrangian submanifold. Various parametric forms of trajectories are associated with different integrals of motion, Lax equations, separated variables and action-angles variables. In this review we will discuss namely these induced transformations instead of the various parametric form of the geometric objects.

1 Introduction

Let us begin with a simple example. Consider an ellipse defined by the standard implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

One can represent this ellipse by the parametric equations

$$x = a\sin(t), \qquad y = b\cos(t), \qquad t \in [0, 2\pi]. \tag{1.1}$$

It is known, there are infinitely many parameterisations of a given curve. For instance, we can reparameterise an ellipse by using another parameter

$$\widetilde{t} = \left(a^2 + b^2\right) t + \left(a^2 - b^2\right) \sin(t). \tag{1.2}$$

Construction of different polynomial, rational and other parameterisations of the plane curves is a subject of classical algebraic geometry.

In classical mechanics the same ellipse may be identified with integral trajectories of various integrable systems on the common phase space. In this case the parameter t is the time variable conjugated to some Hamilton function H. As an example, the first parametric form of the ellipse (1.1) is related to the two-dimensional oscillator, while the second parameterisation (1.2) may be associated to the Kepler model.

In this and several other unexpected situations in mathematics, dynamics is occasionally invading mathematical objects in which time is not present in the definition and yet the object can be endowed with various dynamical system structures, continuous or discrete.

Thus, there is a problem of finding suitable parameterisation of curves, which are associated with various integrable systems. What do we know about this problem?

We consider a mechanical system defined by some Lagrangian function $\mathcal{L}(q,\dot q)$ or a Hamilton function $H(p,q)$ on the $2n$-dimensional phase space $\mathcal{M}$ with local coordinates $\{q_j,p_j\}_{j=1}^n$. According to Maupertuis' principle the extremals $q_j=\gamma_j(t)$ of the action functional

$$\mathcal{S}=\int_A^B \mathcal{L}\left(q(t),\dot q(t)\right)dt,\qquad q=(q_1,\ldots,q_n) \tag{1.3}$$

coincide with the extremals of the reduced action functional $\mathcal{S}_0$ on the fixed energy surface

$$\mathcal{Q}^{2n-1}=\Big(H(p,q)=E\Big). \tag{1.4}$$

Recall that the Maupertuis' principle was at first enunciated in 1744 [20, 19, 10]. The modern interpretation of the Maupertuis' principle may be found in [2, 22, 5]. The reduced action $\mathcal{S}_0=\int p\,dq$ is independent of any evolution parameter. Moreover, even its initial parameter t and the corresponding Hamilton function cannot be restored from the reduced action problem [2]. Nevertheless, solutions of the corresponding variational problem are the initial trajectories $q_j=\gamma_j(t)$ in the common nonparametric form [2]. For instance, trajectories of the Kepler system are conic sections $ar^{-1}=1+b\cos\phi$, which may be ellipses, hyperbolas or parabolas at $E<0$, $E>0$ and $E=0$, respectively.

So, any integral trajectory may be parameterised by using another parameter $\widetilde{t}$, such that

$$q_j=\gamma_j(t)=\beta_j(\widetilde{t}),\qquad t\in[A,B],\qquad \widetilde{t}\in[C,D].$$

According to Maupertuis' principle all the initial trajectories have one nonparametric form on the surface $\mathcal{Q}^{2n-1}$. Therefore, for all these trajectories we could introduce common parametric form $q_j=\beta_j(\widetilde{t})$ and a new local Hamilton function $\widetilde{H}$ defined on $\mathcal{Q}^{2n-1}$. Two main problems are how to find the new parametric form of the initial trajectories and how to obtain the new Hamilton function defined on the whole phase space $\mathcal{M}$.

Consider the corresponding Hamilton–Jacobi equation

$$\frac{\partial\mathcal{S}}{\partial t}+H\left(\frac{\partial\mathcal{S}}{\partial q_1},\ldots,\frac{\partial\mathcal{S}}{\partial q_n}\right)=0.$$

In the invariant geometric Hamilton–Jacobi theory [44] any hyperplane $\mathcal{Q}^{2n-1}$ in $\mathcal{M}$ may be called the Hamilton–Jacobi equation. Its solution is an n-dimensional Lagrangian submanifold $\mathcal{C}^{(n)}$ in $\mathcal{M}$, such that $\mathcal{C}^{(n)}\subset\mathcal{Q}^{2n-1}$. By definition, a Lagrangian submanifold is one where the symplectic form Ω vanishes when restricted to it, i.e. $\Omega|_{\mathcal{C}}=0$. This definition is completely invariant with respect to change of local coordinates and parametric representations of the Lagrangian submanifold [2, 44]. As above, we could consider various parameterisations of a given Lagrangian submanifold. Each new parametric form yields a new Hamiltonian system related to the same geometric surface.

Below we consider integrable systems on $\mathcal{M}$ with n integrals in involution. According to the Liouville theorem [2] for any integrable system the corresponding n-dimensional Lagrangian submanifold depends at least of n-arbitrary constants. So integrability is a geometric property and it does not depend on the choice of the parameterisation of the Lagrangian surface. Starting with a known Lagrangian surface of some integrable system we can try to get new integrable models by using various parametric forms of this surface. In this case, we can expect that the initial and resulting integrable systems have a lot of common properties. The main problem is how to find different parameterisations and the corresponding sets of integrals of motion defined on the whole phase space $\mathcal{M}$. Generally there is no rule for how to proceed. Each case is different.

Usually the Lagrangian submanifold depends on $m \geq n$ arbitrary constants. The n constants $\alpha_1, \ldots, \alpha_n$ are identified with the values of integrals of motion $I_j = \alpha_j$ [44], while the remaining $m - n$ constants $a_1, \ldots, a_{m-n}$ are free parameters. In this case we have a freedom related to the choice of n integrals of motion from the $n+m$ initial constants of motion. This freedom permits us to associate a family of integrable systems with one Lagrangian surface. In this review we discuss a special class of different parametric forms of a given surface, which is associated with mutual permutations of energy E and arbitrary parameter a_k. The corresponding reparameterisation of $\mathcal{C}^{(n)}$ we will call the generalized Kepler change of time.

The aim of this paper is to bring together some old and new examples of integrable systems related with the various parametric forms of the one Lagrangian surface. The passage from a given parameterisation to the another one gives rise to the transformations of all the properties of integrable systems, such as integrals of motion, Lax equations, separated variables and the action-angles variables. In this paper we discuss namely these induced transformations instead of the various parametric form of the geometric objects.

The initial symplectic form Ω is equal to zero on the Lagrangian submanifold $\mathcal{C}^{(n)}$. We can suppose, that the space $\mathcal{M}$ may be equipped with another form $\widetilde{\Omega}$, which is equal to zero on the same surface $\mathcal{C}^{(n)}$. In this case for a given Lagrangian surface time t and symplectic structure may be changed simultaneously. Moreover, we can try to embed this n-dimensional surface into the various phase spaces. For instance, the n-dimensional Kepler problem may be identified with the geodesic flow on the sphere S^n [17, 21]. The Kolosoff transformation maps the Kowalevski top into the two-dimensional Stäckel system [15]. The same top may be related with the geodesic motion on $SO(4)$ [1] or with the Neumann system on the sphere S^2 [11]. Here we will not discuss such composed transformations of time and phase space.

By embedding a Lagrangian submanifold $\mathcal{C}^{(n)}$ into the infinite-dimensional phase space, we can identify $\mathcal{C}^{(n)}$ with an invariant manifold of some hierarchy of nonlinear evolution equations. Such finite dimensional manifolds are invariant with respect to the action of all flows of the hierarchy and they are naturally expected to be integrable since all the flows of hierarchy commute on them. For instance, given Lagrangian submanifolds may be realised via stationary or restricted flows. Here we will not discuss relation of finite-dimensional integrable systems with soliton equations and restrict ourselves to finding new parameterisations of the known trajectories only.

Below we will consider many well known mechanical systems, such as the Stäckel systems, Toda lattices, Hénon–Heiles and Holt systems, integrable systems with the quartic potential and the Goryachev–Chaplygin top. The results we present are, for the most part, not new and we do not provide detailed proofs (these can be found in the papers cited). What may be new and interesting is an exposition of canonical transformations of the extended phase space as different parametric forms of integrable geometric objects and the action of these transformations on the properties of integrable systems.

2 The Maupertuis–Jacobi transformations

Let M be an n-dimensional Riemannian manifold with the metric g_{ij}. On the cotangent bundle $\mathcal{M} = T^*M$ consider a Hamiltonian system with the natural Hamilton function $H(p,q)$

$$H(p,q) = T(p,q) + V(q) = \sum_{i,j}^{n} g_{ij}(q)p_ip_j + V(q). \tag{2.1}$$

On the smooth submanifold $\mathcal{Q}^{2n-1}$ (1.4), integral trajectories of the Hamiltonian vector field $\xi = \mathsf{sgrad}\, H(p,q)$ coincide with integral trajectories of another vector field $\widetilde{\xi} = \mathsf{sgrad}\, \widetilde{H}(p,q)$, where the new Hamilton function is given by

$$\widetilde{H}(p,q) = \widetilde{T}(p,q) = \sum_{i,j}^{n} \frac{g_{ij}(q)}{E - V(q)} p_ip_j. \tag{2.2}$$

Integral trajectories have two different parametric forms $q_j = \gamma_j(t) = \beta_j(\widetilde{t})$ on the surface $\mathcal{Q}^{2n-1}$ only. However, the resulting Hamilton function describes geodesic motion and, therefore, on the whole phase space we can determine the so-called Maupertuis transformation

$$\xi \mapsto \widetilde{\xi},$$

which relates the initial hamiltonian vector field ξ on $\mathcal{M}$ with the other hamiltonian vector field $\widetilde{\xi}$ defined on the same phase space $\mathcal{M}$ [5].

