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Reflectionless Analytic Difference Operators I. Algebraic Framework

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Abstract

We introduce and study a class of analytic difference operators admitting reflectionless eigenfunctions. Our construction of the class is patterned after the Inverse Scattering Transform for the reflectionless self-adjoint Schrödinger and Jacobi operators corresponding to KdV and Toda lattice solitons.

1 Introduction

This paper and its two companion papers Refs. [1] and [2] (from now on referred to as Parts II and III) originate from our previous work on reflectionless analytic difference operators of relativistic Calogero–Moser type, cf. Refs. [3, 4]. In our conference contribution Ref. [5] we already discussed the possible existence of an extensive class of self-adjoint reflectionless analytic difference operators containing those studied in Ref. [4]. The present series of papers serves to enlarge the scenario envisaged in Ref. [5], and in particular confirms some conjectures made there.

In this first part we restrict attention to the algebraic aspects of a huge class of analytic difference operators that admit reflectionless eigenfunctions. In Part II we show that these operators and eigenfunctions can be tied in with a non-local soliton evolution equation, and with integrable (classical) N-body systems of relativistic Calogero–Moser type. Part III is concerned with a restricted class, for which we are able to prove self-adjointness of the analytic difference operators by exploiting various features of the reflectionless eigenfunctions.

We proceed by delineating the analytic difference operators at issue in this paper. They are of the form

$$A = T_i + V_a(x)T_{-i} + V_b(x). \tag{1.1}$$

Here, the coefficients ("potentials") $V_a(x)$ and $V_b(x)$ are functions from the field $\mathcal{M}$ of meromorphic functions, and the translations $T_{\pm i}$ are defined by

$$(T_\alpha f)(x) \equiv f(x-\alpha), \qquad \alpha \in \mathbb{C}^*, \qquad f \in \mathcal{M}. \tag{1.2}$$

We only consider potentials satisfying

$$\lim_{|\mathrm{Re}\, x| \to \infty} V_a(x) = 1, \qquad \lim_{|\mathrm{Re}\, x| \to \infty} V_b(x) = 0. \tag{1.3}$$

Hence the analytic difference operator A becomes equal to the “free” analytic difference operator (henceforth AΔO)

$$A_0 \equiv T_i + T_{-i} \tag{1.4}$$

for $|\mathrm{Re}\, x| \to \infty$.

Obviously, A_0 admits plane wave eigenfunctions $\exp(ixp)$ with eigenvalue $e^p + e^{-p}$ for all $p \in \mathbb{C}$. It is therefore a natural question whether the AΔO A, viewed as a linear operator on $\mathcal{M}$, admits eigenfunctions with eigenvalue $e^p + e^{-p}$ and plane wave asymptotics for $|\mathrm{Re}\, x| \to \infty$. More specifically, this question reads: Do there exist functions $\mathcal{W}(\cdot, p) \in \mathcal{M}$, satisfying

$$A\mathcal{W}(x,p) = (e^p + e^{-p})\mathcal{W}(x,p), \tag{1.5}$$

$$\mathcal{W}(x,p) \sim \exp(ixp), \qquad \mathrm{Re}\, x \to \infty, \tag{1.6}$$

$$\mathcal{W}(x,p) \sim a(p)\exp(ixp) + b(p)\exp(-ixp), \qquad \mathrm{Re}\, x \to -\infty, \tag{1.7}$$

for generic $p \in \mathbb{C}$?

To our knowledge, this question has not been addressed in previous literature. Here we do not answer it either. However, we do present a vast class of potential pairs V_a, V_b for which we answer the question in the affirmative. Moreover, the pertinent eigenfunctions are reflectionless, in the sense that in (1.7) one has $b(p) = 0$. They are of the form

$$\mathcal{W}(x,p) = \exp(ixp)\left(1 - \sum_{n=1}^{N} \frac{R_n(x)}{e^p - z_n}\right), \tag{1.8}$$

where $z_1, \ldots, z_N$ are distinct complex numbers, and $R_1, \ldots, R_N \in \mathcal{M}^*$. (Here and below, $\mathcal{M}^*$ denotes the space of meromorphic functions with the zero function deleted.)

It should be emphasized at the outset that whenever such eigenfunctions exist, they are highly non-unique. Indeed, introducing the infinite-dimensional space

$$\mathcal{P}_\alpha \equiv \{\mu \in \mathcal{M}^* \mid \mu(x+\alpha) = \mu(x)\}, \qquad \alpha \in \mathbb{C}^*, \tag{1.9}$$

of α-periodic multipliers, one verifies first of all that whenever $\mathcal{W}(x, p_0)$ satisfies (1.5) for a certain $p_0 \in \mathbb{C}$, then the function $\mu(x)\mathcal{W}(x, p_0)$ also satisfies (1.5) for all $\mu(x) \in \mathcal{P}_i$. Moreover, introducing

$$\mathcal{P}_\alpha(c) \equiv \left\{\mu \in \mathcal{P}_\alpha \Big| \lim_{|\mathrm{Re}\, x| \to \infty} \mu(x) = c,\ c \in \mathbb{C}\right\}, \tag{1.10}$$

and choosing $\mu \in \mathcal{P}_i(1)$, the function $\mu(x)\mathcal{W}(x, p_0)$ has once more plane wave asymptotics for $|\mathrm{Re}\, x| \to \infty$. Thus, when we supplement the numbers $z_1, \ldots, z_N$ occurring in (1.8) with further complex numbers $z_{N+1}, \ldots, z_M$, such that $z_1, \ldots, z_M$ are distinct, then the function

$$\tilde{\mathcal{W}}(x,p) \equiv \mathcal{W}(x,p)\left(1 - \sum_{n=N+1}^{M} \frac{\nu_n(x)}{e^p - z_n}\right), \qquad \nu_n \in \mathcal{P}_i(0), \tag{1.11}$$

is of the same form, and satisfies (1.5)–(1.7) as well.

For the class of potentials involved, however, this non-uniqueness can be obviated by restricting attention to "residue functions" $R_1(x), \ldots, R_N(x)$ with quite special properties. (This is shown in Lemma 4.2 below.) In point of fact, we will first define functions $R_1, \ldots, R_N \in \mathcal{M}^*$ as the unique solutions to a certain system of N linear equations, given by (2.37). This system involves suitably constrained complex numbers $z_1 = \exp(r_1), \ldots, z_N = \exp(r_N)$, cf. (2.30), (2.32), and multipliers $\mu_1(x), \ldots, \mu_N(x)$ satisfying (2.34). Subsequently, we define a function $\mathcal{W}(x,p)$ by (1.8). Then we show that this function has asymptotics (1.6) and (1.7) with $b(p) = 0$, and prove that there exists a uniquely determined A∆O A of the above type satisfying (1.5) for $\exp(p) \neq z_1, \ldots, z_N$. (For this discrete set of p-values the wave function $\mathcal{W}(x,p)$ (1.8) is of course ill defined.)

The procedure just sketched will be detailed in Section 2. It owes much to the account of reflectionless Schrödinger operators that can be found in Newell's monograph Ref. [6]. More generally, it is inspired by the inverse scattering transform (IST) for one-dimensional self-adjoint Schrödinger and Jacobi operators. As is well known (see, for instance, Refs. [6]–[10]), in the reflectionless case one winds up with a linear system of N equations, where N is the number of bound states. This system involves the Cauchy matrix C, just as the system (2.37). (We have relegated the features of C we need to Appendix A.)

Our class of A∆Os admitting reflectionless eigenfunctions is far more extensive than the reflectionless Schrödinger and Jacobi operators obtained via the IST. This is because we can allow arbitrary multiplier functions $\mu_n(x)$ from the infinite-dimensional space $\mathcal{P}_i(c_n), n = 1 \ldots, N$, cf. (2.34). Choosing $\mu_n(x)$ equal to the constant c_n for all $n \in \{1, \ldots, N\}$, it is still "twice as large", in the sense that we neither require $|z_n| = 1$ nor conditions on c_n.

On the other hand, we do require constant multipliers and reality conditions in Part III of this series, which deals with Hilbert space aspects [2]. Indeed, we need such constraints in order to exploit the wave function $\mathcal{W}(x,p)$ for associating to the A∆O A a bona fide self-adjoint operator $\hat{A}$ on the Hilbert space $L^2(\mathbb{R}, dx)$. (To date, a general self-adjointness theory in which our indirect definition of $\hat{A}$ fits is not available.)

For our present purposes it suffices to mention one prominent fact illustrating the considerable differences between reflectionless self-adjoint A∆Os and reflectionless self-adjoint Schrödinger and Jacobi operators. This is the existence of an infinite-dimensional family of self-adjoint reflectionless A∆Os without *any* bound states. (Actually, a smaller, but still infinite-dimensional family of such A∆Os can already be found in our paper Ref. [4], cf. also Ref. [5].) This circumstance shows that one should not expect a straightforward analog of the direct transform associated with the Schrödinger and Jacobi cases, and indeed we have very little to say about the direct problem for A∆Os.

We continue with a more detailed account of the organization and results of this paper. As mentioned above, Section 2 is concerned with the groundwork for this series of papers. Fixing $N \in \mathbb{N}^*$, the multipliers (2.34) already form an infinite-dimensional family, yet they are by no means the largest class giving rise to A∆Os admitting reflectionless eigenfunctions. To illustrate the freedom involved, we have made a close-up of the $N = 1$ case, where all pertinent objects can be inspected without difficulty.

