UV Manifold and Integrable Systems in Spaces of Arbitrary Dimension

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Abstract

The $2n$ dimensional manifold with two mutually commutative operators of differentiation is introduced. Nontrivial multidimensional integrable systems connected with arbitrary graded (semisimple) algebras are constructed. The general solution of them is presented in explicit form.

1   Introduction

The success in the application of group-theoretical methods to the theory of two-dimensional integrable systems [1, 2] is not accidental. It is connected with the circumstance that the operations of multiplication of the group elements from left and right are mutually commutative. It allows us to associate with two commutative differential operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ of two dimensional space infinitesimal displacement generators dependent upon one parameter from left and right in such a way as to solve trivially the representation of zero curvature type.

The foundation of the whole construction is the group valued function:

$$K = M(x)N(y), \tag{1}$$

where the group valued elements M, N are constructed by definite simple rules, containing only operations in one dimensional spaces (x, y) respectively.

Many different attempts have been made to generalize this construction to the multidimensional case by substituting instead of (x, y) in (1) some multidimensional functions $x = \phi(x_1, \ldots, x_n)$, $y = \psi(y_1, \ldots, y_m)$ and considering the consequences which follow from the associated multidimensional L–A pair formalism. However no interesting nontrivial integrable system has been discovered in this way. Moreover the solutions which arise in such a construction (in the concrete examples considered) are always only particular but not general. The reader can find the corresponding literature in [4].

In the present paper we want exploit the following observation. It is always possible to represent general solutions of integrable systems following from (1) in local form. Or

in other words to obtain the general solutions of such systems only the operation of differentiation is necessary (see [3]). The operation of integration arises only on the middle step of calculations and may be eliminated from the final result. This means that if we have a multidimensional space with two mutually commutative operators of differentiation D_1, D_2 we will be able to use the two dimensional construction to obtain nontrivial multidimensional integrable systems together with their general solutions.

The aim of the present paper is in the explicit construction of multidimensional manifolds with such properties. The paper is organized in the following way. In Section 2 we discuss general properties of the operators of differentiation and obtain the equations they define. In Section 3 we find the solution of the corresponding equations and in this realize the manifold with the necessary properties. Section 4 is devoted to consideration of concrete examples of integrable systems on the multidimensional manifold constructed above. Comments and perspectives for further investigation are collected in Section 5.

2   The main equations defining the manifold

Suppose we have a $2n$ dimensional Euclidian space with a set of independent coordinates y_i, $\bar{y}_i$. We assume the existence of two mutually commutative generators of differentiation D, $\bar{D}$, which satisfy the basic relations:

$$D\bar{y} = 0, \qquad \bar{D}y = 0 \tag{2}$$

which are reminiscent of (anti) holomorphic functions in the theory of the complex variables. So D is a linear combination of space derivatives with respect to the unbarred coordinates, $\bar{D}$ is the same with respect to the barred ones:

$$D = \frac{\partial}{\partial y_n} + \sum u^\mu \frac{\partial}{\partial y_\mu}, \qquad \bar{D} = \frac{\partial}{\partial \bar{y}_n} + \sum v^\nu \frac{\partial}{\partial \bar{y}_\nu}. \tag{3}$$

We choose the coefficient functions of differentiation with respect to y_n, $\bar{y}_n$ to be equal to unity. This is an inessential restriction.

If we demand mutual commutativity of these generators $[D, \bar{D}] = 0$, then as a corollary of (3) we obtain the system of equations, which the coefficient functions must satisfy:

$$Dv^\nu \equiv v^\nu_{y_n} + \sum u^\mu v^\nu_{y_\mu} = 0, \qquad \bar{D}u^\mu = u^\mu_{\bar{y}_n} + \sum v^\nu u^\mu_{\bar{y}_\nu} = 0. \tag{4}$$

We will call (4) the uv system and the corresponding manifold the UV one.

From (4) it follows that an arbitrary function $\bar{f}$ of variables v, $\bar{y}$ are annihilated by differentiation by D, all functions $f(u, y)$ by $\bar{D}$. In this sense we will speak about holomorphic and antiholomrphic functions.

The following proposition arises:

Proposition 1. *Each function $\bar{f}$ annihilated by the operator D is holomorphic; a function f annihilated by the operator $\bar{D}$ is antiholomorpthic.*

Let us add the equations $D\bar{f} = 0$ and $\bar{D}f = 0$ to the system of $(n-1)$ equations (4). Then the n sets of variables $(1, u)$, and $(1, v)$ respectively satisfy a linear system of n

algebraic equations the matrix of which coincides with the Jacobian matrix (we consider the holomorphic case):

$$J = \det_n \begin{vmatrix} v^1 & \cdots & v^{n-1} & f \\ y_1 & \cdots & y_{n-1} & y_n \end{vmatrix} \tag{5}$$

which in the case of a non-zero solution of the linear system must vanish. So Proposition 1 is proved.