If $\widetilde{t}$ be the time along trajectories of the new vector field $\widetilde{\xi}$, then the Maupertuis mapping gives rise to so-called Jacobi transformations [10, 16, 27] of the Hamilton function (2.2) and of the time variable

$$d\widetilde{t} = (E - V(q))dt. \tag{2.3}$$

This Jacobi transformation explicitly describes new parameterisation of the common integral trajectories $q(t) = q(\widetilde{t})$ and determines the new Hamilton function $\widetilde{H}(p,q)$ (2.2).

The Maupertuis transformation maps any integrable system with a natural Hamilton function $H(p,q)$ into the other integrable system on the same phase space $\mathcal{M}$. Namely this property has been used for the search of the new integrable systems (see references within [19, 10, 5, 27, 28]). The Maupertuis principle for integrable systems with a nonnatural Hamilton function is discussed in [22].

The property of integrability is independent on the choice of parametric form of trajectories. However, some criteria of integrability drastically depend upon the parameterisation. For instance, the method of singularity analysis associates integrability with the

Kowalevski–Painlevé property, i.e. the only singularities of the solutions of the equations of motion are movable poles $(t-t_0)^{-m}$ in the complex t-plane [1, 23]. There exist cases of integrable systems with the rational integrals of motion, whose analytic structure permits solutions with algebraic singularities of the type $(t-t_0)^{-m/k}$, (k being a positive integer larger than one). These systems satisfy to the so-called "weak" Painlevé property (see review [23]).

We have some examples of integrable systems related with one Lagrangian submanifold [23, 38, 39], which satisfy to the usual Kowalevski–Painlevé property and the "weak" Painlevé property, respectively. These systems are related with the different parametrisations of one geometric object. So we can see that a change of parametric form leads to a change of the Kowalevski–Painlevé criteria of integrability.

3 Canonical transformation of the extended phase space

To find new parameterisation of the known integral trajectories or of the Lagrangian surfaces one has to introduce a new parameter $\widetilde{t}$ and the corresponding Hamilton function $\widetilde{H}$. Thus, to describe explicitly the following mapping

$$t \mapsto \widetilde{t}, \qquad H(p,q) \mapsto \widetilde{H}(p,q) \tag{3.1}$$

we extend initial phase space $\mathcal{M}$ by adding to it the new coordinate $q_{n+1} = t$ with the corresponding momentum $p_{n+1} = -H$. The resulting $2n+2$-dimensional space $\mathcal{M}_E$ [16, 34] is the so-called extended phase space of the hamiltonian system. We emphasize that $H(p,q)$ is the Hamilton function on $\mathcal{M}$, but H is an independent variable in the space $\mathcal{M}_E$. The energy E is a fixed value of the variable H or the function $H(p,q)$.

To describe evolution on the extended phase space $\mathcal{M}_E$ we introduce the generalized Hamilton function [16, 34]

$$\mathcal{H}(p_1,\dots,p_{n+1}; q_1 \dots, q_{n+1}) = H(p,q) - H. \tag{3.2}$$

The Hamilton equations for the variables $q_{n+1} = t$ and $p_{n+1} = -H$ are

$$\frac{dt}{d\tau} = 1, \qquad \frac{dH}{d\tau} = 0.$$

Here τ is a generalized time (parameter) associated to the generalized Hamilton function $\mathcal{H}$. The time variable t is a cyclic coordinate and the conjugated momentum is a constant of motion. Other $2n$ equations coincide with the initial Hamilton equations on the zero-valued energy surface

$$\mathcal{H}(p,q) = H(p,q) - H = 0. \tag{3.3}$$

Thus, our initial hamiltonian system on $\mathcal{M}$ may be immersed into the hamiltonian system on $\mathcal{M}_E$. Using this immersion and canonical transformations of the extended phase space $\mathcal{M}_E$ [16, 34] we introduce transformations

$$\begin{array}{ccc}
 & (H,t) \quad \text{canonical transformations} \quad (\widetilde{H},\widetilde{t}) & \\
\mathcal{M}_E & \longrightarrow & \mathcal{M}_E \\
\uparrow & & \downarrow \\
\mathcal{M} & & \mathcal{M} \\
\uparrow & & \downarrow \\
H(p,q) & & \widetilde{H}(p,q)
\end{array}$$

which map an initial Hamilton function $H(p,q)\mapsto \widetilde{H}(p,q)$ into another Hamilton function defined on the same phase space. Of course the similar mapping $t\mapsto\widetilde{t}$ permits us to describe new parameterisation of the corresponding integral trajectories.

Note two different classical definitions of the canonical transformations are known [2].

1. *Canonical transformations preserve the canonical form of the Hamilton–Jacobi equations.*
2. *Canonical transformations preserve the differential 2-form,* $\Omega=\sum\limits_{i=1}^{n} dp_i\wedge dq_i$, *on* $\mathcal{M}$.

For instance the first definition is used in textbooks on variational principles of classical mechanics [10, 16, 34, 6]. The second definition was later introduced to consider geometry of the phase space [2, 3, 44].

Below we use the first definition of canonical transformations because the Maupertuis–Jacobi transformation and the Kepler change of the time preserve the canonical form of the Hamilton–Jacobi equation, but it retains the corresponding differential 2-form $\Omega=\sum\limits_{i=1}^{n+1} dp_i\wedge dq_i$ on the level $\mathcal{Q}^{2n-1}$ (1.4) only [5].

We introduce general canonical transformations of the extended phase space $\mathcal{M}_E$

$$\begin{aligned} &t\mapsto\widetilde{t}, \qquad d\widetilde{t}=v(p,q)dt,\\ &H\mapsto\widetilde{H}, \qquad \widetilde{H}=v(p,q)^{-1}H. \end{aligned} \tag{3.4}$$

which change the initial equations of motion

$$\frac{dq_i}{d\widetilde{t}}=v^{-1}(p,q)\left(\frac{dq_i}{dt}-\widetilde{H}\frac{\partial v}{\partial p_i}\right),\qquad \frac{dp_i}{d\widetilde{t}}=v^{-1}(p,q)\left(\frac{dp_i}{dt}+\widetilde{H}\frac{\partial v}{\partial q_i}\right),$$

but preserve the canonical form of the Hamilton–Jacobi equation

$$\frac{\partial\mathcal{S}}{\partial t}+H=0,\qquad \text{where}\qquad \mathcal{S}=\int(p\,dq-H\,dt). \tag{3.5}$$

and retain the corresponding zero-energy surfaces (3.3) at $v(p,q)\neq 0$

$$\widetilde{\mathcal{H}}(p,q)=v(p,q)^{-1}\mathcal{H}(p,q)=0.$$

Zeroes of the function $v(p,q)$ determine the behavior of the system with respect to the inversion of time [6]. Here we do not consider this problem in detail.

The Maupertuis–Jacobi mapping (2.2), (2.3) may be rewritten as such a canonical transformation (3.4) of the extended phase space

$$T(p,q)\mapsto\widetilde{T}(p,q)=v(p,q)^{-1}T(p,q),\qquad v(p,q)=E-V(q),$$

which maps an initial geodesic flow into another geodesic flow. This map preserves integrability, if the function $v(p,q)$ is constructed by any potential $V(q)$, which may be added to the initial kinetic energy $T(p,q)$ without loss of integrability [10].

In contrast with the Maupertuis–Jacobi transformations, even if the general canonical transformation of $\mathcal{M}_E$ (3.4) retains integrability, we have no general method to construct new integrals of motion starting with initial ones. To solve this problem in [38, 39, 41, 42, 43], we used some analogies with the one known example of such transformations due to Kepler.

4 The Kepler change of the time

We begin with brief description of the Kepler change of the time [17]. We commence with two-dimensional oscillator defined by the Hamilton function

$$H_{\rm osc}(p,q) = p_1^2 + p_2^2 + a\left(q_1^2 + q_2^2\right) + b, \qquad a, b \in \mathbb{R}.$$

For this system the Kepler canonical transformation (3.4) of $\mathcal{M}_E$ with the function

$$v(p,q) = q_1^2 + q_2^2 \tag{4.1}$$

preserves integrability. After change of the time (3.4) and the point canonical transformation to other variables

$$x = q_1 q_2, \qquad y = \left(q_1^2 - q_2^2\right)/2 \tag{4.2}$$

integral trajectories of the oscillator come to be trajectories of the Kepler problem defined by

$$\widetilde{H}_{\rm kepl}(p,x) = \frac{H_{\rm osc}(p,q)}{q_1^2 + q_2^2} = p_x^2 + p_y^2 + \frac{b}{2\sqrt{x^2+y^2}} + a. \tag{4.3}$$

Various parametric and nonparametric forms of the common trajectories are discussed in [17, 2, 3]. Coincidence of the integral trajectories may be regarded as a local result, whereas the corresponding canonical transformation of the extended phase space preserves integrability in the whole initial phase space.

As for the Maupertuis–Jacobi transformation, the function $v(p,q)$ (4.1) in the Kepler transformation (4.3) could be identified with the oscillator potential $V(q)$. Below we prove that it is a simple coincidence. Nevertheless canonical transformations of the type (3.4) have been called the coupling constant metamorphoses in [12, 23] because of this casual coincidence.