The restrictions (2.30) and (2.32) are primarily motivated by the applications in Parts II and III, where they will be seen to be quite natural. On the other hand, for the soliton solutions in Part II and for all of Part III we also need the multipliers to be constant. It is a striking feature of the algebraic viewpoint adopted in this first part that the freedom allowed by (2.34) does not give rise to any additional difficulty, as compared to allowing only constants. (By contrast, when we try to relax the requirements (2.30), (2.32) and (2.34), we typically find that some of our arguments break down.) In particular, one obtains the same transmission coefficient (given by (2.48)) for all of these multipliers.

Section 3 contains on the one hand some general insights bearing on the question whether the potential $V_b(x)$ can vanish identically. On the other hand, these results point the way towards special cases where one does have $V_b = 0$. This enables us to show that when all of the numbers $r_1, \ldots, r_N$ in (2.30) have imaginary parts in $(0, \pi)$ or in $(-\pi, 0)$, then the wave function is not only an eigenfunction of A (1.1) with eigenvalue $\exp(p) + \exp(-p)$, but also an eigenfunction with eigenvalue $\exp(p/2) + \delta \exp(-p/2)$ of an AΔO

$$S_\delta = T_{i/2} + \delta V(x) T_{-i/2}, \qquad \delta = +, -. \tag{1.12}$$

Here, one has

$$\lim_{|\mathrm{Re}\, x| \to \infty} V(x) = 1, \tag{1.13}$$

and $\delta = +/-$ corresponds to $\mathrm{Im}\, r_1, \ldots, \mathrm{Im}\, r_N$ belonging to $(0, \pi)/(-\pi, 0)$, resp.

Since we *construct* reflectionless wave functions $\mathcal{W}(x, p)$ and AΔOs A related via (1.5) from given data (r, μ) satisfying (2.30), (2.32) and (2.34), it is an obvious question whether distinct data (r, μ) can give rise to the same $\mathcal{W}(x, p)$ and/or A. Section 4 is devoted to a study of this injectivity problem. As it turns out, it is rather easy to answer the question completely for $\mathcal{W}(x, p)$, cf. Theorem 4.1. For A, however, our results are not complete. But the partial answers we obtain in Lemmas 4.2–4.4 yield considerable evidence for our conjecture that the "generalized IST map" from (r, μ) to A is injective up to permutations.

This paper is concluded with four Appendixes. In Appendix A we collect some information on matrices of Cauchy and Vandermonde type that occur in the main text. Appendix B concerns the Casorati determinants associated to the solutions of the ordinary second-order analytic difference equations at hand. Its results are of interest in itself. They are also a crucial input for our study of the injectivity problem in Section 4.

In Appendix C we obtain alternative representations for various important quantities, including the wave function and potentials. In Appendix D we use these formulas to show that the AΔO A (1.1) is *formally* self-adjoint on $L^2(\mathbb{R}, dx)$, provided the numbers r_n, $n = 1, \ldots, N$, are purely imaginary and the functions $i \exp(-r_n) \mu_n(x)$, $n = 1, \ldots, N$, are real-valued for real x.

2 Reflectionless AΔO-eigenfunctions

Consider a function of the form (1.8), with $R_1, \ldots, R_N \in \mathcal{M}^*$ and $z_1, \ldots, z_N$ distinct numbers in $\mathbb{C}^*$, written as

$$z_n \equiv \exp(-r_n), \qquad \mathrm{Im}\, r_n \in (-\pi, \pi], \qquad n = 1, \ldots, N, \tag{2.1}$$

but without further constraints for the moment. Introducing the auxiliary wave function

$$\mathcal{A}(x, p) = \prod_{n=1}^{N} \left(e^p - e^{-r_n} \right) \cdot \mathcal{W}(x, p), \tag{2.2}$$

it follows from (1.8) that $\mathcal{A}(x,p)$ can be rewritten as

$$\mathcal{A}(x,p) = e^{ixp}\left(e^{Np} + \sum_{k=0}^{N-1} c_k(x)e^{kp}\right), \qquad c_k \in \mathcal{M}. \tag{2.3}$$

Here we have in particular

$$c_0(x) = \prod_{n=1}^{N}\left(-e^{-r_n}\right)\cdot\lambda(x), \qquad \lambda(x) \equiv 1 + \sum_{n=1}^{N} e^{r_n}R_n(x), \tag{2.4}$$

$$c_{N-1}(x) = \sum_{n=1}^{N}\left(-e^{-r_n} - R_n(x)\right). \tag{2.5}$$

Let us now ask: When is $\mathcal{A}(x,p)$ an eigenfunction with eigenvalue e^p+e^{-p} of an AΔO A of the form (1.1)? Clearly, this amounts to $\mathcal{A}(x,p)$ satisfying the analytic difference equation (from now on AΔE)

$$\mathcal{A}(x-i,p) + V_a(x)\mathcal{A}(x+i,p) + \left(V_b(x) - e^p - e^{-p}\right)\mathcal{A}(x,p) = 0, \tag{2.6}$$

for certain functions $V_a, V_b \in \mathcal{M}$. In view of the simple structure (2.3) of $\mathcal{A}(x,p)$, this comes down to a system of $N+2$ equations relating the functions $c_0,\ldots,c_{N-1}$ and potentials V_a, V_b. In particular, the vanishing of the coefficients of e^{Np} and e^{-p} is equivalent to the relations

$$c_{N-1}(x-i) + V_b(x) - c_{N-1}(x) = 0, \tag{2.7}$$

$$V_a(x)c_0(x+i) - c_0(x) = 0. \tag{2.8}$$

Therefore, $V_b(x)$ is uniquely determined by $\mathcal{A}(x,p)$, and so is $V_a(x)$, provided $\lambda(x)$ does not vanish identically.

Of course, for $N=0$ this reasoning yields the unsurprising consequence $V_a(x)=1$, $V_b(x)=0$. But already for $N=1$ one gets in addition to (2.7) and (2.8) the relation

$$V_a(x) + V_b(x)c_0(x) - 1 = 0 \qquad (N=1). \tag{2.9}$$

Using (2.7) and (2.8), this amounts to $c_0(x)$ satisfying

$$\frac{c_0(x)}{c_0(x+i)} + [c_0(x) - c_0(x-i)]c_0(x) = 1 \qquad (N=1). \tag{2.10}$$

Thus, even in this simple case the extra constraint looks forbidding.

More generally, when one starts from a function $\mathcal{A}(x,p)$ of the form (2.3) with $c_0(x) \in \mathcal{M}^*$, and defines V_a and V_b by

$$V_a(x) \equiv c_0(x)/c_0(x+i), \tag{2.11}$$

$$V_b(x) \equiv c_{N-1}(x) - c_{N-1}(x-i), \tag{2.12}$$

it seems hopeless to solve the N remaining nonlinear AΔEs for the coefficients $c_0,\ldots,c_{N-1}$.

Even so, we are going to construct a large class of solutions by viewing $\mathcal{A}(x,p)$ as arising from (1.8) via (2.2). We have already seen that these formulas determine the coefficients c_k in (2.3) in terms of complex numbers $\exp(-r_1),\ldots,\exp(-r_N)$ and "residue functions" $R_1,\ldots,R_N$. As it turns out, $R_1,\ldots,R_N$ can be defined via a linear system of N equations such that (2.6) is obeyed, provided V_a is defined by (2.11) and (2.4), and V_b by (2.12) and (2.5).

It is illuminating to present the details first for $N=1$. Using (2.5) (or (2.4)) to express c_0 in terms of the residue function $R\equiv R_1$ at the pole $p=-r\equiv -r_1$ of $\mathcal{W}(x,p)$, we begin by noting

$$\mathcal{A}(x,-r)=-e^{-irx}R(x), \tag{2.13}$$

$$\mathcal{A}(x,r)=e^{irx}\left(e^{r}-e^{-r}-R(x)\right). \tag{2.14}$$

For the pertinent linear constraint on R we now need to require $e^{2r}\neq 1$, or, equivalently, $r\neq 0,i\pi$. Then it reads

$$\mu(x)e^{-2irx}R(x)+\frac{1}{e^{r}-e^{-r}}R(x)=1, \tag{2.15}$$

where $\mu(x)$ is an arbitrary function in $\mathcal{P}_i$ (1.9). Obviously, (2.15) determines a unique function $R\in\mathcal{M}^*$.

The crux of the constraint (2.15) is that it guarantees that $\mathcal{A}(x,p)$ fulfils

$$\mathcal{A}(x,r)=\left(e^{-r}-e^{r}\right)\mu(x)\mathcal{A}(x,-r),\qquad \mu\in\mathcal{P}_i, \tag{2.16}$$

as is clear from (2.13), (2.14). Next, we observe that a function $\mathcal{E}(x,p)$ of the form

$$\mathcal{E}(x,p)=e^{ixp}\left(e^{p}+c(x)\right), \tag{2.17}$$

is uniquely determined when it satisfies

$$\mathcal{E}(x,r)=\left(e^{-r}-e^{r}\right)\mu(x)\mathcal{E}(x,-r),\qquad \mu\in\mathcal{P}_i. \tag{2.18}$$

Indeed, substituting (2.17) in (2.18), one gets a linear constraint that uniquely determines $c\in\mathcal{M}^*$.