As a corollary we obtain the following

Proposition 2.

$$\bar{D}v^\nu = v^\nu_{\bar{y}_n} + \sum v^\mu v^\nu_{\bar{y}_\mu} = V^\nu(v; \bar{y}), \qquad Du^\mu = u^\mu_{y_n} + \sum u^\nu u^\mu_{y_\nu} = U^\mu(u; y). \tag{6}$$

Indeed operators D, $\bar{D}$ are commutative and so $\bar{D}v$ is solution of the same equation as v satisfies. But each solution of third equation is a holomorphic function, which proves Proposition 2.

If we consider functions U, V as given, then all operations of differentiation applied to the functions $\Phi(u, v; y, \bar{y})$ are well defined.

As was mentioned in the introduction only these operations arise in the theory of integrable systems constructed in the manner of (1). Thus for a realization of the proposed program it is necessary to solve the system of equations (4), (6) with the given holomorphic and antiholomorphic U, V functions.

3   General solution of *uv* system and realisation of the multidimensional *UV* manifold

Consider the system of equations defining implicitly $(n-1)$ unknown functions (ϕ) in $(2n)$ dimensional space $(y, \bar{y})$:

$$Q^\nu(\phi; y) = P^\nu(\phi; \bar{y}) \tag{7}$$

with the convention that all Greek indices take values between 1 and $(n-1)$. The number of equations in (7) coincides with the number of unknown functions ϕ^α. Each arbitrary function Q, P depends on $(2n - 1)$ coordinates.

With the help of the usual rules of differentiation of implicit functions we find from (7):

$$\phi_y = (P_\phi - Q_\phi)^{-1}Q_y, \qquad \phi_{\bar{y}} = -(P_\phi - Q_\phi)^{-1}P_{\bar{y}}. \tag{8}$$

Let us assume, that between n derivatives with respect to barred and unbarred variables the following linear dependence takes place:

$$\sum_1^n c_i \phi^\alpha_{y_i} = 0, \qquad \sum_1^n d_i \phi^\alpha_{\bar{y}_i} = 0$$

and analyse the corollaries following from these facts.

Assuming that $c_n \neq 0$, $d_n \neq 0$, dividing them into each equation of the left and right sets respectively and introducing the notation $u^\alpha = \frac{c_\alpha}{c_n}$, $v^\alpha = \frac{d_\alpha}{d_n}$, we rewrite the last system in the form:

$$\phi^\alpha_{y_n} + \sum_1^{n-1} u^\nu \phi^\alpha_{y_\nu} = 0, \qquad \phi^\alpha_{\bar{y}_n} + \sum_1^{n-1} v^\nu \phi^\alpha_{\bar{y}_\nu} = 0. \tag{9}$$

Substituting values of the derivatives from (8) and multiplying result by the matrix $(P_\phi - Q_\phi)$ on the left we obtain:

$$Q^\alpha_{y_n} + \sum_1^{n-1} u^\nu Q^\alpha_{y_\nu} = 0, \qquad P^\alpha_{\bar{y}_n} + \sum_1^{n-1} v^\nu P^\alpha_{\bar{y}_\nu} = 0. \tag{10}$$

From the last equations it immediately follow that:

$$u = -(Q_y)^{-1} Q_{y_n}, \qquad v = -(P_{\bar{y}})^{-1} P_{\bar{y}_n}. \tag{11}$$

We see that if we augment the initial system (7), by $(n-1)$ vector functions (u, v) defined by (11) then the differential operators D, $\bar{D}$ defined by (3) in connection with (9) annihilate every ϕ and therefore the functions Q, P:

$$D\phi = \bar{D}\phi = DQ = DP = \bar{D}Q = \bar{D}P = 0. \tag{12}$$

This means that $D\bar{f}(\phi, \bar{y}) = \bar{D}f(\phi, y) = 0$. And as a direct corollary of this fact $Dv = \bar{D}u = 0$ and so the generators D, $\bar{D}$ constructed in this way are mutually commutative.

Thus we have found the general solution of the uv system and in such a way realise the manifold with the properties postulated in the previous section.

It is possible to say (the solution of the uv system is general) that this realisation is unique up to possible similarity transformations.

We present below the result of calculations of the functions U, V using the definition of u, v functions (11):

$$U = Du = -Q_y^{-1} \left(DQ_{y_n} + \sum u^\alpha DQ_{y_\alpha} \right),$$

$$V = \bar{D}v = -P_{\bar{y}}^{-1} \left(\bar{D}P_{\bar{y}_n} + \sum^\alpha \bar{D}Q_{\bar{y}_\alpha} \right).$$

4   Examples

Below we would like to consider only two examples clarifying the situation. But really the same is true with respect to each two dimensional system integrable with the help of the formalism of graded algebras [1, 2].