The Kepler change of time has been generalized by Liouville [18]. The Liouville integrable systems are systems with the natural Hamilton function

$$\widetilde{H}(p,q) = \widetilde{T} + \widetilde{V},$$

where the kinetic and potential energies are given by

$$\widetilde{T} = v^{-1}(q)\sum_{i=1}^n a_i p_i^2, \qquad \widetilde{V} = v^{-1}(q)\sum_{i=1}^n U_i, \qquad v(q) = \sum_{i=1}^n v_i.$$

The functions a_i, v_i and U_i depends only on the variable q_i.

For the Liouville system the following quantities

$$\widetilde{I}_j = a_j p_j^2 + U_j - \widetilde{H} v_j, \qquad j = 1, \ldots, n,$$

are integrals of motion in involution and $\sum I_j = 0$. Thus, we have n quadratic integrals of motion including the Hamilton function and consequently the Liouville systems are completely integrable. Note the two-dimensional oscillator and the Kepler model belong to the Liouville family of integrable systems.

We recall how equations of motion were integrated in quadratures by Liouville [18]. From $\widetilde{I}_j = \alpha_j$, one obtains a system of differential equations

$$\frac{dq_j}{\sqrt{a_j\left(\alpha_j + \widetilde{E}v_j - U_j\right)}} = \frac{d\widetilde{t}}{v(q)}, \qquad j = 1, \ldots, n.$$

Here $\widetilde{H} = \widetilde{E}$ and time $\widetilde{t}$ is associated with the Hamilton function $\widetilde{H}$. Choosing a new time variable t according to

$$dt = v^{-1}(q)\widetilde{t}$$

we come down to the system of equations

$$\frac{dq_j}{\sqrt{a_j\left(\alpha_j + \widetilde{E}v_j - U_j\right)}} = dt.$$

It allows one to find integral trajectories $q_j = \gamma_j(t)$. After that the new parameter t may be expressed in terms of the initial time $\widetilde{t}$ by the quadrature

$$\widetilde{t} = \int^t v(q_1(\tau), \ldots, q_n(\tau))d\tau. \tag{4.4}$$

This transformation is related to a new parametric form of the same trajectories $q_j = \beta_j(\widetilde{t})$.

In fact, Liouville tacitly used canonical transformation of the extended phase space (3.4) and considered a new integrable Hamilton function $H = v(q)\,\widetilde{H}$ instead of the initial one. After integration of equations of motion for the new system we have change parametric form of the trajectories in order to describe solution of the initial problem. For instance in parametric form of the ellipse the Kepler time (1.2) is explicitly the Liouville quadrature (4.4).

Note canonical transformations of the extended phase space (3.4) have a natural counterpart in quantum mechanics. Namely, it is known that the standard eigenvalue problem of the Hamiltonian operator $H(p,q)$,

$$H(p,q)\Psi = (H_0 + aV + b)\Psi = E\Psi,$$

may be associated with the eigenvalue problem of the charge operator a

$$\widetilde{H}(p,q)\widetilde{\Psi} = \left(\widetilde{H}_0 + (b - E)V^{-1}\right)\widetilde{\Psi} = -a\widetilde{\Psi}.$$

In quantum mechanics such a duality of the two eigenvalue problems has been used by Schrödinger and many other [25]. Canonical transformation of the extended phase space (3.4) is an analogue of this duality. It is interesting that for the first time this duality has been studied for the quantum Kepler problem. In the Birman–Schwinger formalism function $v(q)$ is called a “sandwich” potential [25]. Recall that in the Birman–Schwinger formalism we can estimate the spectrum and eigenfunctions of the one Hamiltonian $\widetilde{H}$, by using the known spectrum and eigenfunctions of the dual Hamiltonian H. Moreover, for some quantum models it is a single known way to find solutions of the initial Schrödinger equation. Below we briefly discuss a similar property for the quantum Toda lattice.

5 The generalized Kepler change of time

The Maupertuis–Jacobi mapping is traditionally used for the search of new integrable systems. The Kepler change of time and the Liouville reparameterisation of trajectories have been used for integration of equations of motion. In this Section we propose some generalisations of the Kepler–Liouville results, which may be useful for the search of new integrable systems as well.

All the Liouville systems are particular case of Stäckel integrable systems. Therefore let us briefly recall some necessary facts about Stäckel systems [33]. The nondegenerate $n \times n$ Stäckel matrix $\mathbf{S}$, its j-th column of which s_{kj}, depends only on q_j

$$\det \mathbf{S} \neq 0, \qquad \frac{\partial s_{kj}}{\partial q_m} = 0, \qquad j \neq m$$

defines the set of functionally independent integrals of motion, $\{I_k\}_{k=1}^n$, where

$$I_k = \sum_{j=1}^{n} c_{jk} \left(p_j^2 + U_j(q_j) \right), \qquad c_{jk} = \frac{\mathbf{S}^{kj}}{\det \mathbf{S}}, \tag{5.1}$$

which are quadratic in the momenta. Here $\mathbf{C} = [c_{jk}]$ denotes the inverse matrix of $\mathbf{S}$ and $\mathbf{S}^{kj}$ is the cofactor of the element s_{kj}.

Proposition 1 [38]. *If the two Stäckel matrices $\mathbf{S}$ and $\tilde{\mathbf{S}}$ be distinguished by the m-th row only, i.e.*

$$s_{kj} = \tilde{s}_{kj}, \qquad k \neq m,$$

the corresponding Hamilton functions I_m and $\widetilde{I}_m$ (5.1) with a common set of potentials U_j are related by canonical transformations of the extended phase space $\mathcal{M}_E$

$$I_m \mapsto \widetilde{I}_m = v^{-1}(q) I_m(p,q), \qquad d\widetilde{t}_m = v(q) dt_m, \tag{5.2}$$

where

$$v(q_1, \dots, q_n) = \frac{\det \widetilde{\mathbf{S}}(q_1, \dots, q_n)}{\det \mathbf{S}(q_1, \dots, q_n)}. \tag{5.3}$$

The proposed generalization of the Kepler transformation maps one integrable Stäckel system into another integrable Stäckel system. For instance, the Stäckel matrices for the oscillator and the Kepler problem are

$$\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \widetilde{\mathbf{S}} = \begin{pmatrix} q_1^2 & q_2^2 \\ 1 & -1 \end{pmatrix}. \tag{5.4}$$

It is obvious that the Kepler change of time (4.3) coincides with the proposed mapping (5.2).

We return to the Kepler transformation (4.3) of $\mathcal{M}_E$. After permutation of coordinates and momenta ($q_{1,2} \leftrightarrow p_{1,2}$) the Hamilton function for the Kepler problem

$$\widetilde{H}_{\mathrm{kepl}} = a \frac{p_1^2 + p_2^2 + a^{-1}\left(q_1^2 + q_2^2\right) + a^{-1} b}{p_1^2 + p_2^2} = a \frac{H_{\mathrm{osc}}}{H_{\mathrm{free}}}$$

becomes a ratio of the Hamilton function $H_{\rm osc}$ for the oscillator and the Hamilton function $H_{\rm free}$ for the free motion.

So for any two integrable systems the ratio of their Hamilton functions could be the Hamilton function of a third integrable system on the same phase space. The main remaining problem is a search of a complete set of integrals of motion.

We consider two integrable hamiltonian systems on the common phase space $\mathcal{M}$. These systems are defined by the two sets of independent integrals of motion $\{I_j\}_{j=1}^n$, and $\{J_j\}_{j=1}^n$ in involution, i.e.

$$\{I_j, I_k\} = 0 \qquad \text{and} \qquad \{J_j, J_k\} = 0, \qquad j,k = 1,\dots,n.$$

Introduce the antisymmetric matrix $\mathcal{K} = (\mathbf{I} \otimes \mathbf{J})$, which is the inner product of the two independent vectors of integrals $\mathbf{I}$ and $\mathbf{J}$ in $\mathbb{R}^n$. Any column or row of this matrix defines a set of $n-1$ independent functions

$$\mathcal{K}_{ij} = (\mathbf{I} \otimes \mathbf{J})_{ij} = I_i\, J_j - I_j\, J_i, \qquad i,j = 1,\dots,n.$$

Proposition 2 [41]. *If all the differences of integrals of motion $(I_j - J_j)$ with the common index $j = 1,\dots,n$ are in involution, i.e.*

$$\{I_j - J_j, I_k - J_k\} = 0\,, \qquad j,k = 1,\dots,n, \tag{5.5}$$

then the ratio of integrals

$$K_m = \frac{I_m}{J_m} \tag{5.6}$$

and $n-1$ functions K_j, $j \neq m$

$$K_j = \frac{\mathcal{K}_{mj}}{J_m} = \frac{I_m J_j - I_j J_m}{J_m} = K_m J_j - I_j, \qquad m \neq j = 1,\dots,n \tag{5.7}$$

are integrals of motion for new integrable system on the same phase space.

Thus the mapping (5.6) defines a canonical transformation (3.4) of the extended phase space, which preserves the property of integrability. To apply this transformation we have to find two integrable systems satisfying condition (5.5).

We consider a pair of the Stäckel systems with a common Stäckel matrix $\mathbf{S}$ and with different potentials U_j. Namely, in addition to the system with integrals $\{I_k\}$ (5.1), we introduce the second integrable system with the similar integrals of motion,

$$J_k = \sum_{j=1}^n c_{jk} \left(p_j^2 + W_j(q_j)\right), \qquad k = 1,\dots,n. \tag{5.8}$$

At least one potential $U_j(q_j)$ has to be functionally independent of the corresponding potential $W_j(q_j)$.