We are now going to exploit the uniqueness of $\mathcal{E}$. With $R(x)$ determined by (2.15), we define V_a and V_b via (2.11), (2.12) and (2.5). Consider the function $\mathcal{D}(x,p)$ on the lhs of (2.6). By construction, it is of the form

$$\mathcal{D}(x,p)=e^{ixp}d(x),\qquad d\in\mathcal{M}. \tag{2.19}$$

Moreover, on account of (2.16) and the i-periodicity of $\mu(x)$, it obeys

$$\mathcal{D}(x,r)=\left(e^{-r}-e^{r}\right)\mu(x)\mathcal{D}(x,-r). \tag{2.20}$$

But then the function $\mathcal{E}=\mathcal{A}-\mathcal{D}$ has the two features (2.17), (2.18) that uniquely determine $\mathcal{A}$. Therefore, $d(x)$ must vanish identically, so that $\mathcal{A}(x,p)$ solves the AΔE (2.6). (In particular, this entails that $c_0(x)=-\exp(-r)-R(x)$ solves the nonlinear AΔE (2.10).)

The upshot is that the wave function

$$\mathcal{W}(x,p)=e^{ixp}\left(1-\frac{R(x)}{e^{p}-e^{-r}}\right), \tag{2.21}$$

with $\exp(2r) \neq 1$ and $R \in \mathcal{M}^*$ given by (2.15), satisfies the eigenvalue equation (1.5), with V_a and V_b given by

$$V_a(x) = \frac{e^{-r} + R(x)}{e^{-r} + R(x+i)}, \tag{2.22}$$

$$V_b(x) = R(x-i) - R(x). \tag{2.23}$$

But we need further constraints to guarantee the asymptotics (1.3), (1.6) and (1.7).

Clearly, (1.6) and (1.7) are equivalent to

$$\lim_{\operatorname{Re} x \to \infty} R(x) = 0, \tag{2.24}$$

$$\lim_{\operatorname{Re} x \to -\infty} R(x) = \kappa, \qquad \kappa \in \mathbb{C}. \tag{2.25}$$

Moreover, (2.24) and (2.25) imply (1.3), provided $\kappa \neq -\exp(-r)$. But (1.3) can be fulfilled without (2.24) holding true. For example, choosing $\mu(x)$ equal to $1/\cosh(4\pi x)$, one gets $R(x) \to \exp(r) - \exp(-r) \neq 0$ as $|\operatorname{Re} x| \to \infty$, cf. (2.15).

In the example just given the function $\mu(x)\exp(-2irx)$ has limit 0 for $|\operatorname{Re} x| \to \infty$. But we can also let it diverge for $|\operatorname{Re} x| \to \infty$, by choosing for instance $\mu(x) = \cosh(4\pi x)$. Then we obtain $R(x) \to 0$ as $|\operatorname{Re} x| \to \infty$, so that (1.3), (1.6) and (1.7) are satisfied, with $a(p) = 1$ and $b(p) = 0$. As a result, already for $N = 1$ we obtain a huge class of AΔOs of the form (1.1)–(1.3) admitting eigenfunctions $\mathcal{W}(x,p)$ of the form (1.8) with *trivial* scattering.

Consider next the case

$$\mu(x) \in \mathcal{P}_i(c), \qquad c \neq 0, \qquad \operatorname{Im} r \in (0, \pi). \tag{2.26}$$

Then we read off from (2.15) that

$$\lim_{\operatorname{Re} x \to \infty} R(x) = 0, \qquad \lim_{\operatorname{Re} x \to -\infty} R(x) = e^r - e^{-r}. \tag{2.27}$$

Therefore, (1.3), (1.6) and (1.7) are once again satisfied, now with $b(p) = 0$ and

$$a(p) = \frac{e^p - e^r}{e^p - e^{-r}}. \tag{2.28}$$

There is yet another way to obtain (2.27) and hence (1.3), (1.6), (1.7), with $b(p) = 0$ and $a(p)$ (2.28). Indeed, we may choose

$$e^{-2\pi x}\mu(x) \in \mathcal{P}_i(c), \qquad c \neq 0, \qquad \operatorname{Im} r \in (-\pi, 0). \tag{2.29}$$

Observe that for real p the functions $a(p)$ thus obtained are the complex conjugates of the functions $a(p)$ obtained from the choice (2.26).

We are singling out the cases (2.26) and (2.29), since our requirements for *arbitrary* N reduce to (2.26) and (2.29) for $N = 1$. Indeed, turning to the general N case, our conditions on the numbers $r_1, \ldots, r_N$ read as follows. First, one has

$$\operatorname{Im} r_n \in (-\pi, 0) \cup (0, \pi), \qquad n = 1, \ldots, N. \tag{2.30}$$

Thus we have N_+ complex numbers in the strip $\{\mathrm{Im}\, r \in (0,\pi)\}$ and N_- in $\{\mathrm{Im}\, r \in (-\pi,0)\}$, with

$$N_+ \in \{0,1,\ldots,N\}, \qquad N_- = N - N_+. \tag{2.31}$$

Second, we require

$$e^{r_m} \neq e^{\pm r_n}, \qquad 1 \leq m < n \leq N. \tag{2.32}$$

The conditions (2.30) and (2.32) ensure that the Cauchy matrix

$$C(r)_{mn} \equiv \frac{1}{e^{r_m} - e^{-r_n}}, \qquad m,n = 1,\ldots,N, \tag{2.33}$$

is well defined and regular, cf. Appendix A.

Next, we choose multipliers $\mu_1(x),\ldots,\mu_N(x)$, satisfying

$$\mu_n \in \mathcal{P}_i(c_n), \qquad c_n \in \mathbb{C}^*, \qquad n = 1,\ldots,N. \tag{2.34}$$

Then we introduce a diagonal matrix

$$D(r,\mu;x) \equiv \mathrm{diag}\,(d(r_1,\mu_1;x),\ldots,d(r_N,\mu_N;x)), \tag{2.35}$$

where the function d is defined by

$$d(\rho,\nu;x) \equiv \begin{cases} \nu(x)e^{-2i\rho x}, & \mathrm{Im}\,\rho \in (0,\pi), \\ \nu(x)e^{-2i(\rho+i\pi)x}, & \mathrm{Im}\,\rho \in (-\pi,0), \end{cases} \qquad \nu \in \mathcal{P}_i(c), \qquad c \in \mathbb{C}^*. \tag{2.36}$$

We are now prepared to study the linear system of N equations

$$(D(r,\mu;x) + C(r))R(x) = \zeta, \qquad \zeta \equiv (1,\ldots,1)^t, \tag{2.37}$$

for N unknown functions $R_1,\ldots,R_N$. Let us begin by noting that the determinant $|D(x)+C|$ cannot vanish identically. Indeed, we have $|C| \neq 0$ and it is clear from (2.36) that $D(x) \to 0$ for $\mathrm{Re}\, x \to -\infty$. (Here we suppressed the (r,μ)-dependence, as we often do in the sequel.)

Next, we denote by $Z_n(x)$ the matrix obtained upon replacing the nth column of $D(x)+C$ by ζ. Then we deduce that the system (2.37) admits a unique solution, given by Cramer's rule:

$$R_n(r,\mu;x) = |Z_n(r,\mu;x)|/|D(r,\mu;x) + C(r)|, \qquad n = 1,\ldots,N. \tag{2.38}$$

In the following lemma we collect some salient features of $R(x)$.

Lemma 2.1. *The solution $R(r,\mu;x)$ (2.38) to the system (2.37) belongs to $\mathcal{M}^{*N}$. It satisfies*

$$\lim_{\mathrm{Re}\, x\to\infty} R(r,\mu;x) = 0, \tag{2.39}$$

$$\lim_{\mathrm{Re}\, x\to-\infty} R(r,\mu;x) = C(r)^{-1}\zeta, \tag{2.40}$$

and

$$\lim_{\mathrm{Re}\, x\to\infty} c_n e^{-2ir_n x} R_n(r,\mu;x) = 1, \qquad \mathrm{Im}\, r_n \in (0,\pi), \tag{2.41}$$

$$\lim_{\mathrm{Re}\, x\to\infty} c_n e^{-2i(r_n+i\pi)x} R_n(r,\mu;x) = 1, \qquad \mathrm{Im}\, r_n \in (-\pi,0). \tag{2.42}$$

Proof. Obviously, (2.36) yields

$$\lim_{\mathrm{Re}\, x\to-\infty} d(\rho,\nu;x) = 0, \tag{2.43}$$

$$\lim_{\mathrm{Re}\, x\to\infty} c^{-1}e^{2i\rho x}d(\rho,\nu;x) = 1, \qquad \mathrm{Im}\,\rho \in (0,\pi), \tag{2.44}$$

$$\lim_{\mathrm{Re}\, x\to\infty} c^{-1}e^{2i(\rho+i\pi)x}d(\rho,\nu;x) = 1, \qquad \mathrm{Im}\,\rho \in (-\pi,0). \tag{2.45}$$

From (2.43) we deduce (2.40). In view of the identities (A.5), we have $\left(C(r)^{-1}\zeta\right)_n \neq 0$, so that $R_n \in \mathcal{M}^*$. To prove (2.41), (2.42), we rewrite (2.38) as

$$R_n(x) = \left|Z_n(x)D(x)^{-1}\right| / \left|\mathbf{1}_N + CD(x)^{-1}\right|, \qquad n = 1,\ldots,N. \tag{2.46}$$

When $\mathrm{Im}\,r_n \in (0,\pi)$, we now multiply the nth column of $Z_n(x)D(x)^{-1}$ by $c_n\exp(-2ir_nx)$ and use (2.44) to obtain (2.41). Likewise, (2.42) follows from (2.45). Finally, (2.39) is evident from (2.41) and (2.42). ∎

Next, we define the functions

$$\lambda(r,\mu;x) \equiv 1 + \sum_{n=1}^{N} e^{r_n}R_n(r,\mu;x), \tag{2.47}$$

$$a(r;p) \equiv \prod_{n=1}^{N} \frac{e^p - e^{r_n}}{e^p - e^{-r_n}}. \tag{2.48}$$

Their pertinent properties are once more collected in a lemma.

