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Correctors for Some Nonlinear Monotone Operators

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Abstract

In this paper we study homogenization of quasi-linear partial differential equations of the form $-\operatorname{div}(a(x, x/\varepsilon_h, Du_h)) = f_h$ on Ω with Dirichlet boundary conditions. Here the sequence (ε_h) tends to 0 as $h \to \infty$ and the map $a(x, y, \xi)$ is periodic in y, monotone in ξ and satisfies suitable continuity conditions. We prove that $u_h \to u$ weakly in $W_0^{1,p}(\Omega)$ as $h \to \infty$, where u is the solution of a homogenized problem of the form $-\operatorname{div}(b(x, Du)) = f$ on Ω. We also derive an explicit expression for the homogenized operator b and prove some corrector results, i.e. we find (P_h) such that $Du_h - P_h(Du) \to 0$ in $L^p(\Omega, \mathbf{R}^n)$.

1 Introduction

In mathematical models of microscopically non-homogeneous media, various local characteristics are usually described by functions of the form $a(x/\varepsilon_h)$ where $\varepsilon_h > 0$ is a small parameter. The function $a(x)$ can be periodic or belong to some other specific class. To compute the properties of a micro non-homogeneous medium is an extremely difficult task, since the coefficients are rapidly oscillating functions. One way to attack the problem is to apply asymptotic analysis to the problems of microlevel non-homogeneous media, which immediately leads to the concept of homogenization. When the parameter ε_h is very small, the heterogeneous medium will act as a homogeneous medium. To characterize this homogeneous medium is one of the main tasks in the homogenization theory. For more information concerning the homogenization theory, the reader is referred to [2, 10] and [13].

In this paper we consider the homogenization problem for monotone operators and the local behavior of the solutions. Monotone operators are very important in the study of nonlinear partial differential equations. The problem we study here can be used to model different nonlinear stationary conservation laws, e.g. stationary temperature distribution. For a more detailed discussion concerning different applications, see [16].

We will study the limit behavior of the sequence of solutions (u_h) of the Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}\left(a\left(x, \dfrac{x}{\varepsilon_h}, Du_h\right)\right) = f_h \quad \text{on} \quad \Omega, \\ u_h \in W_0^{1,p}(\Omega), \end{cases}$$

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where $f_h \to f$ in $W^{-1,q}(\Omega)$ as $\varepsilon_h \to 0$. Moreover, the map $a(x, y, \xi)$ is defined on $\Omega \times \mathbf{R}^n \times \mathbf{R}^n$ and is assumed to be periodic in y, continuous in ξ and monotone in ξ. We also need some continuity restriction in the first variable of $a(x, y, \xi)$. We will consider two different cases, namely when $a(x, y, \xi)$ is of the form

$$a(x, y, \xi) = \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, a_i(y, \xi),$$

and when $a(x, y, \xi)$ satisfies that

$$|a(x_1, y, \xi) - a(x_2, y, \xi)|^q \leq \omega(|x_1 - x_2|)(1 + |\xi|)^p,$$

where $\omega : \mathbf{R} \to \mathbf{R}$ is continuous, increasing and $\omega(0) = 0$. In both cases we will prove that $u_h \to u$ weakly in $W^{1,p}(\Omega)$ and that u is the solution of the *homogenized* problem

$$\begin{cases} -\mathrm{div}\,(b(x, Du)) = f \quad \text{on} \quad \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

We will prove that the operator b has the same structure properties as a and is given by

$$b(x, \xi) = \int_Y a\left(x, y, \xi + Dv^{\xi,x}(y)\right) dy,$$

where $v^{\xi,x}$ is the solution of the *cell-problem*

$$\begin{cases} -\mathrm{div}\left(a\left(x, y, \xi + Dv^{\xi,x}(y)\right)\right) = 0 \quad \text{on} \quad Y, \\ v^{\xi,x} \in W_{\square}^{1,p}(Y). \end{cases} \tag{1}$$

Here Y is a periodic cell and $W_{\square}^{1,p}(Y)$ is the subset of $W^{1,p}(Y)$ such that u has mean value 0 and u is Y-periodic. The corresponding weak formulation of the cell problem is

$$\begin{cases} \displaystyle\int_Y \left\langle a\left(x, y, \xi + Dv^{\xi,x}(y)\right), Dw\right\rangle dy = 0 \quad \text{for every} \quad w \in W_{\square}^{1,p}(Y), \\ v^{\xi,x} \in W_{\square}^{1,p}(Y). \end{cases} \tag{2}$$

The homogenization problem described above with $p = 2$ was studied in [15]. Others have investigated the case where we have no dependence in x, that is, when a is of the form $a(x, y, \xi) = a(y, \xi)$. Here we mention [9] and [6] where the problems corresponding to single valued and multi valued operators were studied. Moreover, the almost periodic case was treated in [4].

The weak convergence of u_h to u in $W^{1,p}(\Omega)$ implies that $u_h - u \to 0$ in $L^p(\Omega)$, but in general, we only have that $Du_h - Du \to 0$ weakly in $L^p(\Omega, \mathbf{R}^n)$. However, we will prove that it is possible to express Du_h in terms of Du up to a remainder which converges strongly in $L^p(\Omega, \mathbf{R}^n)$. This is done by constructing a family of correctors $P_h(x, \xi, t)$, defined by

$$P_h(x, \xi, t) = P\left(\frac{x}{\varepsilon_h}, \xi, t\right) = \xi + Dv^{\xi,t}\left(\frac{x}{\varepsilon_h}\right). \tag{3}$$

Let (M_h) be a family of linear operators converging to the identity map on $L^p(\Omega, \mathbf{R}^n)$ such that $M_h f$ is a step function for every $f \in L^p(\Omega, \mathbf{R}^n)$. Moreover, let γ_h be a step function approximating the identity map on Ω. We will show that

$$Du_h - P_h(x, M_h Du, \gamma_h) \to 0 \quad \text{in} \quad L^p(\Omega, \mathbf{R}^n).$$

The problem of finding correctors has been studied by many authors, see e.g. [7] where single valued monotone operators of the form $-\text{div}\left(a\left(\frac{x}{\varepsilon_h}, Du_h\right)\right)$ were considered and [3] where the corresponding almost periodic case was considered.

2 Preliminaries and notation

Let Ω be a open bounded subset of $\mathbf{R}^n$, $|E|$ denote the Lebesgue measure in $\mathbf{R}^n$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean scalar product on $\mathbf{R}^n$. Moreover, if X is a Banach space, we let X^* denote its dual space and $\langle\cdot\mid\cdot\rangle$ denote the canonical pairing over $X^* \times X$.

Let $\{\Omega_i \subset \Omega : i = 1, \ldots, N\}$ be a family of disjoint open sets such that $\left|\Omega \backslash \cup_{i=1}^N \Omega_i\right| = 0$ and $|\partial\Omega_i| = 0$. Let (ε_h) be a decreasing sequence of real numbers such that $\varepsilon_h \to 0$ as $h \to \infty$. Furthermore, $Y = (0,1)^n$ is the unit cube in $\mathbf{R}^n$ and we put $Y_h^j = \varepsilon_h(j + Y)$, where $j \in \mathbf{Z}^n$, i.e. the translated image of $\varepsilon_h Y$ by the vector $\varepsilon_h j$. We also define the following index sets:

$$J_h = \left\{j \in \mathbf{Z}^n : \overline{Y}_h^j \subset \Omega\right\}, \qquad J_h^i = \left\{j \in \mathbf{Z}^n : \overline{Y}_h^j \subset \Omega_i\right\},$$

$$B_h^i = \left\{j \in \mathbf{Z}^n : \overline{Y}_h^j \cap \Omega_i \neq \emptyset,\ \overline{Y}_h^j \backslash \Omega_i \neq \emptyset\right\}.$$

Moreover, we define $\Omega_i^h = \cup_{j \in J_h^i} \overline{Y}_h^j$ and $F_i^h = \cup_{j \in B_h^i} Y_h^j$.

In a corresponding way let $\left\{\Omega_i^k \subset \Omega : i \in I_k\right\}$ denote a family of disjoint open sets with diameter less than $\frac{1}{k}$ such that $\left|\Omega \backslash \cup_{i \in I_k} \Omega_i^k\right| = 0$ and $\left|\partial\Omega_i^k\right| = 0$. We also define the following index sets:

$$J_h^{i,k} = \left\{j \in \mathbf{Z}^n : \overline{Y}_h^j \subset \Omega_i^k\right\},$$

$$B_h^{i,k} = \left\{j \in \mathbf{Z}^n : \overline{Y}_h^j \cap \Omega_i^k \neq \emptyset,\ \overline{Y}_h^j \backslash \Omega_i^k \neq \emptyset\right\}.$$

Let $\Omega_i^{k,h} = \cup_{j \in J_h^{i,k}} \overline{Y}_h^j$ and $F_i^{k,h} = \cup_{j \in B_h^{i,k}} Y_h^j$.