Proposition 3 [41]. *Any two integrable systems defined by the same Stäckel matrix* $\mathbf{S}$ *and by functionally independent potentials $U_j(q_j)$ and $W_j(q_j)$ satisfy the necessary condition (5.5) of the previous proposition. Thus, the ratio of the two Stäckel integrable Hamiltonians defines new integrable system*

$$(I_m, J_m) \quad \mapsto \quad K_m = v(p,q)^{-1} I_m = \frac{I_m}{J_m}. \tag{5.9}$$

It is obvious that all the integrals I_k and J_k differe by the potential part

$$(I_k - J_k) = \sum_{j=1}^{2} c_{jk} \left[U_j(q_j) - W_j(q_j)\right]$$

depending on the coordinates q only. Thus systems with a common Stäckel matrix $\mathbf{S}$ satisfy condition (5.5).

The Hamilton function (5.9) has the following form

$$H(p,q) = K_m = \frac{\sum\limits_{j=1}^{n} c_{jm}\left[p_j^2 + U_j(q_j)\right] - \beta_m}{\sum\limits_{j=1}^{n} c_{jm}\left[p_j^2 + W_j(q_j)\right]}, \qquad \beta_m \in \mathbb{R}. \tag{5.10}$$

This Hamiltonian $H(p,q)$ is a rational function in the momenta, but next one can try to use canonical transformations to simplify it. Occasionally, one obtains again a natural type of Hamilton function. For instance, according to [41], integrable systems with the following Hamilton functions

$$\begin{aligned} H_I &= p_x^k p_y^k + a(xy)^{-\frac{k}{k+1}}, \qquad a,k \in \mathbb{R}, \\ H_{II} &= p_x^k + p_y^k + a(xy)^{-\frac{k}{k+1}}, \end{aligned} \tag{5.11}$$

belong to the proposed family of generalized Stäckel systems. At $k = 1$ the first Hamiltonian coincides with the Hamiltonian of the Kepler problem. At $k = 2$ the second integrable Hamiltonian has been found by Fokas and Lagerström (see references within [23]). In this case both initial systems satisfy to Kowalevski–Painlevé criterion, whereas the resulting Fokas–Lagerstrom system admits asymptotic solutions with fractional powers in t [23].

6 Properties of the change of time for the Stäckel systems

For the Stäckel family of integrable systems we proposed two different examples, (5.2) and (5.9), of the canonical transformations (3.4) of the extended phase space $\mathcal{M}_E$. Now we consider some properties of these transformations.

Recall that the common level surface of the Stäckel integrals I_j

$$M_\alpha = \left\{ z \in \mathbb{R}^{2n} : I_i(z) = \alpha_i,\ i = 1, \ldots, n \right\} \tag{6.1}$$

is diffeomorphic to the n-dimensional real torus. We can immediately construct the one-dimensional separated equations

$$p_j^2 = \left(\frac{\partial \mathcal{S}_0}{\partial q_j}\right)^2 = P_j(q_j) = \sum_{i=1}^{n} \alpha_i s_{ij}(q_j) - U_j(q_j, a), \qquad a_k \in \mathbb{R}. \tag{6.2}$$

Here $\mathcal{S}_0$ is a reduced action functional [33]. Integral trajectories $q_j(t, \alpha_1, \ldots, \alpha_n)$ are determined from the following equations

$$\sum_{j=1}^{n} \int \frac{s_{kj}(q_j)\, dq_j}{\sqrt{P_j(q_j)}} = \beta_k, \qquad \beta_1 = t. \tag{6.3}$$

In fact the polynomial $P(\lambda)$ and the contour of integration depend upon the values α_j of the integrals of motion, which are dropped in the notation.

For the rational entries of $\mathbf{S}$ and rational potentials $U_j(q_j)$ the Riemann surfaces (6.2) are isomorphic to the hyperelliptic curves

$$\mathcal{C}_j: \quad \mu_j^2 = P(\lambda_j) = \sum_{k=1}^{2g+1} a_k \lambda_j^k. \tag{6.4}$$

Considered together these curves determine the n-dimensional Lagrangian submanifold in the phase space $\mathcal{M} = \mathbb{R}^{2n}$

$$\mathcal{C}^{(n)}: \quad \mathcal{C}_1(p_1, q_1) \times \mathcal{C}_2(p_2, q_2) \times \cdots \times \mathcal{C}_n(p_n, q_n), \tag{6.5}$$

which is decomposed into plane curves. Applying Arnold's method [2] we find that the action variables have the form

$$\mathfrak{s}_j = \oint_{A_j} \sqrt{P(\lambda_j)}\, d\lambda_j. \tag{6.6}$$

The Abel transformation linearizes the equations of motion on the Lagrangian submanifold $\mathcal{C}^{(n)}$ in terms of abelian differentials of the first kind on the corresponding spectral curves [37, 38].

We consider a pair of the Stäckel systems related by the first generalization of the Kepler mapping (5.2) such that the potentials $U_j(\lambda) = \sum a_k \lambda^k$ depend on arbitrary parameters a_k. According to [38] initial and resulting integrable systems are associated with algebraic hyperelliptic curves (6.4) $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ described by

$$\begin{aligned} \mathcal{C}: &\quad \mu^2 = \sum a_j \lambda^j + a_m \lambda^m + E\lambda^k + \sum \alpha_i \lambda^i, \\ \widetilde{\mathcal{C}}: &\quad \mu^2 = \sum a_j \lambda^j + \widetilde{E}\lambda^m + a_k \lambda^k + \sum \widetilde{\alpha}_i \lambda^i. \end{aligned} \tag{6.7}$$

Here the n coefficients $\{\alpha_j\}$ and $\{\widetilde{\alpha}_j\}$ are values of the integrals of motion such that $E = \alpha_1$ and $\widetilde{E} = \widetilde{\alpha}_1$. The other coefficients $a_j \in \mathbb{R}$ are arbitrary parameters (charges), which define the potential part of the Hamilton function $H(p, q) = T(p) + V(q, a)$. Canonical transformations of $\mathcal{M}_E$ give rise to mutual permutation of the energy E and one of the parameters a_k (charge).

The initial and resulting Riemann surfaces are topologically equivalent. One can prove that the corresponding integrable systems are topologically equivalent too. Moreover, initial curves coincide with the resulting curves at the special values of integrals of motion

$$E = a_m, \qquad \widetilde{E} = a_k, \qquad \alpha_j = \widetilde{\alpha}_j, \qquad j = 2, \ldots, n.$$

In this case we have two different parametric forms of the common Lagrangian submanifold $\mathcal{C}^{(n)}$ (6.5) depending on n values of integrals α_j and constants a_k. On the corresponding submanifolds $\mathcal{M}_\alpha$ and $\widetilde{\mathcal{M}}_\alpha$ (6.1) integral trajectories of the initial system $q_j(t)$ (6.3), coincide with integral trajectories of the resulting system $q_j(\widetilde{t})$. In the neighbourhood of the intersection of these submanifolds we can introduce the common set of the action-angle variables (6.6). In this region the Kepler transformation (5.2) retains the differential

2-form Ω in $\mathcal{M}_E$ and the function $v(p,q) = v(\mathfrak{s})$ depends on the common action variables only.

We consider a pair of the Stäckel systems related by the second generalization of the Kepler mapping (5.9). According to [41] initial and resulting integrable systems are associated with algebraic hyperelliptic curves (6.4) $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ of the form

$$\mathcal{C}: \quad \mu^2 = \sum a_i\lambda^i + \sum \alpha_j\lambda^j \quad \mapsto \quad \widetilde{\mathcal{C}}: \quad \mu^2 = \sum b_k\lambda^k + \sum \widetilde{\alpha}_m\lambda^m.$$

Here the resulting coefficients are rational functions of the initial ones [41]. So canonical transformations of $\mathcal{M}_E$ (5.9) give rise to transformations of the modulus of the curves only.

As mentioned above the corresponding Riemann surfaces and integrable systems are topologically equivalent. At the special choice of integrals $\{\alpha, \widetilde{\alpha}\}$ and parameters of potentials $\{a_i, b\}$ integral trajectories of the initial system coincide with trajectories of the resulting system. In a small region of $\mathcal{M}$ this generalization of the Kepler transformation (5.9) preserves the differential 2-form Ω in $\mathcal{M}_E$ and the function $v(p,q)$ depends on the action variables only.

In an attempt to understand the origin of the conservation of integrability by canonical transformation of $\mathcal{M}_E$ (3.4) one can attempt to rewrite equations of motion in the Lax form

$$\{H(p,q), L(\lambda)\} = [L(\lambda), A(\lambda)].$$

For the some classes of Stäckel systems the Lax matrices have been constructed in [37, 38, 40].

We consider some examples only. One of the simplest Lax matrix $L(\lambda)$ for the one-dimensional Stäckel systems is given by

$$L(\lambda) = \begin{pmatrix} p & \lambda - q \\ \left[\frac{\phi(\lambda)}{\lambda - q}\right]_{MN} & -p \end{pmatrix}. \tag{6.8}$$

Here $\phi(\lambda) = \sum \phi_k\lambda^k$ is a parametric function of the spectral parameter λ and the elements, $[z]_{MN}$, are the linear combinations of the Taylor projections

$$[z]_N = \left[\sum_{k=-\infty}^{+\infty} z_k\lambda^k\right]_N \equiv \sum_{k=0}^{N} z_k\lambda^k, \tag{6.9}$$

or the Laurent projections [36]. The coefficients ϕ_k of the function $\phi(\lambda)$ are the constants of motion, which may be parameters a_k or integrals of motion. So at arbitrary M, N the family of the Lax matrices $L_\phi(\lambda)$ may be associated with a web of algebraic curves instead of one concrete curve.