Lemma 2.2. *The function $a(p)$ (2.48) can be rewritten as*

$$1 - \sum_{n=1}^{N} \frac{\left(C(r)^{-1}\zeta\right)_n}{e^p - e^{-r_n}}. \tag{2.49}$$

The function $\lambda(x)$ (2.47) satisfies

$$\lim_{\mathrm{Re}\, x\to\infty} \lambda(r,\mu;x) = 1, \tag{2.50}$$

$$\lim_{\mathrm{Re}\, x\to-\infty} \lambda(r,\mu;x) = \exp\left(2\sum_{n=1}^{N} r_n\right). \tag{2.51}$$

Proof. The functions (2.49) and (2.48) are $2i\pi$-periodic meromorphic functions of p with finite limits for $\mathrm{Re}\,p \to -\infty$ and limit 1 for $\mathrm{Re}\,p \to \infty$. Therefore, one need only verify equality of residues at the simple poles $p = -r_m$, $m = 1,\ldots,N$, to conclude (by Liouville's theorem) that they coincide. This can be done via the identities (A.5).

Next, we note that (2.39) entails (2.50), whereas (2.40) yields

$$\lim_{\mathrm{Re}\, x\to-\infty} \lambda(r,\mu;x) = 1 + \sum_{n=1}^{N} e^{r_n}\left(C(r)^{-1}\zeta\right)_n. \tag{2.52}$$

Taking $\mathrm{Re}\, p \to -\infty$ in (2.49) and (2.48), we obtain

$$1 + \sum_{n=1}^{N} e^{r_n} \left(C(r)^{-1}\zeta \right)_n = \prod_{n=1}^{N} e^{2r_n}. \tag{2.53}$$

Hence (2.51) results. ■

We now use these lemmas to study the potentials and AΔO

$$V_a(r, \mu; x) \equiv \lambda(r, \mu; x)/\lambda(r, \mu; x + i), \tag{2.54}$$

$$V_b(r, \mu; x) \equiv \sum_{n=1}^{N} (R_n(r, \mu; x - i) - R_n(r, \mu; x)), \tag{2.55}$$

$$A(r, \mu) \equiv T_i + V_a(r, \mu; x)T_{-i} + V_b(r, \mu; x), \tag{2.56}$$

in relation to the wave function

$$\mathcal{W}(r, \mu; x, p) \equiv e^{ixp} \left(1 - \sum_{n=1}^{N} \frac{R_n(r, \mu; x)}{e^p - e^{-r_n}} \right). \tag{2.57}$$

Theorem 2.3. *One has the limits*

$$\lim_{|\mathrm{Re}\, x| \to \infty} V_a(r, \mu; x) = 1, \qquad \lim_{|\mathrm{Re}\, x| \to \infty} V_b(r, \mu; x) = 0, \tag{2.58}$$

$$\lim_{\mathrm{Re}\, x \to \infty} e^{-ixp} \mathcal{W}(r, \mu; x, p) = 1, \tag{2.59}$$

$$\lim_{\mathrm{Re}\, x \to -\infty} e^{-ixp} \mathcal{W}(r, \mu; x, p) = a(r; p), \tag{2.60}$$

where $a(r; p)$ is given by (2.48). Furthermore, the wave function (2.57) satisfies the eigenvalue equation

$$A(r, \mu)\mathcal{W}(r, \mu; x, p) = \left(e^p + e^{-p} \right) \mathcal{W}(r, \mu; x, p), \tag{2.61}$$

for all p with $\exp(p) \neq \exp(-r_1), \ldots, \exp(-r_N)$.

Proof. From (2.39) and (2.50) it is obvious that (2.58) holds for $\mathrm{Re}\, x \to \infty$. Likewise, (2.40) yields $V_b \to 0$ for $\mathrm{Re}\, x \to -\infty$. The rhs of (2.51) is non-zero, entailing $V_a \to 1$ for $\mathrm{Re}\, x \to -\infty$. Thus (2.58) is proved.

The limit (2.59) is immediate from (2.39). From (2.40) we obtain

$$\lim_{\mathrm{Re}\, x \to -\infty} e^{-ixp} \mathcal{W}(r, \mu; x, p) = 1 - \sum_{n=1}^{N} \frac{\left(C(r)^{-1}\zeta \right)_n}{e^p - e^{-r_n}}, \tag{2.62}$$

which equals $a(r; p)$, by virtue of Lemma 2.2. Thus it remains to prove the eigenvalue equation (2.61).

Clearly, (2.61) will follow once we show that the auxiliary wave function $\mathcal{A}(x, p)$ (2.2) satisfies the AΔE (2.6). To this end we calculate

$$\mathcal{A}(x, -r_m) = -e^{-ir_m x} \prod_{n \neq m} \left(e^{-r_m} - e^{-r_n} \right) \cdot R_m(x), \tag{2.63}$$

$$\mathcal{A}(x,r_m)=e^{ir_mx}\prod_{k=1}^{N}\left(e^{r_m}-e^{-r_k}\right)\cdot\left(1-\sum_{n=1}^{N}\frac{R_n(x)}{e^{r_m}-e^{-r_n}}\right), \tag{2.64}$$

and use (2.37) to deduce

$$\mathcal{A}(x,r_m)=\left(e^{-r_m}-e^{r_m}\right)\prod_{n=1,\,n\neq m}^{N}\frac{(e^{r_m}-e^{-r_n})}{(e^{-r_m}-e^{-r_n})}\cdot e^{2ir_mx}d(r_m,\mu_m;x)\mathcal{A}(x,-r_m). \tag{2.65}$$

Recalling the definition (2.36) of d, we see that $\mathcal{A}(x,p)$ satisfies N equations of the form

$$\mathcal{A}(x,r_m)=\alpha_m(x)\mathcal{A}(x,-r_m),\qquad m=1,\ldots,N, \tag{2.66}$$

$$\operatorname{Im} r_m\in\left\{\begin{array}{l}(0,\pi)\\(-\pi,0)\end{array}\right.\quad\Rightarrow\quad\left\{\begin{array}{ll}\alpha_m(x)\in\mathcal{P}_i(a_m), & a_m\in\mathbb{C}^*,\\ e^{-2\pi x}\alpha_m(x)\in\mathcal{P}_i(a_m), & a_m\in\mathbb{C}^*.\end{array}\right. \tag{2.67}$$

Just as for $N=1$, it is therefore enough to show that a function $\mathcal{A}(x,p)$ given by (2.3) and satisfying N equations of this type is uniquely determined. (Indeed, the function $\mathcal{D}(x,p)$ on the lhs of (2.6) is by construction of the form

$$\mathcal{D}(x,p)=e^{ixp}\sum_{k=0}^{N-1}d_k(x)e^{kp},\qquad d_k\in\mathcal{M}, \tag{2.68}$$

and satisfies the same N equations. Thus $\mathcal{A}-\mathcal{D}$ must equal $\mathcal{A}$, by virtue of uniqueness.)

In order to prove uniqueness, we combine (2.3) and (2.66), obtaining

$$\begin{aligned}&e^{2ir_mx}\left(e^{Nr_m}+\sum_{k=0}^{N-1}c_k(x)e^{kr_m}\right)\\&\qquad=\alpha_m(x)\left(e^{-Nr_m}+\sum_{k=0}^{N-1}c_k(x)e^{-kr_m}\right),\qquad m=1,\ldots,N.\end{aligned} \tag{2.69}$$

These equations can be rewritten as

$$\sum_{k=0}^{N-1}M_{mk}(x)c_k(x)=f_m(x),\qquad m=1,\ldots,N, \tag{2.70}$$

where

$$M_{mk}(x)\equiv\left\{\begin{array}{ll}e^{2ir_mx}e^{kr_m}-\alpha_m(x)e^{-kr_m}, & \operatorname{Im} r_m\in(0,\pi),\\ e^{2i(r_m+i\pi)x}e^{kr_m}-e^{-2\pi x}\alpha_m(x)e^{-kr_m}, & \operatorname{Im} r_m\in(-\pi,0),\end{array}\right. \tag{2.71}$$

and where the functions $f_m(x)\in\mathcal{M}$ need not be specified. This system of equations has a unique solution $c(x)\in\mathcal{M}^N$ iff $|M(x)|\in\mathcal{M}^*$. Finally, to verify that $|M(x)|$ cannot vanish identically, one need only take $\operatorname{Re} x\to\infty$ to obtain (cf. (2.67))

$$M_{mk}(x)\to -a_me^{-kr_m},\qquad a_m\in\mathbb{C}^*,\quad m=1,\ldots,N,\quad k=0,\ldots,N-1. \tag{2.72}$$

Since $\exp(-r_m)\neq\exp(-r_n)$ for $m\neq n$, the limit matrix is regular, cf. (A.11). Hence we must have $|M(x)|\in\mathcal{M}^*$. ■

From the proof just given it is readily seen that we could have allowed more general multipliers, just as we have explicitly shown for $N=1$. But the class of AΔOs admitting reflectionless eigenfunctions to which we have restricted attention permits a uniform treatment and contains all of the AΔOs of interest in later sections and in Parts II and III.