Corresponding to $f \in L^p(\Omega, \mathbf{R}^n)$ we define the function $M_h f : \mathbf{R}^n \to \mathbf{R}^n$ by

$$(M_h f)(x) = \sum_{j \in J_h} \chi_{Y_h^j}(x)\, \xi_h^j,$$

where $\xi_h^j = \frac{1}{|Y_h^j|} \int_{Y_h^j} f\, dx$ and χ_E is the characteristic function of the set E (in order to define ξ_h^j for all $j \in \mathbf{Z}^n$, we treat f as $f = 0$ outside Ω). It is well known that

$$M_h f \to f \quad \text{in} \quad L^p(\Omega, \mathbf{R}^n), \tag{4}$$

see [14, p. 129]. We also define the step function $\gamma_h : \Omega \to \Omega$ by

$$\gamma_h(x) = \sum_{j \in J_h} \chi_{Y_h^j}(x)\, x_h^j, \tag{5}$$

where $x_h^j \in Y_h^j$. Finally, C will denote a positive constant that may differ from one place to an other.

Let $a : \Omega \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a function such that $a(x, \cdot, \xi)$ is Lebesgue measurable and Y-periodic for $x \in \Omega$ and $\xi \in \mathbf{R}^n$. Let p be a real constant $1 < p < \infty$ and let q be its dual exponent, $\frac{1}{p} + \frac{1}{q} = 1$. We also assume that a satisfies the following continuity and monotonicity conditions: There exists two constants c_1, $c_2 > 0$, and two constants α and β, with $0 \leq \alpha \leq \min(1, p-1)$ and $\max(p, 2) \leq \beta < \infty$ such that

$$|a(x, y, \xi_1) - a(x, y, \xi_2)| \leq c_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha}, \tag{6}$$

$$\langle a(x, y, \xi_1) - a(x, y, \xi_2), \xi_1 - \xi_2 \rangle \geq c_2 (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^{\beta}, \tag{7}$$

for $x \in \Omega$, a.e. $y \in \mathbf{R}^n$ and every $\xi \in \mathbf{R}^n$. Moreover, we assume that

$$a(x, y, 0) = 0, \tag{8}$$

for $x \in \Omega$, a.e. $y \in \mathbf{R}^n$. Let (f_h) be a sequence in $W^{-1,q}(\Omega)$ that converges to f.

Remark 1. We will use these continuity and monotonicity conditions to show theorems and properties. However, we concentrate on showing the non-trivial cases, for instance when $\beta \neq p$, and omit the simple ones, in this case when $\beta = p$.

The solution $v^{\xi,x}$ of the cell-problem (1) can be extended by periodicity to an element in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$, still denoted by $v^{\xi,x}$, and

$$\int_{\mathbf{R}^n} \left\langle a\left(x, y, \xi + Dv^{\xi,x}(y)\right), D\phi(y) \right\rangle dy = 0 \quad \text{for every} \quad \phi \in C_0^{\infty}(\mathbf{R}^n). \tag{9}$$

3 Some useful lemmas

The following lemma, see e.g. [12], is fundamental to the homogenization theory.

Lemma 1 (Compensated compactness). *Let $1 < p < \infty$. Moreover, let (v_h) be a sequence in $L^q(\Omega, \mathbf{R}^n)$ which converges weakly to v, $(-\operatorname{div} v_h)$ converges to $-\operatorname{div} v$ in $W^{-1,q}(\Omega)$ and let (u_h) be a sequence which converges weakly to u in $W^{1,p}(\Omega)$. Then*

$$\int_{\Omega} \langle v_h, Du_h \rangle \phi \, dx \to \int_{\Omega} \langle v, Du \rangle \phi \, dx,$$

for every $\phi \in C_0^{\infty}(\Omega)$.

We will also use the following estimates, which are proved in [5].

Lemma 2. *Let a satisfy (6), (7) and (8). Then the following inequalities hold:*

$$(a) \quad |a(x, y, \xi)| \leq c_a \left(1 + |\xi|^{p-1}\right), \tag{10}$$

$$(b) \quad |\xi|^p \leq c_b (1 + \langle a(x, y, \xi), \xi \rangle), \tag{11}$$

$$(c) \quad \int_Y \left|\xi + Dv^{\xi,x_i}\right|^p dy \leq c_c (1 + |\xi|^p). \tag{12}$$

Lemma 3. *For every $\xi_1, \xi_2 \in \mathbf{R}^n$ we have*

$$\int_Y \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^p dy \leq C (1 + |\xi_1|^p + |\xi_2|^p)^{\frac{\beta-\alpha-1}{\beta-\alpha}} |\xi_1 - \xi_2|^{\frac{p}{\beta-\alpha}}. \tag{13}$$

4 Some homogenization results

Let $a(x, y, \xi)$ satisfy one of the conditions

1. a is of the form
$$a(x, y, \xi) = \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, a_i(y, \xi). \tag{14}$$
2. There exists a function $\omega : \mathbf{R} \to \mathbf{R}$ that is continuous, increasing and $\omega(0) = 0$ such that
$$|a(x_1, y, \xi) - a(x_2, y, \xi)|^q \le \omega(|x_1 - x_2|)(1 + |\xi|)^p \tag{15}$$
for $x_1,\ x_2 \in \Omega$, a.e. $y \in \mathbf{R}$ and every $\xi \in \mathbf{R}^n$.

Now we consider the weak Dirichlet boundary value problems (one for each choice of h):
$$\begin{cases} \displaystyle\int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), D\phi \right\rangle dx = \langle f_h \mid \phi \rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega), \\ u_h \in W_0^{1,p}(\Omega). \end{cases} \tag{16}$$
By a standard result in the existence theory for boundary value problems defined by monotone operators, these problems have a unique solution for each h, see e.g. [17].

We let $\phi = u_h$ in (16) and use Hölder's reversed inequality, (7), (8) and Poincare's inequality. This implies that
$$\begin{aligned} C\left(\int_\Omega (1 + |Du_h|)^p\, dx\right)^{\frac{p-\beta}{p}} \left(\int_\Omega |Du_h|^p\, dx\right)^{\frac{\beta}{p}} &\le c_2 \int_\Omega (1 + |Du_h|)^{p-\beta} |Du_h|^\beta\, dx \\ &\le \int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), Du_h \right\rangle dx = \langle f_h \mid u_h \rangle \\ &\le \|f_h\|_{W^{-1,q}} \|u_h\|_{W_0^{1,p}} \le C \|Du_h\|_{L^p(\Omega, \mathbf{R}^n)}, \end{aligned} \tag{17}$$
where C does not depend on h. Now if $\|Du_h\|^p_{L^p(\Omega,\mathbf{R}^n)} < |\Omega|$, then clearly $\|u_h\|_{W_0^{1,p}(\Omega)} \le C$ by Poincare's inequality. Hence assume that $\|Du_h\|^p_{L^p(\Omega,\mathbf{R}^n)} \ge |\Omega|$. But then we have by (17) that
$$\begin{aligned} 2^{p-\beta} \int_\Omega |Du_h|^p\, dx &= \left(2^{p-1} \int_\Omega 2\, |Du_h|^p\, dx\right)^{\frac{p-\beta}{p}} \left(\int_\Omega |Du_h|^p\, dx\right)^{\frac{\beta}{p}} \\ &\le \left(\int_\Omega 2^{p-1}(1 + |Du_h|^p)\, dx\right)^{\frac{p-\beta}{p}} \left(\int_\Omega |Du_h|^p\, dx\right)^{\frac{\beta}{p}} \le C\left(\int_\Omega |Du_h|^p\, dx\right)^{\frac{1}{p}}, \end{aligned}$$
that is, $\|Du_h\|_{L^p(\Omega,\mathbf{R}^n)} \le C$. According to Poincare's inequality we thus have that $\|u_h\|_{W_0^{1,p}(\Omega)} \le C$. Summing up, we have that
$$\|u_h\|_{W_0^{1,p}(\Omega)} \le C. \tag{18}$$
Since u_h is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence (h') such that
$$u_{h'} \to u_* \quad \text{weakly in} \quad W_0^{1,p}(\Omega). \tag{19}$$

The following theorems will show that u_* satisfy an equation of the same type as those which are satisfied by u_h.