For instance, we present some one-dimensional Hamilton functions, the functions $\phi(\lambda)$ and the corresponding Lax matrices associated with the first Taylor projection $[z]_{01}$. An application of the pure numeric function $\phi(\lambda)$ yields

$$\begin{gathered} H = p^2 + aq^2 + bq, \qquad \phi = -a\lambda^2 - b\lambda, \\ L = \begin{pmatrix} p & \lambda - q \\ -a(\lambda + q) - b & -p \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}. \end{gathered} \tag{6.10}$$

The corresponding spectral curve is given by

$$\mathcal{C}: \quad \mu^2 = a\lambda^2 + b\lambda - H.$$

Note that in the spectral curve we can substitute integrals of motion, while in the corresponding Lagrangian submanifold we have to substitute the values of integrals of motion.

From (6.7) the first possible change of this curve looks like

$$\mathcal{C} \quad\mapsto\quad \widetilde{\mathcal{C}}:\ \mu^2 = a\lambda^2 + (b - \widetilde{H})\lambda + c.$$

The associated canonical transformation of the extended phase space,

$$\widetilde{H} = v^{-1}(H + c) = q^{-1}(H + c),$$

changes the Lax matrices by the following rule

$$\phi = -a\lambda^2 - (b - \widetilde{H})\lambda, \qquad \widetilde{L} = L + \widetilde{H}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{A} = v^{-1}(q)A. \tag{6.11}$$

The second possible change of the curve

$$\mathcal{C} \quad\mapsto\quad \widehat{\mathcal{C}}:\ \mu^2 = (a - \widehat{H})\lambda^2 + b\lambda + c$$

may be related to the other canonical transformation of the extended phase space,

$$\widehat{H} = v^{-1}(H + c) = q^{-2}(H + c),$$

such that

$$\begin{gathered} \phi = -(a - \widehat{H})\lambda^2 - b\lambda, \\ \widehat{L} = L + \widehat{H}\begin{pmatrix} 0 & 0 \\ l + q & 0 \end{pmatrix}, \qquad \widehat{A} = v^{-1}(q)\left[A + \widehat{H}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]. \end{gathered} \tag{6.12}$$

It is not hard to check that the Poisson bracket relations for all these Lax matrices $L(\lambda)$, $\widetilde{L}(\lambda)$ and $\widehat{L}(\lambda)$ are closed into the linear r-matrix algebra. At the first case the r-matrix is a constant matrix $r = \Pi/(\lambda - \mu)$, whereas the second and third r-matrices depend on dynamical variables. Here Π is a permutation of auxiliary spaces [8].

We turn now to the original change of time in Kepler problem (4.3). The Lax matrices for the two-dimensional oscillator,

$$\mathcal{L}(\lambda) = \begin{pmatrix} L_1(\lambda, p_1, q_1) & 0 \\ 0 & L_2(\lambda, p_2, q_2) \end{pmatrix}, \qquad \mathcal{A}(\lambda) = \begin{pmatrix} A_1(\lambda) & 0 \\ 0 & A_2(\lambda) \end{pmatrix}, \tag{6.13}$$

may be constructed from two independent 2×2 blocks $L_j(\lambda)$ and $A_j(\lambda)$ of (6.10). The corresponding spectral curve $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ is a product of two hyperelliptic curves.

The Kepler mapping (4.3) gives rise to the following transformation of the Lax matrices

$$\begin{gathered} \widetilde{\mathcal{L}}(\lambda) = \mathcal{L}(\lambda) - \widetilde{H}_{\rm kepl}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda + q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda + q_2 & 0 \end{pmatrix}, \\ \widetilde{\mathcal{A}}(\lambda) = v(p, q)^{-1}\left[\mathcal{A}(\lambda) - \widetilde{H}_{\rm kepl}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right]. \end{gathered} \tag{6.14}$$

The spectral curve $\widetilde{\mathcal{C}}$ remains a product of two new hyperelliptic curves. The initial oscillator and the resulting Kepler model are separable in the common variables, which lie on these curves. Note that these separated variables are cartesian coordinates for the oscillator and parabolic coordinates for the Kepler problem.

Similar transformations of the Lax matrices (6.11), (6.12), (6.14) may be proposed for the other two-dimensional Stäckel systems separable in cartesian or parabolic coordinates. For the two-dimensional Stäckel systems separable in elliptic or polar coordinates transformation of the Lax matrices has a more complicated form [38]. These transformation may be constructed by using two different outer automorphisms of infinite-dimensional representations of underlying $sl(2)$ algebra proposed in [40].

In the next Sections we consider similar transformations of the Lax matrices (6.11) and the spectral curves (6.7) for integrable systems associated with non-hyperelliptic algebraic curves and for the non-Stäckel integrable systems.

7 On integrable systems with quartic potential

According to [23, 39, 43], canonical transformations of the extended phase space relate three integrable cases of Henón–Heiles systems with the three integrable cases of the Holt systems. One of these systems admits a 2×2 Lax matrix and the corresponding spectral curve is an hyperelliptic algebraic curve. Another two systems possess 3×3 Lax matrices and the corresponding spectral curves are the trigonal algebraic curves $\mu^3 = P(\lambda)$. As for the Stäckel systems transformations of these 3×3 Lax matrices are shifts of their entries by the element of the extended phase space (6.11).

We consider two integrable systems with quartic potential for which 4×4 Lax matrices were constructed in [4] by applying relations with stationary flows of some known integrable PDEs. The first system belongs to the Stäckel family and is separable in cartesian coordinates. Its Hamilton function is

$$H(p,q) = p_1^2 + p_2^2 + \frac{1}{4}\left(q_1^4 + 6q_1^2 q_2^2 + q_2^4\right) \tag{7.1}$$

As above we could construct some block 4×4 Lax matrices (6.13) for this system in cartesian separated variables. The corresponding spectral curve is a product of hyperelliptic curves. The two pairs of the separated variables lie on these two curves.

Another 4×4 Lax matrix for the same system may be obtained from the Lax representation for the Hirota–Satsuma coupled KdV system [4]. This Lax matrix

$$L(\lambda) = \begin{pmatrix} -p_1 q_1 & q_1^2 & 0 & 1 \\ \lambda - p_1^2 & p_1\, q_1 & -\frac{q_1^2+q_2^2}{2} & 0 \\ 0 & \lambda & -p_2 q_2 & q_2^2 \\ -\lambda\frac{q_1^2+q_2^2}{2} & 0 & \lambda - p_2^2 & p_2 q_2 \end{pmatrix}$$

does not have a pure block-diagonal structure and the corresponding spectral curve $\mu^4 = P(\lambda)$ is not a product of two hyperelliptic curves. The second Lax matrix $A(\lambda)$ may be found in [4].

For this Stäckel system canonical transformation of the extended phase space (5.2)

$$\widetilde{H} = \left(q_1^2 + q_2^2\right)^{-1}(H - b)$$

gives rise to the shift of the first Lax matrix and the rescaling of the second Lax matrix

$$\widetilde{L} = L + \widetilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{A} = v^{-1}A.$$

In contrast with the Stäckel systems (6.7), transformation of the corresponding spectral curves acts on the both side of the equation defining the curve

$$\begin{aligned} &\mathcal{C}: \quad \mu^4 = \lambda^3 - H\lambda^2 + J\lambda, \\ &\widetilde{\mathcal{C}}: \quad \mu^4 - 2\mu^2\lambda\widetilde{H} = \lambda^3 - \left(\widetilde{H}^2 - b\right)\lambda^2 + \widetilde{J}\lambda. \end{aligned} \tag{7.2}$$

Note that after canonical transformations of the other variables (p, q) the new Hamilton function $\widetilde{H}$ may be rewritten in the natural form

$$\widetilde{H} = p_x^2 + p_y^2 + \frac{x^2 + 2y^2 - b}{2\sqrt{x^2 + y^2}}.$$

The second integrable system with the quartic potential is the non-Stäckel system defined by the following Hamilton function

$$H = p_1^2 + p_2^2 - \frac{1}{8}\left(q_1^4 + 6q_1^2q_2^2 + 8q_2^4\right). \tag{7.3}$$

This system is separable after a so-called quasi-point canonical transformation [24].

The second system possesses 4×4 Lax matrix

$$L = \begin{pmatrix} \frac{q_2q_1^2}{2} + q_1p_1 & -q_1^2 & 2q_2 & 2 \\ \frac{q_1^2q_2^2}{4} + p_1^2 + q_1q_2p_1 + \frac{\lambda}{2} & -\frac{q_2q_1^2}{2} - q_1p_1 & p_2 + q_2^2 + \frac{q_1^2}{4} & 0 \\ -2q_2\lambda & 2\lambda & -\frac{q_2q_1^2}{2} + q_1p_1 & -q_1^2 \\ \lambda\left(-p_2 + q_2^2 + \frac{q_1^2}{4}\right) & 0 & \frac{q_1^2q_2^2}{4} + p_1^2 - q_1q_2p_1 + \frac{\lambda}{2} & \frac{q_2q_1^2}{2} - q_1p_1 \end{pmatrix}$$

which was obtained using a gauge transformation of the Hirota–Satsuma coupled KdV system [4].