3 Special cases: $V_b = 0$ vs $N_\delta = 0$

In this section we are mainly concerned with features of the potential $V_b(r,\mu;x)$ (2.55) and of the AΔO $A(r,\mu)$ (2.56) for the special cases $N_- = 0$ and $N_+ = 0$, cf. (2.31). Indeed, as will transpire, these two topics are closely related.

Our first result is of a general nature: It asserts that V_b cannot vanish identically for generic r.

Theorem 3.1. *Suppose $r = (r_1, \ldots, r_N)$ satisfies*

$$e^{r_m} \neq -e^{r_n}, \qquad 1 \leq m < n \leq N. \tag{3.1}$$

Then one has

$$V_b(r,\mu;x) \in \mathcal{M}^*. \tag{3.2}$$

Proof. Let us assume $V_b(x)$ vanishes identically. Then (2.55) entails

$$\mathrm{Col}\,(R(x)-R(x-i), R(x-i)-R(x-2i), \ldots, R(x-(N-1)i)-R(x-Ni))^t\zeta = 0, \tag{3.3}$$

where $\mathrm{Col}\,(\gamma_1,\ldots,\gamma_N)$ denotes the matrix with columns $\gamma_1,\ldots,\gamma_N \in \mathbb{C}^N$. Therefore the determinant of

$$\mathrm{Col}\,(R(x)-R(x-i), R(x)-R(x-2i), \ldots, R(x)-R(x-Ni)) \tag{3.4}$$

vanishes. This remains true when we multiply the nth row by $c_n \exp(-2ir_n x)$ for $\mathrm{Im}\, r_n \in (0,\pi)$ and by $c_n \exp(-2i(r_n + i\pi)x)$ for $\mathrm{Im}\, r_n \in (-\pi, 0)$. Due to (2.41) and (2.42), the resulting matrix has $\mathrm{Re}\, x \to \infty$ limit

$$\mathrm{Row}\,(\eta_1,\ldots,\eta_N), \quad \eta_n \equiv \left(1-e^{2r_n}, 1-e^{4r_n}, \ldots, 1-e^{2Nr_n}\right), \quad n = 1,\ldots,N. \tag{3.5}$$

This limit matrix is of the form (A.12), so it is regular iff $\exp(2r_m) \neq \exp(2r_n)$ for all pairs $m \neq n$, cf. Lemma A.2. In view of our standing assumption $\exp(r_m) \neq \exp(r_n)$, this amounts to (3.1). Therefore we obtain a contradiction. ■

As a corollary of this general result, one infers that V_b cannot vanish when either $N_- = 0$ or $N_+ = 0$. (Indeed, in the first/second case all of the numbers $\exp(r_1),\ldots,\exp(r_N)$ belong to the upper/lower half plane, so (3.1) holds true.) The following result is of a general nature as well, but as a corollary it yields a special case in which V_b does vanish. As a preparation we define an involution on the space of (r,μ) by

$$\mathcal{C} \,:\, (r,\mu) \mapsto (r^*, \mu^*), \tag{3.6}$$

$$r_n^* \equiv \begin{cases} r_n - i\pi, & \mathrm{Im}\, r_n \in (0,\pi), \\ r_n + i\pi, & \mathrm{Im}\, r_n \in (-\pi,0), \end{cases} \qquad \mu_n^* \equiv -\mu_n, \qquad n = 1,\ldots,N. \tag{3.7}$$

Theorem 3.2. *One has*

$$R(r^*,\mu^*;x) = -R(r,\mu;x), \tag{3.8}$$

$$\lambda(r^*,\mu^*;x) = \lambda(r,\mu;x), \tag{3.9}$$

$$V_a(r^*,\mu^*;x) = V_a(r,\mu;x), \tag{3.10}$$

$$V_b(r^*,\mu^*;x) = -V_b(r,\mu;x), \tag{3.11}$$

$$\mathcal{W}(r^*,\mu^*;x,p) = e^{\pi x}\mathcal{W}(r,\mu;x,p+i\pi). \tag{3.12}$$

Proof. From (2.35) and (2.36) we deduce

$$D(r^*, \mu^*; x) = -D(r, \mu; x). \tag{3.13}$$

Likewise, (2.33) yields

$$C(r^*) = -C(r). \tag{3.14}$$

Therefore, (3.8) follows from (2.37). Using (3.8) it is straightforward to check the remaining relations, cf. (2.47), (2.54), (2.55) and (2.57). ■

We proceed by observing that V_a, V_b and $\mathcal{W}$ are invariant under arbitrary permutations

$$\pi \,:\, (r, \mu) \mapsto (\pi(r), \pi(\mu)), \qquad \pi \in S_N. \tag{3.15}$$

In view of (3.11), we therefore have the implication

$$(r^*, \mu^*) = (\pi(r), \pi(\mu)), \qquad \pi \in S_N \Rightarrow V_b(r, \mu; x) = 0. \tag{3.16}$$

We conjecture that this sufficient condition is also necessary:

$$V_b(r, \mu; x) = 0 \Rightarrow (r^*, \mu^*) = (\pi(r), \pi(\mu)), \qquad \pi \in S_N. \qquad (?) \tag{3.17}$$

From substantial later developments (cf. (4.6) below), we will be able to conclude that the sufficient condition (3.1) for $V_b \neq 0$ can be improved to $r^* \neq \pi(r), \forall\, \pi \in S_N$, but a proof of our expectation (3.17) has not materialized thus far.

In view of permutation invariance, we are free to choose a convenient ordering for the special case just considered. Since the involution $\mathcal{C}$ (3.6) maps the set $\{\exp(r_1), \ldots, \exp(r_N)\}$ to $\{-\exp(r_1), \ldots, -\exp(r_N)\}$, this case can only arise when $N = 2M$, $M \in \mathbb{N}$, and $N_+ = N_- = M$. Choosing $r_1, \ldots, r_M$ in the strip $\{\mathrm{Im}\, r \in (0, \pi)\}$, we may choose r_{M+n} equal to $r_n - i\pi$. Setting

$$\pi_M \equiv \begin{pmatrix} 0 & \mathbf{1}_M \\ \mathbf{1}_M & 0 \end{pmatrix}, \tag{3.18}$$

we then have

$$r^* = \pi_M r, \qquad \mu^* = \pi_M \mu, \tag{3.19}$$

$$V_b((r_1, \ldots, r_M, r_1 - i\pi, \ldots, r_M - i\pi, \mu_1, \ldots, \mu_M, -\mu_1, \ldots, -\mu_M; x) = 0, \tag{3.20}$$

where $\mathrm{Im}\, r_n \in (0, \pi)$ for $n = 1, \ldots, M$.

We proceed by exploiting the above insights to obtain some striking properties of the special cases $N_\delta = 0$, $\delta = +, -$. First, we introduce the “square root” AΔOs

$$S_\pm(r, \mu) \equiv T_{i/2} \pm V(r, \mu; x) T_{-i/2}, \tag{3.21}$$

where

$$V(r, \mu; x) \equiv \lambda(r, \mu; x)/\lambda(r, \mu; x + i/2). \tag{3.22}$$

From (2.54) we then deduce

$$S_\delta(r, \mu)^2 = T_i + V_a(r, \mu; x) T_{-i} + \delta V_s(r, \mu; x), \qquad \delta = +, -, \tag{3.23}$$

where we have set

$$V_s(r, \mu; x) \equiv V(r, \mu; x - i/2) + V(r, \mu; x). \tag{3.24}$$

It is understood here that r is arbitrary. But only when $N_\delta = 0$ we can show that $S_{-\delta}(r, \mu)$ has a special significance, as detailed in our next theorem.