Theorem 4. *Let a satisfy (6), (7), (8) and (14). Let (u_h) be solutions of (16). Then we have that*

$$\begin{aligned} &u_h \to u \quad \text{weakly in} \quad W_0^{1,p}(\Omega), \\ &a\left(x, \frac{x}{\varepsilon_h}, Du_h\right) \to b(x, Du) \quad \text{weakly in} \quad L^q(\Omega, \mathbf{R}^n), \end{aligned} \tag{20}$$

where u is the unique solution of the homogenized problem

$$\begin{cases} \displaystyle\int_\Omega \langle b(x, Du), D\phi\rangle \, dx = \langle f \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega), \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{21}$$

The operator $b : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is defined a.e. as

$$b(x, \xi) = \sum_{i=1}^N \chi_{\Omega_i}(x) \int_Y a_i\left(y, \xi + Dv^{\xi, x_i}(y)\right) dy = \sum_{i=1}^N \chi_{\Omega_i}(x)\, b_i(\xi),$$

where $x_i \in \Omega_i$, $b_i(\xi) = \int_Y a_i\left(y, \xi + Dv^{\xi,x_i}(y)\right) dy$ and v^{ξ,x_i} is the unique solution of the cell problem

$$\begin{cases} \displaystyle\int_Y \left\langle a_i\left(y, \xi + Dv^{\xi,x_i}(y)\right), D\phi(y)\right\rangle dy = 0 \quad \text{for every} \quad \phi \in W_\square^{1,p}(Y), \\ v^{\xi,x_i} \in W_\square^{1,p}(Y). \end{cases} \tag{22}$$

Remark 2. An equivalent formulation of the equations (21) and (22) above can be given in the following unified manner

$$\begin{cases} \displaystyle\int_\Omega \int_Y \langle a(x, y, Du(x) + Dv(x, y)), D\overline{u}(x) + D\overline{v}(x, y)\rangle \, dxdy = \int_\Omega f\overline{u}\, dx, \\ (u, v) \in W_0^{1,p}(\Omega) \times L^q\left(\Omega; W_\square^{1,p}(Y)\right), \end{cases}$$

for all $(\overline{u}, \overline{v}) \in W_0^{1,p}(\Omega) \times L^q\left(\Omega; W_\square^{1,p}(Y)\right)$. This formulation often occurs in the notion of two-scale limit introduced in e.g. [1].

Proof. We have shown that $u_{h'} \to u_*$ weakly in $W_0^{1,p}(\Omega)$ for a subsequence (h') since u_h is bounded in $W_0^{1,p}(\Omega)$. We define

$$\psi_{h'}^i = a_i\left(\frac{x}{\varepsilon_{h'}}, Du_{h'}\right).$$

Then according to (6), (8), Hölder's inequality, Poincare's inequality and (18), we find that

$$\begin{aligned} \int_{\Omega_i} \left|\psi_{h'}^i\right|^q dx &= \int_{\Omega_i} \left|a_i\left(\frac{x}{\varepsilon_{h'}}, Du_{h'}\right)\right|^q dx \le c_1^q \int_{\Omega_i} (1 + |Du_{h'}|)^{q(p-1-\alpha)} |Du_{h'}|^{\alpha q} dx \\ &\le C\left(\int_{\Omega_i} (1 + |Du_{h'}|)^p dx\right)^{\frac{p-1-\alpha}{p-1}} \left(\int_{\Omega_i} |Du_{h'}|^p dx\right)^{\frac{\alpha}{p-1}} \\ &\le C\int_{\Omega_i} (1 + |Du_{h'}|)^p dx \le C, \end{aligned}$$

that is, $\psi^i_{h'}$ is bounded in $L^q(\Omega_i,\mathbf{R}^n)$. Hence there is a subsequence (h'') of (h') such that

$$\psi^i_{h''} \to \psi^i_* \quad \text{weakly in} \quad L^q(\Omega_i,\mathbf{R}^n).$$

From our original problem (16) we conclude that

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_{h''}}, Du_{h''}\right), D\phi\right\rangle dx = \langle f_{h''} \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega), \\ u_{h''} \in W_0^{1,p}(\Omega). \end{cases}$$

In the limit we have

$$\sum_{i=1}^N \int_{\Omega_i} \langle \psi^i_*, D\phi\rangle\, dx = \langle f \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega).$$

If we could show that

$$\psi^i_* = b_i(Du_*) \quad \text{for a.e.} \quad x \in \Omega_i, \tag{23}$$

then it follows by uniqueness of the homogenized problem (21) that $u_* = u$. Let

$$w_h^{\xi,x_i}(x) = \langle \xi, x\rangle + \varepsilon_h v^{\xi,x_i}\left(\frac{x}{\varepsilon_h}\right)$$

for a.e. $x \in \mathbf{R}^n$. By the monotonicity of a_i we have for a fix ξ that

$$\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_{h''}}, Du_{h''}(x)\right) - a_i\left(\frac{x}{\varepsilon_{h''}}, Dw_{h''}^{\xi,x_i}(x)\right), Du_{h''}(x) - Dw_{h''}^{\xi,x_i}(x)\right\rangle \phi(x)\, dx \geq 0$$

for every $\phi \in C_0^\infty(\Omega_i)$, $\phi \geq 0$. We now note that by (16) it also holds that

$$\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_{h''}}, Du_{h''}(x)\right), D\phi\right\rangle dx = \langle f_{h''} \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega_i).$$

Since $f_h \to f$ in $W^{-1,q}(\Omega)$, this implies that

$$-\operatorname{div}\left(a_i\left(\frac{x}{\varepsilon_{h''}}, Du_{h''}(x)\right)\right) = -\operatorname{div}\psi^i_{h''} \to -\operatorname{div}\psi^i_* \quad \text{in} \quad W^{-1,q}(\Omega_i). \tag{24}$$

We also have that

$$-\operatorname{div}\left(a_i\left(\frac{x}{\varepsilon_{h''}}, Dw_{h''}^{\xi,x_i}(x)\right)\right) = 0 \quad \text{on} \quad \Omega_i. \tag{25}$$

By the compensated compactness lemma (Lemma 1) we then get in the limit

$$\int_{\Omega_i} \langle \psi^i_* - b_i(\xi), Du_*(x) - \xi\rangle\, \phi(x)\, dx \geq 0$$

for every $\phi \in C_0^\infty(\Omega_i)$, $\phi \geq 0$. Hence for our fixed $\xi \in \mathbf{R}^n$ we have that

$$\langle \psi^i_* - b_i(\xi), Du_*(x) - \xi\rangle \geq 0 \quad \text{for a.e.} \quad x \in \Omega_i. \tag{26}$$

In particular, if (ξ_m) is a countable dense set in $\mathbf{R}^n$, then (26) implies that

$$\left\langle \psi_*^i - b_i(\xi_m), Du_*(x) - \xi_m \right\rangle \geq 0 \quad \text{for a.e.} \quad x \in \Omega_i. \tag{27}$$

By the continuity of b (see Remark 3) it follows that

$$\left\langle \psi_*^i - b_i(\xi), Du_*(x) - \xi \right\rangle \geq 0 \quad \text{for a.e.} \quad x \in \Omega_i \quad \text{and for every} \quad \xi \in \mathbf{R}^n. \tag{28}$$

Since b is monotone and continuous (see Remark 3) we have that b is maximal monotone and hence (23) follows. We have proved the theorem up to a subsequence $(u_{h''})$ of (u_h). By uniqueness of the solution to the homogenized equation it follows that this holds for the whole sequence. ■

Theorem 5. *Let a satisfy (6), (7), (8) and (15). Let (u_h) be solutions of (16). Then we have that*

$$u_h \to u \quad \textit{weakly in} \quad W_0^{1,p}(\Omega),$$

$$a\left(x, \frac{x}{\varepsilon_h}, Du_h\right) \to b(x, Du) \quad \textit{weakly in} \quad L^q(\Omega, \mathbf{R}^n),$$

where u is the unique solution of the homogenized problem

$$\begin{cases} \displaystyle\int_\Omega \langle b(x, Du), D\phi \rangle \, dx = \langle f \mid \phi \rangle \quad \textit{for every} \quad \phi \in W_0^{1,p}(\Omega), \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The operator $b : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is defined a.e. as

$$b(x, \xi) = \int_Y a\left(x, y, \xi + Dv^{\xi,x}(y)\right) dy,$$

where $v^{\xi,x}$ is the unique solution of the cell problem

$$\begin{cases} \displaystyle\int_Y \left\langle a\left(x, y, \xi + Dv^{\xi,x}(y)\right), D\phi(y) \right\rangle dy = 0 \quad \textit{for every} \quad \phi \in W_\square^{1,p}(Y), \\ v^{\xi,x} \in W_\square^{1,p}(Y). \end{cases} \tag{29}$$

Before we prove this theorem, we make some definitions that will be useful in the proof. Define the function

$$a^k(x, y, \xi) = \sum_{i \in I_k} \chi_{\Omega_i^k}(x) \, a\left(x_i^k, y, \xi\right),$$

where $x_i^k \in \Omega_i^k$. Consider the boundary value problems

$$\begin{cases} \displaystyle\int_\Omega \left\langle a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right), D\phi \right\rangle dx = \langle f_h \mid \phi \rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega), \\ u_h^k \in W_0^{1,p}(\Omega). \end{cases} \tag{30}$$