For this system we propose the canonical transformation of the extended phase space given by

$$\widetilde{H} = \left(q_1^2 + 4q_2^2\right)^{-1}(H - b),$$

which preserves integrability. As for the Stäckel systems, transformation of the Lax matrices retains a shift of the first Lax matrix and rescaling of the second Lax matrix

$$\widetilde{L} = L + 2\widetilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{A} = v^{-1}A.$$

The spectral curves are changed by the rule

$$\begin{aligned} \mathcal{C}: \quad & \mu^4 = \lambda^3 + 4H\lambda^2 + J\lambda, \\ \widetilde{\mathcal{C}}: \quad & \mu^4 - 8\mu^2\lambda\widetilde{H} = \lambda^3 + 4\left(b - 4\widetilde{H}^2\right)\lambda^2 + \widetilde{J}\lambda. \end{aligned} \tag{7.4}$$

Recall that for Stäckel systems with quadratic integrals of motion canonical transformations of the extended phase space (5.2), (5.9) preserve separated variables lying on the common hyperelliptic curve (6.7).

For the Hénon–Heiles systems and systems with quartic potentials transformations of the trigonal spectral curve $\mu^3 = P(\lambda)$ [39] and the curve $\mu^4 = P(\lambda)$ (7.2), (7.4) have a more complicated form. In this case separated variables for the initial and resulting systems are different [39]. It means that the proposed canonical transformation of the extended phase space preserve integrability, but seriously changes other properties of the systems.

8 The Toda lattices

Before proceeding further, it is useful to recall some known facts about the Toda lattices (all details may be founded in the review [26]).

Let $\mathfrak{g}$ be a real, split, simple Lie algebra of rank $\mathfrak{g} = n$ and let $K(\cdot,\cdot)$ be its Killing form and P be a system of simple roots. we identify the phase space with the coadjoint algebra $\mathcal{M} \simeq \mathfrak{g}_R^* = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^*$. Here $\mathfrak{g}_+$ is a Borel subalgebra and $\mathfrak{g}_-$ is the opposite nilpotent subalgebra of $\mathfrak{g}$.

The Lax matrices for the Toda lattices are

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{n} p_i h_i + \sum_{\alpha\in P} \cdot \exp K(\alpha, q) \cdot e_{-\alpha} + \sum_{\alpha\in P} a_\alpha e_\alpha, \\ \mathcal{A} &= -\sum_{\alpha\in P} \exp K(\alpha, q) \cdot e_{-\alpha}, \qquad a_\alpha \in \mathbb{R}. \end{aligned} \tag{8.1}$$

The Hamilton function is given by

$$H(p,q) = \frac{1}{2}K(\mathcal{L},\mathcal{L}) = \frac{1}{2}K(p,p) + \sum_{\alpha\in P} a_\alpha e^{\alpha(q)}. \tag{8.2}$$

Recall that in a shifted version of the Adler–Kostant–Symes scheme in order to construct the Toda orbit (8.1) in $\mathcal{M}$ we have to translate a dynamical orbit living in $\mathfrak{g}_+^*$ by adding to it a constant vector $e = \sum a_\alpha e_\alpha$ from the remaining part of $\mathcal{M}$. This vector e has to be a character and has to be a constant.

We replace now the phase space $\mathcal{M}$ by the extended phase space $\mathcal{M}_E$. Roughly speaking, to consider the Stäckel systems we exchange the pure numerical function ϕ (6.10) by the function with coefficients from $\mathcal{M}_E$ (6.11), (6.12). By a similar reasoning we try to construct the same modification of the Adler–Kostant–Symes scheme. Namely we translate the Toda orbit in $\mathcal{M} \simeq \mathfrak{g}_R^*$ by adding to it a constant vector from the remaining part of the whole space $\mathcal{M}_E$. As above this vector has to be a character and has to be a constant with respect to the new time.

In addition we impose a constraint on the possible change of parametric form of trajectories. As for the Stäckel systems (6.7) the initial invariant polynomial has to generate the arbitrary constant

$$K(\widetilde{\mathcal{L}},\widetilde{\mathcal{L}})=-b \tag{8.3}$$

instead of the Hamilton function (8.2). This condition together with the form of transformations of the Lax matrices dictates to us a very special choice of the functions $v(p,q)$ in (3.4) for the Toda lattices.

Proposition 4 [41]. *For each simple root $\beta\in P$ and for any constant $b_\beta\in\mathbb{R}$ the following canonical transformation of the extended phase space $\mathcal{M}_E$*

$$\begin{aligned} d\widetilde{t}&=e^{\beta(q)}\cdot dt,\\ \widetilde{H}_\beta&=e^{-\beta(q)}\cdot(H+b_\beta) \end{aligned} \tag{8.4}$$

maps the Toda lattice into the other integrable system. This canonical transformation induces the following transformation of the Lax matrices

$$\widetilde{\mathcal{L}}_\beta=\mathcal{L}-\widetilde{H}_\beta\cdot e_\beta,\qquad \widetilde{\mathcal{A}}=e^{-\beta(q)}\cdot\mathcal{A}. \tag{8.5}$$

Here H, $\mathcal{L}$ and $\mathcal{A}$ are the Hamiltonian (8.2) and the Lax matrices (8.1) for the corresponding Toda lattice.

In this proposition we explicitly determine the new parameter $\widetilde{t}$ and the new associated Hamilton function $\widetilde{H}$. The corresponding spectral curves depend on the choice of a representation of $\mathfrak{g}$. We prove that the Toda lattice and the new integrable system relate to a common geometric object in the one example only.

The number of the functional independent Hamilton functions $\widetilde{H}_\beta$, $\beta\in P$ depends upon the symmetries of the associated root system. For closed Toda lattices associated with the affine Lie algebras canonical time transformation has a similar form. Similar canonical transformations may be applied to the relativistic and discrete time Toda lattices.

We describe explicitly some new integrable systems related to the standard three-particle Toda lattice and two-particle Toda lattices associated to the affine algebras $X_2^{(1)}$. After an appropriate point transformation of coordinates (similar to (4.2), see [41]) all the Hamilton functions have the common form

$$\widetilde{H}=p_xp_y+\frac{b}{xy}+ax^{z_1}y^{z_2}+cx^{s_1}y^{s_2}+d,\qquad a,b,c,d\in\mathbb{R}, \tag{8.6}$$

where $z_{1,2}$ and $s_{1,2}$ are the roots of the different quadratic equations and are related to the angles of the corresponding Dynkin diagrams. Below we show these equations explicitly

$$\begin{array}{lll} A_3^{(1)}: & z^2+3z+3=0 & s^2+3s+3=0\\[2ex] B_2^{(1)}C_2^{(1)}: & \begin{array}{l} z^2+4z+5=0\\ z^2+2z+2=0\end{array} & \begin{array}{l} s^2+4s+5=0\\ s^2+3s+5/2=0\end{array}\\[3ex] D_2^{(1)}: & \begin{array}{l} z^2+2z+2=0\\ z^2+2z+2=0\end{array} & \begin{array}{l} s^2+2s+2=0\\ (s+2)^2=0\end{array} \end{array}$$

$$G_2^{(1)}: \quad \begin{array}{ll} z^2+2z+4=0 & s^2+5s+7=0 \\ z^2+2z+4=0 & s^2+3s+3=0 \\ z^2+3z+7/3=0 & s^2+3s+3=0 \end{array}$$

Originally the integrable system with the Hamilton function $\widetilde{H}$ (8.6) associated to the root system $A_3^{(1)}$ was found by Drach [7].

The corresponding second integrals of motion K are polynomials of the third, fourth and sixth order in momenta. Note that for the algebra $A_3^{(1)}$ all the three Hamiltonians H_β, $\beta \in P$ are equivalent. Two different Hamilton functions are associated with the algebras $B_2^{(1)}$, $C_2^{(1)}$ and $D_2^{(1)}$. For the $G_2^{(1)}$ algebra we have three independent potentials in (8.6).

9 On the common properties of the Toda lattices and the dual systems

For the periodic Toda lattice associated with the root system A_n the Hamilton function is

$$H(p,q)=\sum_{i=1}^{n}\left(\frac{1}{2}p_i^2+a_i e^{q_i-q_{i+1}}\right), \qquad a_i \in \mathbb{R}. \tag{9.1}$$

Here $\{p_i, q_i\}$ are canonical variables and the periodicity conventions $q_{i+n}=q_i$ and $p_{i+n}=p_i$ are always assumed for the indices of q_i and p_i.

The exact solution of the equations of motion is due to existence of a Lax equation with the following $n\times n$ Lax matrices [13, 9]

$$\begin{aligned} \mathcal{L}^{(n)}(\mu) &= \sum_{i=1}^{n} p_i E_{i,i} + \sum_{i=1}^{n-1}\left(e^{q_i-q_{i+1}}E_{i+1,i}+a_iE_{i,i+1}\right)+\mu e^{q_n-q_1}E_{1,n}+a_n\mu^{-1}E_{n,1}, \\ \mathcal{A}^{(n)}(\mu,q) &= \sum_{i=1}^{n-1} e^{q_i-q_{i+1}}E_{i+1,i}+\mu e^{q_n-q_1}E_{1,n}, \end{aligned} \tag{9.2}$$

where $E_{i,k}$ stands for the $n\times n$ matrix with unity on the intersection of the ith row and the kth column as the only nonzero entry.

According to [41, 42] canonical transformation of the extended phase space (3.4) by

$$v(p,q)=\exp(q_j-q_{j+1}), \qquad \widetilde{H}=e^{q_{j+1}-q_j}(H-b), \qquad b\in\mathbb{R} \tag{9.3}$$

maps the Toda lattice into the dual integrable system with the following equations of motion

$$\frac{dq_i}{d\tilde{t}}=v^{-1}(q)\frac{dq_i}{dt}, \qquad \frac{dp_i}{d\tilde{t}}=v^{-1}(q)\frac{dp_i}{dt}+\widetilde{H}(\delta_{i,j}-\delta_{i,j+1}). \tag{9.4}$$

Associated with the different indexes j canonical mappings (9.3) are related with each other by canonical transformations of the other variables (p,q).