Theorem 3.3. *Fix N distinct numbers $r_1, \ldots, r_N$ in the strip $\{\mathrm{Im}\, r \in (0, \pi)\}$ and multipliers $\mu_1(x), \ldots, \mu_N(x)$ satisfying (2.34), and set*

$$r^+ \equiv (r_1, \ldots, r_N), \qquad r^- \equiv (r_1 - i\pi, \ldots, r_N - i\pi). \tag{3.25}$$

Then one has the AΔEs

$$S_\delta\left(r^\delta, \mu\right) \mathcal{W}(r^\delta, \mu; x, p) = \left(e^{p/2} + \delta e^{-p/2}\right) \mathcal{W}\left(r^\delta, \mu; x, p\right), \qquad \delta = +, -, \tag{3.26}$$

and identities

$$V_s\left(r^+, \mu; x\right) = V_b\left(r^+, \mu; x\right) + 2, \tag{3.27}$$

$$V\left(r^+, \mu; x\right) = \sum_{n=1}^{N} \left(R_n\left(r^+, \mu; x - i/2\right) - R_n\left(r^+, \mu; x\right)\right) + 1. \tag{3.28}$$

Moreover, introducing

$$r_s \equiv (r_1/2, \ldots, r_N/2, r_1/2 - i\pi, \ldots, r_N/2 - i\pi), \tag{3.29}$$

$$\mu_r(x) \equiv \left(2e^{-r_1/2}\mu_1(x/2), \ldots, 2e^{-r_N/2}\mu_N(x/2)\right), \tag{3.30}$$

one has the relations

$$2e^{-r_n/2} R_n(r_s, \mu_r, -\mu_r; 2x) = R_n\left(r^+, \mu; x\right), \qquad n = 1, \ldots, N, \tag{3.31}$$

$$\mathcal{W}(r_s, \mu_r, -\mu_r; 2x, p/2) = \mathcal{W}\left(r^+, \mu; x, p\right), \tag{3.32}$$

$$\lambda(r_s, \mu_r, -\mu_r; 2x) = \lambda\left(r^+, \mu; x\right), \tag{3.33}$$

$$V_a(r_s, \mu_r, -\mu_r; 2x) = V\left(r^+, \mu; x\right), \tag{3.34}$$

$$V_b(r_s, \mu_r, -\mu_r; 2x) = 0. \tag{3.35}$$

Proof. We begin by proving (3.31). To this end we observe

$$D(r_s, \mu_r, -\mu_r; 2x) = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \qquad E \equiv D\left(r^+/2, \mu_r; 2x\right), \tag{3.36}$$

cf. (2.35), (2.36). Similarly, we have

$$C(r_s) = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \tag{3.37}$$

with

$$A_{mn} \equiv \left(e^{r_m/2} - e^{-r_n/2}\right)^{-1}, \quad B_{mn} \equiv \left(e^{r_m/2} + e^{-r_n/2}\right)^{-1}, \quad m, n = 1, \ldots, N, \tag{3.38}$$

cf. (2.33). Using obvious notation, the system (2.37) with $r \to r_s$, $\mu \to (\mu_r, -\mu_r)$ can therefore be written as

$$\left(\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} + \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\right) \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} \zeta^+ \\ \zeta^- \end{pmatrix}. \tag{3.39}$$

Now when we multiply this by π_N (given by (3.18) with $M \to N$), we obtain

$$\left(\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} + \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\right)\begin{pmatrix} -R^- \\ -R^+ \end{pmatrix} = \begin{pmatrix} \zeta^+ \\ \zeta^- \end{pmatrix}. \tag{3.40}$$

Comparing (3.39) and (3.40), we deduce

$$R^- = -R^+. \tag{3.41}$$

Substituting this in (3.39), the first N equations become

$$(E + A - B)R^+ = \zeta^+. \tag{3.42}$$

Next, we use (3.38) to get

$$A_{mn} - B_{mn} = 2e^{-r_n/2}\left(e^{r_m} - e^{-r_n}\right)^{-1}, \qquad m, n = 1\ldots, N. \tag{3.43}$$

Furthermore, we have from (3.30) and (2.36)

$$d\left(r_n/2, \mu_{r,n}; 2x\right) = 2e^{-r_n/2} d\left(r_n, \mu_n; x\right), \qquad n = 1, \ldots, N. \tag{3.44}$$

Hence (3.42) entails (3.31).

Using (3.41) and (3.31), we now obtain

$$\begin{aligned} \mathcal{W}(r_s, \mu_r, -\mu_r; 2x, p/2) &= e^{ixp}\left(1 - \sum_{n=1}^{N}\left(\frac{R_n(r_s, \mu_r, -\mu_r; 2x)}{e^{p/2} - e^{-r_n/2}} - \frac{R_n(r_s, \mu_r, -\mu_r; 2x)}{e^{p/2} + e^{-r_n/2}}\right)\right) \\ &= e^{ixp}\left(1 - \sum_{n=1}^{N}\left(e^p - e^{-r_n}\right)^{-1} R_n\left(r^+, \mu; x\right)\right) \\ &= \mathcal{W}\left(r^+, \mu; x, p\right), \end{aligned} \tag{3.45}$$

which proves (3.32). Next, we combine (2.47), (3.31) and (3.29) to obtain (3.33). Then (3.34) follows from (2.54), (3.33) and (3.22). Moreover, recalling (3.20), we deduce (3.35). (Alternatively, (3.35) can be inferred directly from (3.41) and (2.55).)

Consider now the AΔE

$$\begin{aligned} &\mathcal{W}(r_s, \mu_r, -\mu_r; 2x - i, p/2) + V_a(r_s, \mu_r, -\mu_r; 2x)\mathcal{W}(r_s, \mu_r, -\mu_r; 2x + i, p/2) \\ &\quad = \left(e^{p/2} + e^{-p/2}\right)\mathcal{W}(r_s, \mu_r, -\mu_r; 2x, p/2), \end{aligned} \tag{3.46}$$

satisfied by the lhs of (3.32) (due to (3.35)). Rewriting it for the rhs of (3.32) and using (3.34), this yields (3.26) with $\delta = +$.

To arrive at (3.26) with $\delta = -$, we invoke Theorem 3.2. Specifically, from (3.9) and (3.22) we deduce

$$V\left(r^-, -\mu; x\right) = V\left(r^+, \mu; x\right), \tag{3.47}$$

and from (3.12) we have

$$\mathcal{W}\left(r^-, -\mu; x, p\right) = e^{\pi x}\mathcal{W}\left(r^+, \mu; x, p + i\pi\right). \tag{3.48}$$

Therefore, the above AΔE (3.26) with $\delta = -$ readily follows from its $\delta = +$ counterpart.

We continue by proving (3.27). On account of (3.23) we have

$$S_+\left(r^+,\mu\right)^2 - A\left(r^+,\mu\right) - 2 = V_s\left(r^+,\mu;x\right) - V_b\left(r^+,\mu;x\right) - 2. \tag{3.49}$$

Since S_+^2 has eigenvalue $(\exp(p/2)+\exp(-p/2))^2$ and A has eigenvalue $\exp(p)+\exp(-p)$ on $\mathcal{W}\left(r^+,\mu;x,p\right)$, the AΔO difference on the lhs annihilates $\mathcal{W}\left(r^+,\mu;x,p\right)$. Hence the rhs vanishes, which amounts to (3.27).

Finally, we prove (3.28). To this end we use the AΔE (3.26) with $\delta = +$ for the auxiliary wave function (2.2). It entails

$$\begin{aligned}(e^p+1)\left[e^{Np}+\sum_{k=0}^{N-1}c_k(x-i/2)e^{kp}\right] + \left(1+e^{-p}\right)V(x)\left[e^{Np}+\sum_{k=0}^{N-1}c_k(x+i/2)e^{kp}\right] \\ -\left(e^{p/2}+e^{-p/2}\right)^2\left[e^{Np}+\sum_{k=0}^{N-1}c_k(x)e^{kp}\right] = 0.\end{aligned} \tag{3.50}$$

The vanishing of the coefficient of $\exp(Np)$ now amounts to (3.28), cf. (2.5). ■

We would like to add that we have not found a direct proof of the main results (3.26)–(3.28) of this theorem. (That is, a proof that avoids the detour via the doubled-up system involving (3.29), (3.30).)

In this connection we point out that (3.26)–(3.28) would also follow from the identity

$$\sum_{n=1}^{N}\left(R_n\left(r^+,\mu;x-i/2\right) - R_n\left(r^+,\mu;x\right)\right) + 1 = \frac{1+\sum_{n=1}^{N} e^{r_n}R_n\left(r^+,\mu;x\right)}{1+\sum_{n=1}^{N} e^{r_n}R_n\left(r^+,\mu;x+i/2\right)}. \tag{3.51}$$

To see this, note first that this identity is equivalent to (3.28), cf. (3.22). Now assume one could prove (3.51) directly. Then one easily obtains (3.27) by using (3.24) and (2.55). Moreover, the AΔE (3.26) with $\delta = +$ follows from the uniqueness argument in the proof of Theorem 2.1, with the lhs of (3.50) playing the role of the lhs of (2.6). Finally, the $\delta = -$ counterpart follows as before from Theorem 3.2, cf. (3.47), (3.48).