The conditions for Theorem 4 are satisfied and the theorem implies that there exists a u_*^k such that

$$u_h^k \to u_*^k \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \quad \text{as} \quad h \to \infty,$$

where u_*^k is the unique solution of

$$\begin{cases} \int_\Omega \left\langle b^k\left(x, Du_*^k\right), D\phi\right\rangle dx = \langle f \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega), \\ u_*^k \in W_0^{1,p}(\Omega). \end{cases} \tag{31}$$

Here

$$b^k(x,\xi) = \sum_{i\in I_k} \chi_{\Omega_i^k}(x) \int_Y a\left(x_i^k, y, \xi + Dv^{\xi,x_i^k}(y)\right) dy = \sum_{i\in I_k} \chi_{\Omega_i^k}(x)\, b\left(x_i^k, \xi\right),$$

and v^{ξ,x_i^k} is the solution of

$$\begin{cases} \int_Y \left\langle a\left(x_i^k, y, \xi + Dv^{\xi,x_i^k}(y)\right), D\phi(y)\right\rangle dy = 0 \quad \text{for every} \quad \phi \in W_\square^{1,p}(Y), \\ v^{\xi,x_i^k} \in W_\square^{1,p}(Y). \end{cases}$$

Proof. First we prove that $u_h \to u$ weakly in $W_0^{1,p}(\Omega)$. If $g \in W^{-1,q}(\Omega)$, we have that

$$\begin{aligned} \lim_{h\to\infty} \langle g \mid u_h - u\rangle &= \lim_{k\to\infty}\lim_{h\to\infty} \langle g \mid u_h - u\rangle \\ &= \lim_{k\to\infty}\lim_{h\to\infty} \left\langle g \mid u_h - u_h^k + u_h^k - u_*^k + u_*^k - u\right\rangle \\ &\leq \lim_{k\to\infty}\lim_{h\to\infty} \|g\|_{W^{-1,q}(\Omega)} \left\|u_h - u_h^k\right\|_{W_0^{1,p}(\Omega)} + \lim_{k\to\infty}\lim_{h\to\infty} \left\langle g \mid u_h^k - u_*^k\right\rangle \\ &\quad + \lim_{k\to\infty} \|g\|_{W^{-1,q}(\Omega)} \left\|u_*^k - u\right\|_{W_0^{1,p}(\Omega)}. \end{aligned}$$

We need to show that all terms on the right hand side are zero.

Term 1. Let us prove that

$$\lim_{k\to\infty}\lim_{h\to\infty} \left\|u_h - u_h^k\right\|_{W_0^{1,p}(\Omega)} = 0. \tag{32}$$

By the definition we have that

$$\int_\Omega \left\langle a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right), D\phi\right\rangle dx = \langle f_h \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega),$$

$$\int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), D\phi\right\rangle dx = \langle f_h \mid \phi\rangle \quad \text{for every} \quad \phi \in W_0^{1,p}(\Omega).$$

This implies that we with $\phi = u_h^k - u_h$ have

$$\begin{aligned} &\int_\Omega \left\langle a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right) - a^k\left(x, \frac{x}{\varepsilon_h}, Du_h\right), Du_h^k - Du_h\right\rangle dx \\ &\quad = \int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right) - a^k\left(x, \frac{x}{\varepsilon_h}, Du_h\right), Du_h^k - Du_h\right\rangle dx. \end{aligned}$$

By the monotonicity of a (7), Schwarz' inequality and Hölder's inequality it follows that

$$\begin{aligned}
& c_2 \int_\Omega \left(1+\left|Du_h^k\right|+|Du_h|\right)^{p-\beta}\left|Du_h^k-Du_h\right|^\beta dx \\
& \quad \leq \int_\Omega \left\langle a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right),Du_h^k-Du_h\right\rangle dx \\
& \quad = \int_\Omega \left\langle a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right),Du_h^k-Du_h\right\rangle dx \\
& \quad \leq \left(\int_\Omega \left|a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right)\right|^q dx\right)^{\frac{1}{q}}\left(\int_\Omega \left|Du_h^k-Du_h\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Now since $\|Du_h^k\|^p_{L^p(\Omega,\mathbf{R}^n)} \leq C$ and $\|Du_h\|^p_{L^p(\Omega,\mathbf{R}^n)} \leq C$, we have, according to Hölder's reversed inequality, that

$$\begin{aligned}
& C\left(\int_\Omega \left|Du_h^k-Du_h\right|^p dx\right)^{\frac{\beta}{p}} \\
& \quad \leq c_2\left(\int_\Omega \left(1+\left|Du_h^k\right|+|Du_h|\right)^p dx\right)^{\frac{p-\beta}{p}}\left(\int_\Omega \left|Du_h^k-Du_h\right|^p dx\right)^{\frac{\beta}{p}} \qquad (33)\\
& \quad \leq c_2\int_\Omega \left(1+\left|Du_h^k\right|+|Du_h|\right)^{p-\beta}\left|Du_h^k-Du_h\right|^\beta dx.
\end{aligned}$$

Hence we see that

$$\left\|Du_h^k-Du_h\right\|^p_{L^p(\Omega,\mathbf{R}^n)} \leq C\left(\int_\Omega \left|a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right)\right|^q dx\right)^{\frac{p-1}{\beta-1}}.$$

Thus in view of the continuity condition (15) it yields that

$$\begin{aligned}
& \left\|Du_h^k-Du_h\right\|^p_{L^p(\Omega,\mathbf{R}^n)} \\
& \quad \leq C\left(\omega\left(\frac{1}{k}\right)\int_\Omega (1+|Du_h|)^p dx\right)^{\frac{p-1}{\beta-1}} \leq C\left(\omega\left(\frac{1}{k}\right)\right)^{\frac{p-1}{\beta-1}}, \qquad (34)
\end{aligned}$$

where we in the last inequality have used the fact that there exists a constant C independent of h such that $\|Du_h\|_{L^p(\Omega,\mathbf{R}^n)} \leq C$. Since $\|D\cdot\|_{L^p(\Omega,\mathbf{R}^n)}$ is an equivalent norm on $W_0^{1,p}(\Omega)$, we have that

$$\left\|u_h-u_h^k\right\|_{W_0^{1,p}(\Omega)} \to 0$$

as $k \to \infty$ uniformly in h. This means that we can change order in the limit process in (32) and (32) follows by taking (34) into account.

Term 2. We observe that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g \mid u_h^k-u_*^k\right\rangle = 0$$

as a direct consequence of Theorem 4.

Term 3. Let us prove that

$$\lim_{k\to\infty}\left\|u_*^k-u\right\|_{W_0^{1,p}(\Omega)}=0. \tag{35}$$

By the definition we have that

$$\int_\Omega \left\langle b^k\left(x,Du_*^k\right),D\phi\right\rangle dx=\langle f\mid\phi\rangle \quad \text{for every} \quad \phi\in W_0^{1,p}(\Omega),$$
$$\int_\Omega \langle b(x,Du),D\phi\rangle\,dx=\langle f\mid\phi\rangle \quad \text{for every} \quad \phi\in W_0^{1,p}(\Omega).$$

Hence

$$\int_\Omega \left\langle b^k\left(x,Du_*^k\right)-b^k(x,Du),D\phi\right\rangle dx=\int_\Omega\left\langle b(x,Du)-b^k(x,Du),D\phi\right\rangle dx$$

for every $\phi\in W_0^{1,p}(\Omega)$. Now let $\phi=u_*^k-u$ and use the strict monotonicity of b^k (see Remark 3) on the left hand side and use Schwarz' and Hölder's inequalities on the right hand side. We get

$$\begin{aligned}
&c_2\int_\Omega\left(1+\left|Du_*^k\right|+|Du|\right)^{p-\beta}\left|Du_*^k-Du\right|^\beta dx\\
&\qquad\le\int_\Omega\left\langle b^k\left(x,Du_*^k\right)-b^k(x,Du),Du_*^k-Du\right\rangle dx\\
&\qquad=\int_\Omega\left\langle b(x,Du)-b^k(x,Du),Du_*^k-Du\right\rangle dx\\
&\qquad\le\left(\int_\Omega\left|b(x,Du)-b^k(x,Du)\right|^q dx\right)^{\frac1q}\left(\int_\Omega\left|Du_*^k-Du\right|^p dx\right)^{\frac1p}.
\end{aligned}$$

Inequality (33) with Du_*^k and Du instead of Du_h^k and Du_h, respectively, then yields that

$$\left\|Du_*^k-Du\right\|^p_{L^p(\Omega,\mathbf{R}^n)}\le C\left(\omega\left(\frac1k\right)\int_\Omega(1+|Du|)^p\,dx\right)^{\frac{p-1}{\beta-1}}\le C\left(\omega\left(\frac1k\right)\right)^{\frac{p-1}{\beta-1}}. \tag{36}$$

The right hand side tends to 0 as $k\to\infty$. We now obtain (35) by noting that $\|D\cdot\|_{L^p(\Omega,\mathbf{R}^n)}$ is an equivalent norm on $W_0^{1,p}(\Omega)$.