The mapping (9.3) gives rise to the following transformation of the Lax matrices

$$\widetilde{\mathcal{L}}(\mu)=\mathcal{L}(\mu)-\widetilde{H}E_{j,j+1}, \qquad \widetilde{\mathcal{A}}(\mu)=v^{-1}(q)\mathcal{A}(\mu). \tag{9.5}$$

The corresponding transformation of the spectral curves is

$$\begin{aligned}
\mathcal{C}:\quad & -\mu-\frac{\prod_{i=1}^{n} a_i}{\mu}=P(\lambda)=\lambda^n+\lambda^{n-1}\mathbf{p}+\lambda^{n-2}\left(\frac{\mathbf{p}^2}{2}-H\right)+\sum_{i=1}^{n-3}J_i\lambda^i,\\
\widetilde{\mathcal{C}}:\quad & -\mu-\frac{(a_j-\widetilde{H})\prod_{i\neq j}^{n} a_i}{\mu}=\widetilde{P}(\lambda)=\lambda^n+\lambda^{n-1}\mathbf{p}+\lambda^{n-2}\left(\frac{\mathbf{p}^2}{2}-b\right)+\sum_{i=1}^{n-3}\widetilde{J}_i\lambda^i.
\end{aligned} \tag{9.6}$$

Here $\mathbf{p}=J_1=\sum p_i$ is the total momentum, H and $\widetilde{H}$ are the corresponding Hamilton functions and J_i, $\widetilde{J}_i$ are integrals of motion. Substituting the fixed values of integrals of motion in the product of curves one can construct the corresponding Lagrangian submanifold.

Applying Arnold's method [2, 9] to the standard form of the hyperelliptic curves C and $\widetilde{C}$ (9.6) one construct the action variables

$$\begin{aligned}
\mathfrak{s}_i &= \oint_{A_i}\frac{1}{2}\left(P(\lambda)+\sqrt{P(\lambda)^2-4\prod_{i=1}^{n}a_i}\right)d\lambda,\\
\widetilde{\mathfrak{s}}_i &= \oint_{\widetilde{A}_i}\frac{1}{2}\left(\widetilde{P}(\lambda)+\sqrt{\widetilde{P}(\lambda)^2-4(a_j-\widetilde{H})\prod_{i\neq j}^{n}a_i}\right)d\lambda,
\end{aligned} \tag{9.7}$$

where A_i and $\widetilde{A}_i$ are A-cycles of the Jacobi variety of the algebraic curves (9.6), respectively [9]. In fact the polynomials $P(\lambda)$, $\widetilde{P}(\lambda)$ and A-cycles depend on the values of constants of motion, which are dropped in the notation. The Abel transformation linearizes equations of motion in terms of first kind abelian differentials on the corresponding spectral curves.

Let parameters a_i determine potential of the Toda lattice (9.1) and parameters $\widetilde{a}_i$ and b define the potential of the dual system (9.3). At the special choice of the values of integrals of motion

$$H=b,\qquad \widetilde{H}=\widetilde{a}_j-\frac{\prod^{n}a_i}{\prod_{i\neq j}^{n}\widetilde{a}_i},\qquad J_i=\widetilde{J}_i=\alpha_i \tag{9.8}$$

the initial curve $\mathcal{C}$ is equal to the resulting curve $\widetilde{\mathcal{C}}$ (9.6). Thus, as for the Maupertuis–Jacobi mapping [5], integral trajectories of the Toda lattice coincide with the trajectories of the dual system on the intersection of the corresponding common levels of integrals $\mathcal{M}_\alpha$ and $\widetilde{\mathcal{M}}_\alpha$. In the neighbourhood of this intersection we can introduce the common set of the action variables (9.7) for the both systems. In this small subvariety of the phase space the function $v(p,q)=v(\mathfrak{s})$ depends on the action variables only.

At $a_i=1$ the Poisson bracket relations for the $n\times n$ Lax matrices can be expressed in the r-matrix form

$$\left\{\overset{1}{\mathcal{L}}(\mu),\overset{2}{\mathcal{L}}(\nu)\right\}=\left[r_{12}(\mu,\nu),\overset{1}{\mathcal{L}}(\mu)\right]+\left[r_{21}(\mu,\nu),\overset{2}{\mathcal{L}}(\nu)\right].$$

Here we used the standard notations

$$\overset{1}{\mathcal{L}}(\mu) = \mathcal{L}(\mu) \otimes I, \qquad \overset{2}{\mathcal{L}}(\nu) = I \otimes \mathcal{L}(\nu), \qquad r_{21}(\mu,\nu) = -\Pi r_{12}(\nu,\mu)\Pi,$$

and Π is the permutation operator in $\mathbb{C}^n \times \mathbb{C}^n$ [8]. Canonical transformation of the extended phase space (9.3) maps the constant r-matrix for the Toda lattice

$$r_{12}(\mu,\nu) = r_{12}^{\text{const}}(\mu,\nu) = \frac{1}{\mu-\nu}\left(\nu \sum_{m\geq i} + \mu \sum_{m<i}\right) E_{im} \otimes E_{mi}$$

into the following dynamical r-matrix

$$\widetilde{r}_{12}(\mu,\nu) = r_{12}^{\text{const}}(\mu,\nu) + \widetilde{\mathcal{A}}(\nu,q) \otimes E_{j,j+1},$$

where the second Lax matrix $\widetilde{\mathcal{A}}(\nu,q)$ and, therefore, the dynamical r-matrix $\widetilde{r}_{ij}(\mu,\nu)$ depend on coordinates only.

Another known 2×2 Lax representation [8, 31] for the same Toda lattice is equal to

$$T(\lambda) = L_1(\lambda)\cdots L_n(\lambda), \qquad \frac{dT}{dt} = \left[T(\lambda), A_n(\lambda)\right], \tag{9.9}$$

where

$$L_i = \begin{pmatrix} \lambda + p_i & e^{q_i} \\ -a_{i-1}e^{-q_i} & 0 \end{pmatrix}, \qquad A_i = \begin{pmatrix} \lambda & e^{q_i} \\ -a_i e^{-q_{i-1}} & 0 \end{pmatrix}. \tag{9.10}$$

Canonical transformation of the extended phase space (9.3) gives rise to the following transformation of the Lax matrices

$$\begin{aligned} \widetilde{T}(\lambda) &= L_1\cdots L_{j-1}\cdot\left[L_j\, L_{j+1} + \begin{pmatrix} H-b & 0 \\ 0 & 0 \end{pmatrix}\right]\cdot L_{j+2}\cdots L_n, \\ \widetilde{A}_n(\lambda) &= v^{-1}(q)A_n(\lambda). \end{aligned} \tag{9.11}$$

At $a_i = 1$ the Poisson brackets relations for the 2×2 Lax matrices $T(\lambda)$ (9.9) satisfy the following Sklyanin r-matrix relation

$$\left\{\overset{1}{T}(\lambda), \overset{2}{T}(\nu)\right\} = \left[R(\lambda-\nu), \overset{1}{T}(u)\overset{2}{T}(\nu)\right], \qquad R(\lambda-\nu) = \frac{\Pi}{\lambda-\nu}. \tag{9.12}$$

The mapping (9.3) transforms these quadratic relations into the following polylinear ones

$$\left\{\overset{1}{\widetilde{T}}(\lambda), \overset{2}{\widetilde{T}}(\nu)\right\} = \left[R(\lambda-\nu), \overset{1}{\widetilde{T}}(\lambda)\overset{2}{\widetilde{T}}(\nu)\right] + \left[r_{12}^{\text{dyn}}(\lambda,\nu), \overset{1}{\widetilde{T}}(\lambda)\right] + \left[r_{21}^{\text{dyn}}(\lambda,\nu), \overset{2}{\widetilde{T}}(\nu)\right].$$

The corresponding dynamical r-matrix is given by

$$r_{12}^{\text{dyn}}(\lambda,\nu) = A_n(\lambda,q) \otimes \left(L_1\cdots L_{j-1}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\cdot L_{j+1}\otimes L_n\right).$$

Here all matrices L_k depend on the spectral parameter ν and $A_n(\lambda, q)$ is the second Lax matrix (9.9).