4 The injectivity problem

In this section we return to the general setting of Theorem 2.1. Thus we start from complex numbers $r_1,\ldots,r_N$ satisfying (2.30) and (2.32), and multipliers $\mu_1,\ldots,\mu_N$ satisfying (2.34). For such (r,μ) we have defined a wave function $\mathcal{W}(x,p)$ via (2.57) and an AΔO A via (2.56), with A having eigenvalue $\exp(p)+\exp(-p)$ on $\mathcal{W}(x,p)$. As already observed (cf. (3.15)), we obtain the same $\mathcal{W}(x,p)$ and A when we permute r and μ. Let us denote the above space of (r,μ) with N varying over $\mathbb{N}$, divided by the action of the symmetric groups $S_N, N\in\mathbb{N}$, by $\mathcal{D}_{\mathrm{IST}}$. Then we have well-defined maps

$$\Phi_{\mathcal{W}} : \mathcal{D}_{\mathrm{IST}} \to \{\text{wave functions}\}, \qquad (r,\mu)\mapsto \mathcal{W}(r,\mu;x,p), \tag{4.1}$$

$$\Phi_A : \mathcal{D}_{\mathrm{IST}} \to \{\text{A}\Delta\text{Os}\}, \qquad (r,\mu)\mapsto A(r,\mu). \tag{4.2}$$

Within the algebraic framework of this paper, a natural question now arises. It reads: Are the above "generalized IST" maps $\Phi_{\mathcal{W}}$ and Φ_A one-to-one? It is not hard to see that the answer is "yes" for $\Phi_{\mathcal{W}}$, cf. Theorem 4.1. We conjecture that the answer is "yes" for Φ_A as well, and we go a long way towards proving this, cf. Lemmas 4.2–4.4.

Before embarking on the details, let us point out that the injectivity conjecture just made is closely related to our expectation (3.17). Indeed, if (3.17) is false, then there exists (r_0, μ_0) such that $V_b(r_0, \mu_0; x) = 0$, yet (r_0^*, μ_0^*) is not obtained from (r_0, μ_0) via a permutation. Now from (3.10) and (3.11) we have $A(r_0, \mu_0) = A(r_0^*, \mu_0^*)$, so Φ_A would not be 1-1. As a consequence, our injectivity conjecture is stronger than (3.17).

Theorem 4.1. *The map $\Phi_{\mathcal{W}}$ (4.1) is an injection.*

Proof. Assuming $\mathcal{W}(r_0, \mu_0; x, p)$ equals $\mathcal{W}(r, \mu; x, p)$ for some (r, μ), we compare the poles of these wave functions (given by (2.57)) in the strips $\operatorname{Im} p \in (0, \pi)$ and $\operatorname{Im} p \in (-\pi, 0)$ to deduce that $N = N_0$ and that r is related to r_0 by a permutation. Reordering, we may as well assume $r = r_0$. Then the residue vectors $R(r_0, \mu_0; x)$ and $R(r_0, \mu; x)$ coincide. Using (2.37), we now obtain

$$[D(r_0, \mu_0; x) - D(r_0, \mu; x)]R(x) = 0. \tag{4.3}$$

Since $R \in \mathcal{M}^{*N}$, this yields $\mu = \mu_0$, cf. (2.35), (2.36). ■

We now turn to Φ_A (4.2). We begin by pointing out that it is already quite unclear whether for a given $\mathcal{W}(x, p) \equiv \mathcal{W}(r_0, \mu_0; x, p)$ there might not be infinitely many $\tilde{\mathcal{W}}(x, p)$ of the form $\mathcal{W}(r, \mu; x, p)$ that are related to $\mathcal{W}(x, p)$ via (1.11). Indeed, whenever (r, μ) has this property, (1.11) implies $A(r, \mu) = A(r_0, \mu_0)$. (To check the asserted implication, note that i-periodicity of $\nu_1(x), \ldots, \nu_M(x)$ entails $\tilde{\mathcal{W}}(x, p)$ is an $A(r_0, \mu_0)$-eigenfunction with eigenvalue $\exp(p) + \exp(-p)$, and recall the conclusion below (2.8).) In the next lemma we exclude in particular such ambiguities.

Lemma 4.2. *Suppose that for a given (r_0, μ_0) there exists (r, μ) such that*

$$A(r, \mu) = A(r_0, \mu_0). \tag{4.4}$$

Then one has $N = N_0$ and r equals $\pi(r_0)$ with π a permutation. Moreover, for all $p \in \mathbb{C}$ the auxiliary wave functions are related by

$$\mathcal{A}(r, \mu; x, p) = \nu_+(x, p)\mathcal{A}(r_0, \mu_0; x, p), \qquad \nu_+(\cdot, p) \in \mathcal{P}_i. \tag{4.5}$$

Finally, for all $(r, \mu) \in \mathcal{D}_{\mathrm{IST}}$ one has the implication

$$r^* \neq \pi(r), \qquad \forall\, \pi \in S_N \Rightarrow V_b(r, \mu; x) \neq 0. \tag{4.6}$$

Proof. Our reasoning makes extensive use of Casorati determinants. In Appendix B we have summarized the pertinent general features. Moreover, we have explicitly determined some relevant Casorati determinants. As a consequence, we deduce that the auxiliary wave functions $\mathcal{A}(r_0, \mu_0; x, \pm p)$ yield a basis (over $\mathcal{P}_i$) for the space of all meromorphic solutions to the A∆E

$$F(x - i) + V_a(r_0, \mu_0; x)F(x + i) + \left[V_b(r_0, \mu_0; x) - e^p - e^{-p}\right] F(x) = 0, \tag{4.7}$$

provided

$$\exp(p) \neq \pm 1, \exp(\pm r_{0,1}), \ldots, \exp(\pm r_{0,N_0}). \tag{4.8}$$

Now the assumption (4.4) entails that the functions $\mathcal{A}(r, \mu; x, \pm p)$ solve (4.7) as well. With (4.8) in force, we then have

$$\mathcal{A}(r, \mu; x, p) = \nu_+(x, p)\mathcal{A}(r_0, \mu_0; x, p) + \nu_-(x, p)\mathcal{A}(r_0, \mu_0; x, -p). \tag{4.9}$$

Here, the i-periodic multipliers are quotients of Casorati determinants, cf. (B.4), (B.5). We claim that $\nu_-(x, p)$ vanishes.

To prove this claim, we need only show that

$$D(x, p) \equiv \mathcal{D}(\mathcal{A}(r, \mu; x, p), \mathcal{A}(r_0, \mu_0; x, p)) \tag{4.10}$$

is zero. To this end we recall $D(x, p)$ satisfies the AΔE

$$D(x, p) = V_a(x)D(x + i, p), \tag{4.11}$$

cf. (B.3). Now $D(x, p)$ is of the form

$$D(x, p) = e^{2ixp}e^p \sum_{l=0}^{N_0+N-1} e^{lp}d_l(x), \tag{4.12}$$

cf. (B.2) and (2.3). Therefore, (4.11) yields

$$\sum_{l=0}^{N_0+N-1} e^{(l+1)p}d_l(x) = \sum_{l=0}^{N_0+N-1} e^{(l-1)p}d_l(x + i)V_a(x). \tag{4.13}$$

Thus we obtain recursively $d_l(x) = 0, l = N_0 + N - 1, \ldots, 0$, which proves our claim.

As a consequence, (4.5) holds true for all p satisfying (4.8). Since the auxiliary wave functions are entire in p and belong to $\mathcal{M}^*$ for all $p \in \mathbb{C}$, (4.5) holds for arbitrary p.

Now (4.5) yields in particular

$$\mathcal{A}(r, \mu; x, \pm r_n) = \nu_+(x, \pm r_n)\mathcal{A}(r_0, \mu_0; x, \pm r_n), \tag{4.14}$$

and we also have

$$\mathcal{A}(r, \mu; x, r_n) = \alpha_n(x)\mathcal{A}(r, \mu; x, -r_n), \tag{4.15}$$

with α_n i-periodic, cf. (2.66), (2.67). Hence, we get

$$\mathcal{A}(r_0, \mu_0; x, r_n) = \alpha_n(x)\frac{\nu_+(x, -r_n)}{\nu_+(x, r_n)}\mathcal{A}(r_0, \mu_0; x, -r_n), \tag{4.16}$$

so $\mathcal{A}(r_0, \mu_0; x, r_n)/\mathcal{A}(r_0, \mu_0; x, -r_n)$ is i-periodic. But then the Casorati determinant of $\mathcal{A}(r_0, \mu_0; x, r_n)$ and $\mathcal{A}(r_0, \mu_0; x, -r_n)$ vanishes. Since $\exp(r_n) \neq \pm 1$, this implies that r_n or $-r_n$ belongs to $\{r_{0,1}, \ldots, r_{0,N_0}\}$, on account of (B.13). Likewise, we deduce that $r_{0,k}$ or $-r_{0,k}$ belongs to $\{r_1, \ldots, r_N\}$. Clearly, this entails $N = N_0$ and the existence of a permutation $\pi \in S_N$ such that

$$r_n = s_n\pi(r_0)_n, \qquad s_n \in \{-1, 1\}, \qquad n = 1, \ldots, N. \tag{4.17}$$

Next, we show that all of the signs s_n are positive. Indeed, let us assume $s_k = -1$, so as to derive a contradiction. Reordering, we may and will choose $\pi = \mathrm{id}$ in (4.17). Now from (2.37) we deduce

$$\mathcal{W}(r, \mu; r_k) = e^{ir_k x} d(r_k, \mu_k; x) R_k(r, \mu; x). \tag{4.18}$$

Likewise, our assumption entails

$$\mathcal{W}(r_0, \mu_0; -r_k) = e^{-ir_k x} d(-r_k, \mu_{0,k}; x) R_k(r_0, \mu_0; x). \tag{4.19}$$

Recalling the definition (2.2) of the auxiliary wave function and the relations (2.66), it readily follows from (4.5) that the quotient of the functions (4.18) and (4.19) is i-periodic. Using the definition (2.36) of d, we then deduce

$$R_k(r, \mu; x)/R_k(r_0, \mu_0; x) = e^{2ir_k x} \nu(x), \qquad \nu \in \mathcal{P}_i. \tag{4.20}$$

From this equation we now obtain the desired contradiction. Indeed, the $\mathrm{Re}\, x \to -\infty$ limit of the quotient on the lhs is a non-zero constant q, due to (2.40). Hence we obtain

$$\lim_{\mathrm{Re}\, x \to -\infty} e^{2ir_k x} \nu(x) = q \neq 0. \tag{4.21}$$

But since $\nu(x)$ is i-periodic, we also have

$$q = \lim_{\mathrm{Re}\, x \to -\infty} e^{2ir_k (x-i)} \nu(x - i) = e^{2r_k} q. \tag{4.22}$$

As $\exp(2r_k) \neq 1$, this entails $q = 0$, a contradiction.