Next we prove that $a\left(x,\frac{x}{\varepsilon_h},Du_h\right)\to b(x,Du)$ weakly in $L^q(\Omega,\mathbf{R}^n)$. Now if $g\in(L^q(\Omega,\mathbf{R}^n))^*$, then

$$\begin{aligned}
&\lim_{h\to\infty}\left\langle g\mid a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-b(x,Du)\right\rangle=\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g\mid a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-b(x,Du)\right\rangle\\
&\quad=\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g\mid a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)\right\rangle\\
&\quad+\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g\mid a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)-b^k\left(x,Du_*^k\right)\right\rangle\\
&\quad+\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g\mid b^k\left(x,Du_*^k\right)-b(x,Du)\right\rangle\\
&\quad\le\lim_{k\to\infty}\lim_{h\to\infty}\|g\|_{(L^q(\Omega,\mathbf{R}^n))^*}\left\|a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)\right\|_{L^q(\Omega,\mathbf{R}^n)}\\
&\quad+\lim_{k\to\infty}\lim_{h\to\infty}\left\langle g\mid a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)-b^k\left(x,Du_*^k\right)\right\rangle\\
&\quad+\lim_{k\to\infty}\|g\|_{(L^q(\Omega,\mathbf{R}^n))^*}\left\|b^k\left(x,Du_*^k\right)-b(x,Du)\right\|_{L^q(\Omega,\mathbf{R}^n)}.
\end{aligned}$$

We now prove that the three terms on the right hand side are zero.

Term 1. Let us show that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\|a\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)\right\|_{L^q(\Omega,\mathbf{R}^n)}=0. \tag{37}$$

By Minkowski's inequality we have that

$$\begin{aligned}
&\int_\Omega\left|a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)-a\left(x,\frac{x}{\varepsilon_h},Du_h\right)\right|^q dx\\
&\quad\le C\int_\Omega\left|a^k\left(x,\frac{x}{\varepsilon_h},Du_h^k\right)-a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right)\right|^q dx\\
&\quad+C\int_\Omega\left|a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a\left(x,\frac{x}{\varepsilon_h},Du_h\right)\right|^q dx.
\end{aligned}$$

The second integral on the right hand side is bounded by the continuity condition (15) since

$$\begin{aligned}
&\int_\Omega\left|a^k\left(x,\frac{x}{\varepsilon_h},Du_h\right)-a\left(x,\frac{x}{\varepsilon_h},Du_h\right)\right|^q dx\\
&\quad\le\omega\left(\frac{1}{k}\right)\int_\Omega(1+|Du_h|)^p\,dx\le C\omega\left(\frac{1}{k}\right),
\end{aligned} \tag{38}$$

where the last inequality follows since (Du_h) is bounded in $L^p(\Omega,\mathbf{R}^n)$. For the first integral on the right hand side we use the continuity restriction (6), Hölder's inequality and (34) to derive that

$$\begin{aligned}
&\int_\Omega \left| a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right) - a^k\left(x, \frac{x}{\varepsilon_h}, Du_h\right)\right|^q dx \\
&\quad \le c_1^q \int_\Omega \left(1 + \left|Du_h^k\right| + \left|Du_h\right|\right)^{q(p-1-\alpha)} \left|Du_h^k - Du_h\right|^{\alpha q} dx \\
&\quad \le C\left(\int_\Omega \left(1 + \left|Du_h^k\right| + \left|Du_h\right|\right)^p dx\right)^{\frac{p-1-\alpha}{p-1}} \left(\int_\Omega \left|Du_h^k - Du_h\right|^p dx\right)^{\frac{\alpha}{p-1}} \\
&\quad \le C\left(\int_\Omega \left|Du_h^k - Du_h\right|^p dx\right)^{\frac{\alpha}{p-1}} \le C\left(\omega\left(\frac{1}{k}\right)\right)^{\frac{\alpha}{\beta-1}}
\end{aligned} \tag{39}$$

since (Du_h) and $\left(Du_h^k\right)$ are bounded in $L^p(\Omega, \mathbf{R}^n)$. Hence by (38) and (39) we have that

$$\int_\Omega \left| a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right) - a\left(x, \frac{x}{\varepsilon_h}, Du_h\right)\right|^q dx \le C\left(\omega\left(\frac{1}{k}\right)\right)^{\frac{\alpha}{\beta-1}} + C\omega\left(\frac{1}{k}\right). \tag{40}$$

By the properties of ω, it follows that

$$\left\| a\left(x, \frac{x}{\varepsilon_h}, Du_h\right) - a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right)\right\|_{L^q(\Omega, \mathbf{R}^n)} \to 0$$

as $k \to \infty$ uniformly in h. This implies that we may change the order in the limit process in (37) and (37) follows.

Term 2. We immediately see that

$$\lim_{k\to\infty} \lim_{h\to\infty} \left\langle g \mid a^k\left(x, \frac{x}{\varepsilon_h}, Du_h^k\right) - a^k\left(x, \frac{x}{\varepsilon_h}, Du_*^k\right)\right\rangle = 0$$

as a direct consequence of Theorem 4.

Term 3. Let us show that

$$\lim_{k\to\infty} \left\| b^k\left(x, Du_*^k\right) - b(x, Du)\right\|_{L^q(\Omega, \mathbf{R}^n)} = 0. \tag{41}$$

We note that

$$\begin{aligned}
&\int_\Omega \left| b^k\left(x, Du_*^k\right) - b(x, Du)\right|^q dx \\
&\quad = \int_\Omega \left| b^k\left(x, Du_*^k\right) - b^k(x, Du) + b^k(x, Du) - b(x, Du)\right|^q dx \\
&\quad \le C\left(\int_\Omega \left| b^k\left(x, Du_*^k\right) - b^k(x, Du)\right|^q dx + \int_\Omega \left| b^k(x, Du) - b(x, Du)\right|^q dx\right).
\end{aligned}$$

By applying the continuity conditions in Remark 3, Theorem 6 and Hölder's inequality, we see that

$$\int_\Omega \left|b^k\left(x, Du_*^k\right) - b\left(x, Du\right)\right|^q dx$$

$$\leq C\left(\int_\Omega \left(1 + \left|Du_*^k\right| + |Du|\right)^p dx\right)^{\frac{p-1-\gamma}{p-1}} \left(\int_\Omega \left|Du_*^k - Du\right|^p dx\right)^{\frac{\gamma}{p-1}}$$

$$+C\widetilde{\omega}\left(\frac{1}{k}\right)\int_\Omega 1 + |Du|^p\, dx \leq C\left(\int_\Omega \left|Du_*^k - Du\right|^p dx\right)^{\frac{\gamma}{p-1}} + C\widetilde{\omega}\left(\frac{1}{k}\right).$$

Now (41) follows by taking (36) into account. ∎

5 Properties of the homogenized operator

In this section we prove some properties of the homogenized operator. In particular, these properties imply the existence and uniqueness of the solution of the homogenized problem.

Theorem 6. *Let b be the homogenized operator defined in Theorem 5. Then*

(a) $b(\cdot, \xi)$ *satisfies the continuity condition*

$$|b(x_1, \xi) - b(x_2, \xi)|^q \leq \widetilde{\omega}\left(|x_1 - x_2|\right)\left(1 + |\xi|\right)^p,$$

where $\widetilde{\omega} : \mathbf{R} \to \mathbf{R}$ *is a function that is continuous, increasing and* $\widetilde{\omega}(0) = 0$.

(b) $b(x, \cdot)$ *is strictly monotone. In particular, we have that*

$$\langle b(x, \xi_1) - b(x, \xi_2), \xi_1 - \xi_2\rangle \geq \widetilde{c}_2\left(1 + |\xi_1| + |\xi_2|\right)^{p-\beta}|\xi_1 - \xi_2|^\beta$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

(c) $b(x, \cdot)$ *is continuous. In particular, we have for* $\gamma = \frac{\alpha}{\beta-\alpha}$ *that*

$$|b(x, \xi_1) - b(x, \xi_2)| \leq \widetilde{c}_1\left(1 + |\xi_1| + |\xi_2|\right)^{p-1-\gamma}|\xi_1 - \xi_2|^\gamma$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

(d) $b(x, 0) = 0$ *for* $x \in \Omega$.

Proof.