Now we look at the separation of variables in the framework of the traditional consideration of the Toda lattice. A complete list of references can be found in [13, 9, 31]. Below we put a_i=1 and $j = 1$ without loss of generality, so that

$$\widetilde{H} = \exp(q_2 - q_1)(H + b), \qquad \widetilde{T} = \left[L_1 L_2 + \begin{pmatrix} H+b & 0 \\ 0 & 0 \end{pmatrix}\right] \cdot L_3 \cdots L_n.$$

This transformation changes the first row of the Lax matrix $T(\lambda)$ only

$$T = \begin{pmatrix} \mathbb{A}(\lambda) & \mathbb{B}(\lambda) \\ \mathbb{C}(\lambda) & \mathbb{D}(\lambda) \end{pmatrix} \quad \mapsto \quad \widetilde{T} = \begin{pmatrix} \widetilde{\mathbb{A}}(\lambda) & \widetilde{\mathbb{B}}(\lambda) \\ \mathbb{C}(\lambda) & \mathbb{D}(\lambda) \end{pmatrix}. \tag{9.13}$$

The separated variables $\{\lambda_1\, \lambda_2\, \ldots, \lambda_{n-1}\}$ for both systems are zeroes of the nondiagonal common entry

$$\mathbb{C}(\lambda) = \gamma \cdot \prod_{i=1}^{n-1} (\lambda - \lambda_i). \tag{9.14}$$

An additional set of variables is defined by the second common entry

$$\mu_i = \mathbb{D}(\lambda_i) \qquad \text{such that} \qquad \{\lambda_i, \log \mu_k\} = \delta_{ik}, \qquad i, k = 1, \ldots, n-1.$$

From $\det T(\lambda) = 1$ and $\det \widetilde{T}(\lambda) = (1 - \widetilde{H})$ one immediately obtains the one-dimensional equations

$$\begin{aligned} &\mathbb{A}(\lambda_i) = \mu_i^{-1}, && \mu_i + \mu_i^{-1} = P(\lambda_i), \\ &\widetilde{\mathbb{A}}(\lambda_i) = \left(1 - \widetilde{H}\right) \mu_i^{-1}, && \mu_i + \left(1 - \widetilde{H}\right) \mu_i^{-1} = \widetilde{P}(\lambda_i). \end{aligned} \tag{9.15}$$

At the special choice of parameters and values of integrals (9.8) initial separated equations coincide with the resulting ones.

Thus we prove that the initial and resulting systems have a common set of separated variables. On the other hand canonical transformation of the extended phase space (9.3) changes the form of the Bäcklund transformation [42] and the bihamiltonian structure, which are known for the Toda lattice.

Having obtained a simple change of the separated equations (9.15), one can hope that there is also a simple modification of the one-dimensional Baxter equations in quantum mechanics. Recall that from the works of Sklyanin [31, 14] one knows that the eigenfunctions of the quantum Toda lattice Hamiltonian are given by

$$\psi_E(q) = \int C(\lambda, E) \psi_\lambda(q)\, d\lambda, \qquad C(\lambda, E) = \prod_{j=1}^{n-1} c(\lambda_j, E).$$

Here ψ_λ are renormalized Whittaker functions and the functions $c(\lambda, E)$ satisfy to one-dimensional Baxter equation

$$P(\lambda) c(\lambda, E) = i^n c(\lambda + i\hbar, E) + i^{-n} c(\lambda - i\hbar, E),$$

where $P(\lambda)$ is a trace of the quantum monodromy matrix $T(\lambda)$. In the classical limit the polynomial $P(\lambda)$ enters in the spectral curve (9.6).

By using a similar approach [31, 14] we can suppose that the eigenfunctions for the dual system are expressed in terms of the same Whittaker functions

$$\widetilde{\psi}_{\widetilde{E}}(q) = \int \widetilde{C}\left(\lambda, \widetilde{E}\right) \psi_\lambda(q)\, d\lambda,$$

whereas the corresponding one-dimensional Baxter equation has to be changed

$$\widetilde{P}(\lambda)\widetilde{c}\left(\lambda, \widetilde{E}\right) = i^n \left(1 - \widetilde{E}\right) \widetilde{c}\left(\lambda + i\hbar, \widetilde{E}\right) + i^{-n}\widetilde{c}\left(\lambda - i\hbar, \widetilde{E}\right),$$

in accordance with the corresponding classical separated equations (9.15). In the classical limit the polynomial $\widetilde{P}(\lambda)$ enters in the spectral curve (9.6).

Recall that in the Birman–Schwinger formalism we can estimate spectrum and eigenvalues of the one Hamiltonian $\widetilde{H}$ by using known spectrum and eigenvalues of the dual Hamiltonian H. So it is interesting to study such a duality in framework of the quantum $\mathbb{Q}$-operator theory as an example for the Toda lattice.

10 The Goryachev–Chaplygin top

The construction of canonical transformations of the extended phase space proposed for the Stäckel systems was inspired by the Kepler and Liouville results. These transformations give rise to the shift of the Lax matrices (6.11) and it allowed us to introduce an integrable system dual to the Toda lattice. As above we can try to construct another integrable systems starting with known ones by using the mapping of the 2×2 Lax matrices for the Toda lattice (9.11).

We start with any Lax matrix $T(\lambda)$ in the form (9.9). Substituting the known Hamilton function H into the mapping (9.11) one calculates a new matrix $\widetilde{T}$ and a new Hamilton function, which may be tested on integrability. Note that to construct new Lax matrices by the rule (6.11) (8.5) we have to predict the new Hamiltonian $\widetilde{H}$ from some external reasons.

As an example here we consider the Goryachev–Chaplygin top. We introduce coordinates on the dual space to the Lie algebra $e(3)$ with the standard Lie–Poisson brackets

$$\begin{aligned} &\{l_i, l_j\} = \varepsilon_{ijk} l_k, \qquad \{l_i, g_j\} = \varepsilon_{ijk} g_k, \\ &\{g_i, g_j\} = 0, \qquad i, j, k = 1, 2, 3. \end{aligned} \tag{10.1}$$

The orbits on $e(3)^*$ are fixed by values of the two Casimir operators $C_1 = (g, g)$; $C_2 = (l, g)$. The Hamilton function for the Goryachev–Chaplygin top is equal to

$$H = \frac{1}{2}\left(l_1^2 + l_2^2 + 4l_3^2\right) - pl_3 + g_2, \qquad p \in \mathbb{R}. \tag{10.2}$$

It is a completely integrable top on the one-parameter subset of orbits $\mathcal{O}$ ($C_1 = \text{const}$, $C_2 = 0$) in $e(3)^*$. The corresponding 2×2 Lax matrix $T(\lambda)$ was obtained by Sklyanin [30].

According to [35] this matrix is closely related to the Lax matrix for the three-particle Toda lattice and it may be factored as

$$T(\lambda) = T_1(\lambda)T_{23}(\lambda), \qquad \text{where} \quad T_1 = \begin{pmatrix} \lambda - p + 2l_3 & e^q \\ -e^{-q} & 0 \end{pmatrix},$$

and

$$T_{23} = \begin{pmatrix} \lambda^2 - 2l_3\lambda - l_1^2 - l_2^2 & ie^q[\lambda(g_1 - ig_2) - g_3(l_1 - il_2)] \\ ie^q[\lambda(g1 + ig_2) - g_3(l_1 + il_2)] & g_3^2 \end{pmatrix}. \tag{10.3}$$

By using the transformation of the Lax matrices (9.11) proposed for the Toda lattices one constructs another Lax matrices

$$\widetilde{T}(\lambda) = T_1 \cdot \left[T_{23} + \begin{pmatrix} H - b & 0 \\ 0 & 0 \end{pmatrix} \right], \qquad \widetilde{A} = v^{-1}A$$

which describes a new integrable system on the same one-parameter subset of orbits $\mathcal{O}$. So the canonical transformation of the extended phase space (3.4) by $v = g_3^{-2}$, i.e.

$$\widetilde{H} = g_3^2(H - b), \tag{10.4}$$

preserves integrability. Moreover, initial and resulting systems are separable in the common system of separated variables. By using Maupertuis' principle a similar transformation of the extended phase space (10.4) has been obtained in [28].

Transformation of the corresponding spectral curves has the expected form

$$\mathcal{C}: \quad \mu - \frac{\lambda^2(g,g)}{\mu} = \lambda^3 - p\lambda^2 - 2H\lambda - K,$$

$$\widetilde{\mathcal{C}}: \quad \mu - \frac{\lambda^2(g,g) + 2\widetilde{H}}{\mu} = \lambda^3 - p\lambda^2 - 2b\lambda - \widetilde{K}.$$

As above, these integrable systems are topologically equivalent.

Starting from the matrix $T_{23}(\lambda)$ (10.3) we can construct 2×2 Lax matrix for the so-called Kowalevski–Goryachev–Chaplygin top (see references within [35]). This Lax matrix is closely related to the Lax matrix for the Toda lattice associated with the root system BC_2. Starting with the induced transformation of the 2×2 Lax matrices for the BC_n Toda lattices we can construct a new integrable system related to the Kowalevski–Goryachev–Chaplygin top. Similar transformations of the extended phase space have been proposed in [29] directly from Maupertuis' principle.

11 Conclusion

The modest aim of this review was to collect some old and new examples of canonical transformations of the extended phase space, which map a given integrable system into the other integrable system.

The Sections 4–10 together show that all the examples have many common properties. Analysis of these common properties could allow us to join different integrable systems into the classes of topologically equivalent systems and to study these classes instead of considering of the individual systems. Each of this class of integrable system may be related to the class of topologically equivalent n-dimensional Lagrangian submanifolds, which are diffeomorphic to the n-dimensional torus. In this approach the different integrable systems are associated with the various parametric forms of the common integrable manifold.

It remains unclear how to construct canonical transformations of the extended phase space $\mathcal{M}_E$ for a given integrable system. Moreover, up to now one does not know all consequences of the action of canonical transformations on the Stäckel systems, Toda lattices and other known integrable systems.

There is still much to do: to describe modification of bi-hamiltonian structures and the Bäcklund transformations for the Stäckel systems and the Toda lattices, to transform the corresponding stationary flows of hierarchies of nonlinear evolution equations and to consider composed transformations similar to the Kolosoff mapping for the Kowalevski top.

For all the examples, after canonical transformations of $\mathcal{M}_E$, one usually gets dynamical r-matrices instead of constant ones. We have to check that these resulting r-matrices satisfy to the classical dynamical Yang–Baxter equation and to interpret properly these dynamical matrices in a general theory of dynamical r-matrices.

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