We want to emphasize that in all cases (particulary those considered below) the UV manifold by itself is defined by $2(n-1)$ arbitrary functions Q, P each of one of which depends upon $2n-1$ independent arguments. All these functions occur as coefficient functions (via operators of differentiation D, $\bar{D}$) in equations of multidimensional integrable systems. Naturally the general solution depends upon them. Apart from these functions the general solution of an integrable system depends upon additional arbitrary functions the number of which and their functional dependence has to be sufficient for the statement of Cauchy or Gursat initial data problems.

4.1 Multidimensional Liouville equation

By this term we understand the equation:

$$D\bar{D}\phi = \exp 2\phi.$$

By direct calculation one can become convinced that its general solution has the form:

$$\exp \phi = \frac{(Dc)^{1/2}(\bar{D}\bar{c})^{1/2}}{c + \bar{c}}.$$

For instance in the four-dimensional case $(y_1, y_2; \bar{y}_1, \bar{y}_2)$:

$$D\bar{D} = \bar{D}D = \frac{\partial^2}{\partial y_2 \partial \bar{y}_2} + v\frac{\partial^2}{\partial y_2 \partial \bar{y}_1} + u\frac{\partial^2}{\partial \bar{y}_2 \partial y_1} + uv\frac{\partial^2}{\partial y_1 \partial \bar{y}_1}$$

and

$$u = -\frac{Q_{y_2}}{Q_{y_1}}, \qquad u = -\frac{P_{\bar{y}_2}}{P_{\bar{y}_1}}, \qquad Q(\phi; y_1, y_2) = P(\phi; \bar{y}_1, \bar{y}_2).$$

The functions $c(u; y)$, $\bar{c}(v; \bar{y})$ are arbitrary holomrphic, antiholomorphic functions.

4.2 Multidimensional Toda system

The equations of the Toda lattice in two dimensions have the form (one of the many possible ones):

$$x^i_{z,\bar{z}} = \exp\left(-x^{i-1} + 2x^i - x^{i+1}\right), \qquad 1 \le i \le n, \qquad x_{-1} = x_{n+1} = 0.$$

The general solution may be represented as:

$$\exp\left(-x^i\right) = \det_i\left\{V^0\right\}, \qquad \exp(-x_0) = V^0,$$

where matrix elements of the determinant matrix V^0 are as follows:

$$V^0_{i,j} = \frac{\partial^{i+j-2}}{\partial z^{i-1} \partial \bar{z}^{j-1}} V^0$$

and the single function V^0 has the form:

$$V^0 = \frac{W_n(\theta, z)W_n(\bar{\theta}, \bar{z})}{1 + \sum\limits_1^n \theta_i \bar{\theta}_i}.$$

$W_n(\phi, x)$ is the determinant of the Wronskian matrix with elements:

$$W_{i,j} = \frac{\partial^{i-1}\theta^j}{\partial z^{j-1}}.$$

To obtain the explicit general solution of multidimensional Toda lattice system:

$$\bar{D}D = D\bar{D}x^i = \exp\left(-x^{i-1} + 2x^i - x^{i+1}\right)$$

only the following changes in all the corresponding formulae above are necessary:

$$\frac{\partial}{\partial z} \to D, \qquad \frac{\partial}{\partial \bar{z}} \to \bar{D}$$

and consider arbitrary functions θ, $\bar{\theta}$ as arbitrary holomorphic, antiholomorphic functions on the manifold UV of the corresponding dimension.

5 Outlook

The main results of the present paper are in the general solution of the uv system (4), a re-alisation of a multidimensional manifold with two commutative operators of differentiation (Section 3) and the integrable systems in the spaces of arbitrary dimensions constructed in this framework (Section 4).

It turns out that with the help of UV manifold it is possible to find general solution of such interesting from the point of viw physical applications systems as homogeneous Complex Monge–Amphér and Bateman equations in the spaces of arbitrary dimensions [5]. This was achieved by reducing of the definite kind the general solution of uv system on subclasses in which solution of it is functionally depends only on two arbitrary functions of the necessary number of independent arguments.

It is obviousl that formalism of evolution type systems (integrable with the help of the old inverse scattering method) remains without any changings in the spaces of arbitrary dimension. The exactly integrable systems considered above are the symmetries of evolu-tion type equations. The knowledge of the general solution of the first allow represent in explicit form multisoliton solutions of the last.

We should like to finish this outlook with some speculative comments about the possible application of the proposed construction to the problems of the physics.

If the manifold constructed would have any relation to the real four dimensional world, then something similar to Einstein's General Relativity would occur. Indeed, in both cases the general (geometrical) properties of the world be determined by some fundamen-tal physical objects, the metrical tensor g_{ij} in the case of General Relativity, Einstein's equations for which takes into account the presence of all forms of matter and the equa-tions uv describing the manifold UV in the case considered above (which of course must be modified to take into account all other physical fields). To be optimistic, it may happen that equations of General Relativity have a solution on the manifold of the kind described above or something similar to it. Of course all this is only an attractive speculation and only a deeper investigation of the problem may clarify the situation.

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