(a) By the definition of b we have that

$$|b(x_1, \xi) - b(x_2, \xi)|^q = \left|\int_Y a\left(x_1, y, \xi + Dv^{\xi, x_1}\right) dy - \int_Y a\left(x_2, y, \xi + Dv^{\xi, x_2}\right) dx\right|^q$$

$$\leq C\int_Y \left|a\left(x_1, y, \xi + Dv^{\xi, x_1}\right) - a\left(x_2, y, \xi + Dv^{\xi, x_1}\right)\right|^q dy$$

$$+C\int_Y \left|a\left(x_2, y, \xi + Dv^{\xi, x_1}\right) - a\left(x_2, y, \xi + Dv^{\xi, x_2}\right)\right|^q dy,$$

where the last inequality follows from Jensen's inequality. By the continuity conditions (15) and (6), it then follows that

$$
\begin{aligned}
&|b(x_1,\xi)-b(x_2,\xi)|^q \le C\omega(|x_1-x_2|)\int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|\right)^p dy \\
&\quad + C\left(\int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|^p+\left|\xi+Dv^{\xi,x_2}\right|^p\right)dy\right)^{\frac{p-1-\alpha}{p-1}} \\
&\quad \times \left(\int_Y \left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^p dy\right)^{\frac{\alpha}{p-1}}.
\end{aligned}
\tag{42}
$$

We now study the two terms above separately. For the first term we have by (12) that

$$
\int_Y \left(1+\left|\xi+Dv^{\xi,x_i}\right|\right)^p dy \le C\left(1+|\xi|^p\right). \tag{43}
$$

For the second term we have by the definition that

$$
\begin{aligned}
&\int_Y \left\langle a\left(x_1,y,\xi+Dv^{\xi,x_1}(y)\right),D\phi\right\rangle dy = 0 \quad \text{for every} \quad \phi\in W^{1,p}_{\square}(\Omega),\\
&\int_Y \left\langle a\left(x_2,y,\xi+Dv^{\xi,x_2}(y)\right),D\phi\right\rangle dy = 0 \quad \text{for every} \quad \phi\in W^{1,p}_{\square}(\Omega).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_Y \left\langle a\left(x_1,y,\xi+Dv^{\xi,x_1}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right),D\phi\right\rangle dy\\
&\quad=\int_Y \left\langle a\left(x_2,y,\xi+Dv^{\xi,x_2}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right),D\phi\right\rangle dy
\end{aligned}
$$

for every $\phi\in W^{1,p}_{\square}(\Omega)$. In particular, for $\phi=v^{\xi,x_1}-v^{\xi,x_2}$ we have that

$$
\begin{aligned}
&\int_Y \left\langle a\left(x_1,y,\xi+Dv^{\xi,x_1}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right),Dv^{\xi,x_1}-Dv^{\xi,x_2}\right\rangle dy\\
&\quad=\int_Y \left\langle a\left(x_2,y,\xi+Dv^{\xi,x_2}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right),Dv^{\xi,x_1}-Dv^{\xi,x_2}\right\rangle dy.
\end{aligned}
$$

We apply (7) on the left hand side and Schwarz' and Hölder's inequalities together with (15) and (43) on the right hand side. This yields

$$\begin{aligned}
& c_2 \int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|+\left|\xi+Dv^{\xi,x_2}\right|\right)^{p-\beta}\left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^{\beta} dy \\
&\quad \leq \int_Y \left\langle a\left(x_1,y,\xi+Dv^{\xi,x_1}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right), Dv^{\xi,x_1}-Dv^{\xi,x_2}\right\rangle dy \\
&\quad = \int_Y \left\langle a\left(x_2,y,\xi+Dv^{\xi,x_2}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right), Dv^{\xi,x_1}-Dv^{\xi,x_2}\right\rangle dy \\
&\quad \leq \left(\int_Y \left|a\left(x_2,y,\xi+Dv^{\xi,x_2}(y)\right)-a\left(x_1,y,\xi+Dv^{\xi,x_2}(y)\right)\right|^q dy\right)^{\frac{1}{q}} \\
&\quad \times \left(\int_Y \left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^p dy\right)^{\frac{1}{p}} \\
&\quad \leq \omega\left(\left|x_1-x_2\right|\right)^{\frac{1}{q}} C\left(1+|\xi|^p\right)^{\frac{1}{q}}\left(\int_Y \left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^p dy\right)^{\frac{1}{p}}.
\end{aligned}$$

The reversed Hölder inequality then ensures that

$$\begin{aligned}
& \left(\int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|+\left|\xi+Dv^{\xi,x_2}\right|\right)^p dy\right)^{\frac{p-\beta}{p}}\left(\int_Y \left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^p dy\right)^{\frac{\beta}{p}} \\
&\quad \leq \int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|+\left|\xi+Dv^{\xi,x_2}\right|\right)^{p-\beta}\left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^{\beta} dy,
\end{aligned}$$

that is,

$$\begin{aligned}
& \left(\int_Y \left|Dv^{\xi,x_1}-Dv^{\xi,x_2}\right|^p dy\right)^{\frac{\alpha}{p-1}} \leq C\omega\left(\left|x_1-x_2\right|\right)^{\frac{\alpha}{\beta-1}}\left(1+|\xi|^p\right)^{\frac{\alpha}{\beta-1}} \\
&\quad \times \left(\int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|+\left|\xi+Dv^{\xi,x_2}\right|\right)^p dy\right)^{\frac{\alpha(\beta-p)}{(\beta-1)(p-1)}} \\
&\quad \leq C\omega\left(\left|x_1-x_2\right|\right)^{\frac{\alpha}{\beta-1}}\left(1+|\xi|^p\right)^{\frac{\alpha}{\beta-1}}\left(1+|\xi|^p\right)^{\frac{\alpha(\beta-p)}{(\beta-1)(p-1)}} \\
&\quad = C\omega\left(\left|x_1-x_2\right|\right)^{\frac{\alpha}{\beta-1}}\left(1+|\xi|^p\right)^{\frac{\alpha}{p-1}},
\end{aligned} \tag{44}$$

where the last inequality follows from (43). By combining (43) and (44), we can write (42) as

$$\begin{aligned}
& \left|b\left(x_1,\xi\right)-b\left(x_2,\xi\right)\right|^q \leq C\omega\left(\left|x_1-x_2\right|\right)\left(1+|\xi|^p\right) \\
&\quad +C\left(\int_Y \left(1+\left|\xi+Dv^{\xi,x_1}\right|^p+\left|\xi+Dv^{\xi,x_2}\right|^p\right) dy\right)^{\frac{p-1-\alpha}{p-1}} \\
&\quad \times\omega\left(\left|x_1-x_2\right|\right)^{\frac{\alpha}{\beta-1}}\left(1+|\xi|^p\right)^{\frac{\alpha}{p-1}} \\
&\quad \leq C\left(1+|\xi|^p\right)\left(\omega\left(\left|x_1-x_2\right|\right)+\omega\left(\left|x_1-x_2\right|\right)^{\frac{\alpha}{\beta-1}}\right)=\widetilde{\omega}\left(\left|x_1-x_2\right|\right)\left(1+|\xi|^p\right),
\end{aligned} \tag{45}$$

where $\widetilde{\omega} : \mathbf{R} \to \mathbf{R}$ is a function that is continuous, increasing and $\widetilde{\omega}(0) = 0$.

(b) Let $\xi_j \in \mathbf{R}^n$, $j = 1, 2$ and define for a.e. $x \in \mathbf{R}^n$

$$w_h^{\xi_j,x} = (\xi_j, y) + \varepsilon_h v^{\xi_j,x}\left(\frac{y}{\varepsilon_h}\right).$$

By the periodicity of $v^{\xi_j,x}$ we have that

$$w_h^{\xi_j,x} \to (\xi_j, y) \quad \text{weakly in} \quad W^{1,p}(Y), \tag{46}$$

$$Dw_h^{\xi_j,x} \to \xi_j \quad \text{weakly in} \quad L^p(Y, \mathbf{R}^n), \tag{47}$$

$$a\left(x, \frac{y}{\varepsilon_h}, Dw_h^{\xi_j,x}\right) \to b(x, \xi_j) \quad \text{weakly in} \quad L^p(Y, \mathbf{R}^n). \tag{48}$$

Moreover, the monotonicity condition (7) on a implies that

$$\begin{aligned}
&\int_Y \left\langle a\left(x, \frac{y}{\varepsilon_h}, Dw_h^{\xi_1,x}\right) - a\left(x, \frac{y}{\varepsilon_h}, Dw_h^{\xi_2,x}\right), Dw_h^{\xi_1,x} - Dw_h^{\xi_2,x}\right\rangle \phi(y)\, dy \\
&\qquad \geq c_2 \int_Y \left(1 + \left|Dw_h^{\xi_1,x}\right| + \left|Dw_h^{\xi_2,x}\right|\right)^{p-\beta} \left|Dw_h^{\xi_1,x} - Dw_h^{\xi_2,x}\right|^{\beta} \phi(y)\, dy
\end{aligned}$$

for every $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$. The reversed Hölder inequality then yields

$$\begin{aligned}
&\int_Y \left\langle a\left(x, \frac{y}{\varepsilon_h}, Dw_h^{\xi_1,x}\right) - a\left(x, \frac{y}{\varepsilon_h}, Dw_h^{\xi_2,x}\right), Dw_h^{\xi_1,x} - Dw_h^{\xi_2,x}\right\rangle \phi(y)\, dy \\
&\qquad \geq c_2 \left(\int_Y \left(1 + \left|Dw_h^{\xi_1,x}\right| + \left|Dw_h^{\xi_2,x}\right|\right)^{p} \phi(y)\, dy\right)^{\frac{p-\beta}{p}} \\
&\qquad \times \left(\int_Y \left|Dw_h^{\xi_1,x} - Dw_h^{\xi_2,x}\right|^{p} \phi(y)\, dy\right)^{\frac{\beta}{p}}.
\end{aligned}$$

We apply $\liminf\limits_{h\to\infty}$ on both sides of this inequality and obtain

$$\begin{aligned}
&\langle b(x, \xi_1) - b(x, \xi_2), \xi_1 - \xi_2 \rangle \int_Y \phi(y)\, dy = \int_Y \langle b(x, \xi_1) - b(x, \xi_2), \xi_1 - \xi_2 \rangle \phi(y)\, dy \\
&\qquad \geq c_2 \liminf_{h\to\infty} \left(\int_Y \left(1 + \left|Dw_h^{\xi_1,x}\right| + \left|Dw_h^{\xi_2,x}\right|\right)^{p} \phi(y)\, dy\right)^{\frac{p-\beta}{p}} \\
&\qquad \times \left(\int_Y \left|Dw_h^{\xi_1,x} - Dw_h^{\xi_2,x}\right|^{p} \phi(y)\, dy\right)^{\frac{\beta}{p}}
\end{aligned}$$

$$\geq C \liminf_{h\to\infty} \left(\int_Y \left(1 + \left| Dw_h^{\xi_1,x} \right|^p + \left| Dw_h^{\xi_2,x} \right|^p \right) \phi(y)\, dy \right)^{\frac{p-\beta}{p}}$$

$$\times \liminf_{h\to\infty} \left(\int_Y \left| Dw_h^{\xi_1,x} - Dw_h^{\xi_2,x} \right|^p \phi(y)\, dy \right)^{\frac{\beta}{p}}$$

$$\geq C \left(\int_Y \left(1 + |\xi_1|^p + |\xi_2|^p\right) \phi(y)\, dy \right)^{\frac{p-\beta}{p}} \left(\int_Y |\xi_1 - \xi_2|^p \phi(y)\, dy \right)^{\frac{\beta}{p}}$$

$$\geq C \left(1 + |\xi_1| + |\xi_2|\right)^{p-\beta} |\xi_1 - \xi_2|^\beta \int_Y \phi(y)\, dy,$$

for every $\phi \in C_0^\infty(\Omega)$. Here we have used (47), (48) and Lemma 1 on the left hand side, and on the right hand side we have used (47) and the fact that for a weakly convergent sequence (x_n) converging to x, we have that $\|x\| \leq \liminf_{h\to\infty} \|x_n\|$. This implies that

$$\langle b(x,\xi_1) - b(x,\xi_2), \xi_1 - \xi_2 \rangle \geq \widetilde{c}_2 \left(1 + |\xi_1| + |\xi_2|\right)^{p-\beta} |\xi_1 - \xi_2|^\beta .$$

Since ξ_1, ξ_2 were arbitrarily chosen, (b) is proved.

(c) Fix $\xi_1, \xi_2 \in \mathbf{R}^n$. According to Jensen's inequality, (6), Hölder's inequality, (12) and (13) we have that

$$|b(x,\xi_1) - b(x,\xi_2)|^q = \left| \int_Y a\left(x, y, \xi_1 + Dv^{\xi_1,x}\right) dy - \int_Y a\left(x, y, \xi_2 + Dv^{\xi_2,x}\right) dy \right|^q$$

$$\leq \int_Y \left| a\left(x, y, \xi_1 + Dv^{\xi_1,x}\right) - a\left(x, y, \xi_2 + Dv^{\xi_2,x}\right) \right|^q dy$$

$$\leq \int_Y c_1^q \left(1 + \left| \xi_1 + Dv^{\xi_1,x} \right| + \left| \xi_2 + Dv^{\xi_2,x} \right| \right)^{(p-1-\alpha)q}$$

$$\times \left| \xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x} \right|^{\alpha q} dy$$

$$\leq C \left(\int_Y \left(1 + \left| \xi_1 + Dv^{\xi_1,x} \right| + \left| \xi_2 + Dv^{\xi_2,x} \right| \right)^p dy \right)^{\frac{p-1-\alpha}{p-1}}$$

$$\times \left(\int_Y \left| \xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x} \right|^p dy \right)^{\frac{\alpha}{p-1}}$$

$$\leq C \left(1 + |\xi_1|^p + |\xi_2|^p\right)^{\frac{p-1-\alpha}{p-1}} \left(\left(1 + |\xi_1|^p + |\xi_2|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} |\xi_1 - \xi_2|^{\frac{p}{\beta-\alpha}} \right)^{\frac{\alpha}{p-1}}$$

$$\leq C \left(1 + |\xi_1| + |\xi_2|\right)^{q\left(p-1-\frac{\alpha}{\beta-a}\right)} |\xi_1 - \xi_2|^{q\frac{\alpha}{\beta-a}} .$$

Hence

$$|b(x,\xi_1) - b(x,\xi_2)| \leq \widetilde{c}_1 \left(1 + |\xi_1| + |\xi_2|\right)^{p-1-\gamma} |\xi_1 - \xi_2|^\gamma ,$$

where

$$\gamma = \frac{\alpha}{\beta - a}.$$

(d) Since $a(x,y,0)=0$ we have that the solution of the cell problem (29) corresponding to $\xi=0$ is $v^{0,x}=0$. This implies that

$$b(x,0)=\int_Y a(x,y,0)\,dy=0.$$

The proof is complete. ■

Remark 3. By using similar arguments as in the proof above we find that (b), (c) and (d) holds, up to boundaries, also for the homogenized operator b in Theorem 4.

6 Some corrector results

We have proved in both Theorem 4 and Theorem 5 that we for the corresponding solutions have that u_h-u converges to 0 weakly in $W_0^{1,p}(\Omega)$. By the Rellich imbedding theorem, we have that u_h-u converges to 0 in $L^p(\Omega)$. In general, we do not have strong convergence of Du_h-Du to 0 in $L^p(\Omega,\mathbf{R}^n)$. However, we will prove that it is possible to express Du_h in terms of Du, up to a remainder which converges to 0 in $L^p(\Omega,\mathbf{R}^n)$.

Theorem 7. *Let u and u_h be defined as in Theorem 4 and let P_h be given by (3). Then*

$$Du_h-\sum_{i=1}^{N}\chi_{\Omega_i}(x)\,P_h(x,M_hDu,x_i)\to 0\quad in\quad L^p(\Omega,\mathbf{R}^n).$$

Proof. In [7] the case $N=1$ was considered and in [15] the case $p=2$ was considered. By using these ideas and making the necessary adjustments the proof follows. For the details, see [5]. ■

Theorem 8. *Let u and u_h be defined as in Theorem 5. Moreover, let P_h be given by (3) and γ_h by (5). Then*

$$Du_h-P_h(x,M_hDu,\gamma_h)\to 0\quad in\quad L^p(\Omega,\mathbf{R}^n).$$

Proof. We have that

$$\begin{aligned}\left\|Du_h-M_hDu-Dv^{M_hDu,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^p(\Omega,\mathbf{R}^n)}&\le\left\|Du_h-Du_h^k\right\|_{L^p(\Omega,\mathbf{R}^n)}\\&+\left\|Du_h^k-M_hDu_*^k-\sum_{i\in I_k}\chi_{\Omega_i^k}(x)\,Dv^{M_hDu_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^p(\Omega,\mathbf{R}^n)}\\&+\left\|M_hDu_*^k+\sum_{i\in I_k}\chi_{\Omega_i^k}(x)\,Dv^{M_hDu_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right)-M_hDu-Dv^{M_hDu,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^p(\Omega,\mathbf{R}^n)}.\end{aligned}\tag{49}$$

As in the proof of Theorem 5 we have that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\|Du_h-Du_h^k\right\|_{L^p(\Omega,\mathbf{R}^n)}=0,$$

and, by Theorem 7, this implies that

$$\lim_{k\to\infty}\lim_{h\to\infty}\left\|Du_h^k - M_h Du_*^k - \sum_{i\in I_k}\chi_{\Omega_i^k}(x)\,Dv^{M_h Du_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^p(\Omega,\mathbf{R}^n)} = 0.$$

This means that the theorem would be proved if we prove that $\lim\limits_{k\to\infty}\lim\limits_{h\to\infty}$ acting on the last term in (49) is equal to 0. In order to prove this fact we first make the following elementary estimates:

$$\begin{aligned}
&\left\|M_h Du_*^k + \sum_{i\in I_k}\chi_{\Omega_i^k}(x)\,Dv^{M_h Du_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right) - M_h Du - Dv^{M_h Du,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^p(\Omega,\mathbf{R}^n)}^p \\
&\quad = \sum_{i\in I_k}\int_{\Omega_i^k}\left|M_h Du_*^k + Dv^{M_h Du_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right) - M_h Du - Dv^{M_h Du,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\quad \le C\sum_{i\in I_k}\int_{\Omega_i^k}\left|M_h Du_*^k + Dv^{M_h Du_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right) - M_h Du - Dv^{M_h Du,x_i^k}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\quad + C\sum_{i\in I_k}\int_{\Omega_i^k}\left|Dv^{M_h Du,x_i^k}\left(\frac{x}{\varepsilon_h}\right) - Dv^{M_h Du,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx.
\end{aligned} \tag{50}$$

We will study the two terms on the right hand side of (50) separately, but first we define

$$\xi_{h,*}^{j,k} = \frac{1}{\left|Y_h^j\right|}\int_{Y_h^j} Du_*^k\,dx.$$

By using a change of variables in (13) and Hölder's inequality, we find that we for the first term in (50) have the following estimate

$$\begin{aligned}
&\int_{\Omega_i^k}\left|M_h Du_*^k + Dv^{M_h Du_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right) - M_h Du - Dv^{M_h Du,x_i^k}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\quad \le \sum_{j\in J_h^{i,k}}\int_{Y_h^j}\left|\xi_{h,*}^{j,k} + Dv^{\xi_{h,*}^{j,k},x_i^k}\left(\frac{x}{\varepsilon_h}\right) - \xi_h^j - Dv^{\xi_h^j,x_i^k}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\quad + \sum_{j\in B_h^{i,k}}\int_{Y_h^j}\left|\xi_{h,*}^{j,k} + Dv^{\xi_{h,*}^{j,k},x_i^k}\left(\frac{x}{\varepsilon_h}\right) - \xi_h^j - Dv^{\xi_h^j,x_i^k}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\quad \le \sum_{j\in J_h^{i,k}} C\left(1 + \left|\xi_{h,*}^{j,k}\right|^p + \left|\xi_h^j\right|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left|\xi_{h,*}^{j,k} - \xi_h^j\right|^{\frac{p}{\beta-\alpha}}\left|Y_h^j\right| \\
&\quad + \sum_{j\in B_h^{i,k}} C\left(1 + \left|\xi_{h,*}^{j,k}\right|^p + \left|\xi_h^j\right|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left|\xi_{h,*}^{j,k} - \xi_h^j\right|^{\frac{p}{\beta-\alpha}}\left|Y_h^j\right| \\
&\quad = C\int_{\Omega_i^{k,h}}\left(1 + \left|M_h Du_*^k\right|^p + \left|M_h Du\right|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left|M_h Du_*^k - M_h Du\right|^{\frac{p}{\beta-\alpha}} dx
\end{aligned} \tag{51}$$

$$+C\int_{F_i^{k,h}}\left(1+\left|M_hDu_*^k\right|^p+|M_hDu|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left|M_hDu_*^k-M_hDu\right|^{\frac{p}{\beta-\alpha}}dx$$

$$\leq C\left(\int_{\Omega_i^{k,h}}\left(1+\left|M_hDu_*^k\right|+|M_hDu|\right)^p dx\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left(\int_{\Omega_i^{k,h}}\left|M_hDu_*^k-M_hDu\right|^p dx\right)^{\frac{1}{\beta-\alpha}}$$

$$+C\left(\int_{F_i^{k,h}}\left(1+\left|M_hDu_*^k\right|+|M_hDu|\right)^p dx\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left(\int_{F_i^{k,h}}\left|M_hDu_*^k-M_hDu\right|^p dx\right)^{\frac{1}{\beta-\alpha}}$$

$$\leq C\left(\int_{\Omega_i^k}\left(1+\left|Du_*^k\right|+|Du|\right)^p dx\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left(\int_{\Omega_i^k}\left|Du_*^k-Du\right|^p dx\right)^{\frac{1}{\beta-\alpha}}$$

$$+C\left(\int_{F_i^{k,h}}\left(1+\left|Du_*^k\right|+|Du|\right)^p dx\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left(\int_{F_i^{k,h}}\left|Du_*^k-Du\right|^p dx\right)^{\frac{1}{\beta-\alpha}},$$

where we used Jensen's inequality in the last step. Moreover, by using (44) and Jensen's inequality, we obtain that

$$\begin{aligned}
&\int_{\Omega_i^k}\left|Dv^{M_hDu,x_i^k}\left(\frac{x}{\varepsilon_h}\right)-Dv^{M_hDu,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx\\
&\quad\leq\sum_{j\in J_h^{i,k}}\int_{Y_h^j}\left|Dv^{\xi_h^j,x_i^k}\left(\frac{x}{\varepsilon_h}\right)-Dv^{\xi_h^j,x_h^j}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx\\
&\quad+\sum_{j\in B_h^{i,k}}\int_{Y_h^j}\left|Dv^{\xi_h^j,x_i^k}\left(\frac{x}{\varepsilon_h}\right)-Dv^{\xi_h^j,x_h^j}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx\\
&\quad\leq\sum_{j\in J_h^{i,k}}C\omega\left(\frac{1}{k}\right)\left(1+\left|\xi_h^j\right|\right)^p\left|Y_h^j\right|+\sum_{j\in B_h^{i,k}}C\omega\left(\frac{1}{k}+n\varepsilon_h\right)\left(1+\left|\xi_h^j\right|\right)^p\left|Y_h^j\right|\\
&\quad=\omega\left(\frac{1}{k}\right)C\int_{\Omega_i^{k,h}}\left(1+|M_hDu|^p\right)dx+\omega\left(\frac{1}{k}+n\varepsilon_h\right)C\int_{F_i^{k,h}}\left(1+|M_hDu|^p\right)dx\\
&\quad\leq\omega\left(\frac{1}{k}\right)C\int_{\Omega_i^k}\left(1+|Du|^p\right)dx+\omega\left(\frac{1}{k}+n\varepsilon_h\right)C\int_{F_i^{k,h}}\left(1+|Du|^p\right)dx.
\end{aligned}\tag{52}$$

By combining (50), (51) and (52) we obtain that

$$\left\|M_hDu_*^k+\sum_{i\in I_k}\chi_{\Omega_i^k}Dv^{M_hDu_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right)-M_hDu-Dv^{M_hDu,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^p(\Omega,\mathbf{R}^n)}^p$$

$$\leq C\left(\int_\Omega\left(1+\left|Du_*^k\right|+|Du|\right)^p dx\right)^{\frac{\beta-\alpha-1}{\beta-a}}\left(\int_\Omega\left|Du_*^k-Du\right|^p dx\right)^{\frac{1}{\beta-\alpha}}$$

$$+C\sum_{i\in I_k}\left(\int_{F_i^{k,h}}\left(1+\left|Du_*^k\right|+|Du|\right)^p dx\right)^{\frac{\beta-\alpha-1}{\beta-a}}\left(\int_{F_i^{k,h}}\left|Du_*^k-Du\right|^p dx\right)^{\frac{1}{\beta-\alpha}}$$

$$+C\omega\left(\frac{1}{k}+n\varepsilon_h\right)\left(\int_\Omega\left(1+|Du|^p\right)dx+\sum_{i\in I_k}\int_{F_i^{k,h}}\left(1+|Du|^p\right)dx\right).$$

Moreover, by noting that $\left|F_i^{k,h}\right|\to 0$ as $h\to\infty$, $\|Du\|_{L^p(\Omega,\mathbf{R}^n)}\le C$, $\left\|Du_*^k\right\|_{L^p(\Omega,\mathbf{R}^n)}\le C$, and taking (35) into account, we obtain that

$$\lim_{k\to\infty}\lim_{h\to\infty}\Bigg\|M_hDu_*^k+\sum_{i\in I_k}\chi_{\Omega_i^k}Dv^{M_hDu_*^k,x_i^k}\left(\frac{x}{\varepsilon_h}\right)$$

$$-M_hDu-Dv^{M_hDu,\gamma_h}\left(\frac{x}{\varepsilon_h}\right)\Bigg\|_{L^p(\Omega,\mathbf{R}^n)}=0,$$

and the theorem is proved. ■

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