The implication (4.6) is now a simple corollary: If one has $V_b = 0$, yet $r^* \neq \pi(r)$ for all $\pi \in S_N$, then one gets $A(r, \mu) = A(r^*, \mu^*)$ from (3.10) and (3.11), which contradicts the first assertion of the lemma. ■

The remaining problem is to show that the assumption

$$A(r, \mu) = A(r, \hat{\mu}) \tag{4.23}$$

entails $\hat{\mu} = \mu$. We believe this is true, but have no complete proof. From (4.23) and (4.5) it is obvious that the two pertinent wave functions

$$\mathcal{W}(r, \mu; x, p) \equiv e^{ixp} \left(1 - \sum_{n=1}^{N} \frac{R_n(x)}{e^p - e^{-r_n}} \right), \tag{4.24}$$

$$\mathcal{W}(r, \hat{\mu}; x, p) \equiv e^{ixp} \left(1 - \sum_{n=1}^{N} \frac{\hat{R}_n(x)}{e^p - e^{-r_n}} \right), \tag{4.25}$$

are related by an i-periodic multiplier. This is equivalent to the identity

$$\begin{aligned} &\left(1 - \sum_{m=1}^{N} \frac{R_m(x)}{e^p - e^{-r_m}} \right) \left(1 - \sum_{n=1}^{N} \frac{\hat{R}_n(x+i)}{e^p - e^{-r_n}} \right) \\ &\qquad = \left(1 - \sum_{m=1}^{N} \frac{\hat{R}_m(x)}{e^p - e^{-r_m}} \right) \left(1 - \sum_{n=1}^{N} \frac{R_n(x+i)}{e^p - e^{-r_n}} \right). \end{aligned} \tag{4.26}$$

Multiplying by $(\exp(p) - \exp(-r_k))^2$ and letting $p \to -r_k$, we get

$$R_k(x)\hat{R}_k(x+i) = \hat{R}_k(x)R_k(x+i). \tag{4.27}$$

From this we deduce

$$\hat{R}_k(x) = \nu_k(x)R_k(x), \qquad \nu_k \in \mathcal{P}_i, \qquad k = 1, \ldots, N. \tag{4.28}$$

Taking now the difference of the AΔEs satisfied by (4.24) and (4.25), we obtain the identity

$$e^p H(x-i,p) + e^{-p}V_a(x)H(x+i,p) + \left[V_b(x) - e^p - e^{-p}\right] H(x,p) = 0, \tag{4.29}$$

where V_a, V_b denote the potentials in $A(r,\mu) = A(r,\hat{\mu})$, and where we have introduced

$$H(x,p) \equiv \sum_{n=1}^{N} \frac{1}{e^p - e^{-r_n}}[1 - \nu_n(x)]R_n(x). \tag{4.30}$$

Clearly, the vanishing residues for $p = -r_k$ yield no new information. But the limit for $\mathrm{Re}\, p \to \infty$ yields

$$\sum_{n=1}^{N}[1 - \nu_n(x)][R_n(x-i) - R_n(x)] = 0. \tag{4.31}$$

We are now going to use the consequence (4.31) of (4.23) to prove our conjecture $\hat{\mu} = \mu$ under an additional hypothesis, which is however generically satisfied.

Lemma 4.3. *Suppose N_+ or N_- vanishes. Then one has the implication*

$$A(r,\mu) = A(r,\hat{\mu}) \quad \Rightarrow \quad \hat{\mu} = \mu. \tag{4.32}$$

More generally, suppose $r = (r_1, \ldots, r_N)$ satisfies

$$e^{r_m} \neq -e^{r_n}, \qquad 1 \leq m < n \leq N. \tag{4.33}$$

Then (4.32) holds true as well.

Proof. We need only show that (4.33) entails (4.32). To this end we exploit (4.31). Since $\nu_n(x)$ is i-periodic, it entails that (3.3) holds true, with ζ replaced by $h(x) \equiv (1 - \nu_1(x), \ldots, 1 - \nu_N(x))^t$. Just as in the proof of Theorem 3.1, a contradiction now arises when $h(x)$ does not vanish. Thus we must have $\nu_n(x) = 1$, $n = 1, \ldots, N$. But then the two wave functions (4.24), (4.25) are equal, so that Theorem 4.1 yields $\hat{\mu} = \mu$. ■

Another approach to the injectivity question that may be useful in further studies results from our last lemma in this section. Specifically, we assume that (4.23) holds true, so that (4.28) follows. Now we introduce the i-periodic matrices

$$\nu(x) \equiv \mathrm{diag}\,(\nu_1(x), \ldots, \nu_N(x)), \tag{4.34}$$

$$Q(x) \equiv D(r,\mu;x)D(r,\hat{\mu};x)^{-1} = \mathrm{diag}\,(\mu_1(x)/\hat{\mu}_1(x), \ldots, \mu_N(x)/\hat{\mu}_N(x)), \tag{4.35}$$

$$P(x) \equiv C(r) - Q(x)\nu(x)^{-1}C(r)\nu(x). \tag{4.36}$$

Lemma 4.4. *One has*

$$P(x)(R(x) - R(x-i)) = 0. \tag{4.37}$$

In particular, $P(x)$ has vanishing determinant.

Proof. From (2.37) we obtain

$$D(x)R(x) - D(x-i)R(x-i) + C(R(x) - R(x-i)) = 0. \tag{4.38}$$

Next, we exploit the i-periodicity of $\nu(x)$ to rewrite the analog of (4.38) for $\hat{R}(x) = \nu(x)R(x)$ as

$$D(x)R(x) - D(x-i)R(x-i) + Q(x)\nu(x)^{-1}C\nu(x)(R(x) - R(x-i)) = 0. \tag{4.39}$$

Subtracting (4.39) from (4.38), we get

$$P(x)(R(x) - R(x-i)) = 0. \tag{4.40}$$

Now from (2.37) and (2.35) it is evident that $R(x)$ is not i-periodic. Therefore the lemma follows. ■

Obviously, the expected equality $\hat{\mu} = \mu$ is equivalent to all diagonal elements of $P(x)$ being zero. Introducing the determinant

$$\mathcal{D}(r,\mu;x) \equiv |\mathrm{Col}\,(R(r,\mu;x) - R(r,\mu;x-i), \ldots, R(r,\mu;x) - R(r,\mu;x-Ni))|, \tag{4.41}$$

one readily sees that $P(x)$ actually vanishes identically when $\mathcal{D}(x) \in \mathcal{M}^*$. Unfortunately, the $\mathrm{Re}\,x \to \infty$ asymptotics of the matrix on the rhs of (4.41) leads for the third time to the condition (4.33), cf. the proof of Theorem 3.1.

In this connection it should be pointed out that $\mathcal{D}(x)$ does not alway belong to $\mathcal{M}^*$. Indeed, the arguments in the proof of Theorem 3.3 leading to the relation $R^- = -R^+$ (3.41) give rise to the implication

$$(r^*, \mu^*) = (\pi(r), \pi(\mu)), \pi \in S_N \Rightarrow \mathcal{D}(r,\mu;x) = 0. \tag{4.42}$$

(Note that (3.29) does not yield all of the r allowed in (4.42), since the imaginary parts of the numbers $r_{s,n}$ are all in $(-\pi, -\pi/2) \cup (0, \pi/2)$. But this restriction is not used to arrive at (3.41).)

It may well be that the converse of (4.42) is also valid. It would be worthwile to study this issue further, since it clearly has a bearing on the above-mentioned conjectures. It is also connected to another natural question we leave open, namely: Do there exist $(r,\mu) \in \mathcal{D}_{\mathrm{IST}}$ such that $V_a(r,\mu;x) = 1$? Indeed, from (2.54) and (2.47) one sees that this is equivalent to vanishing of

$$\sum_{n=1}^{N} e^{r_n}(R_n(r,\mu;x) - R_n(r,\mu;x-i)). \tag{4.43}$$

Therefore, $\mathcal{D}(r,\mu;x) \in \mathcal{M}^*$ implies $V_a(r,\mu;x) - 1 \in \mathcal{M}^*$. (In particular, one has $V_a(x) \neq 1$ when (4.33) holds true.)

Appendix A. Cauchy and Vandermonde matrices

As is well known, the Cauchy matrix

$$C_{ij} \equiv \frac{1}{x_i - y_j}, \qquad i, j = 1, \ldots, N, \tag{A.1}$$

with $x_1, \ldots, x_N, y_1, \ldots, y_N$ distinct complex numbers, has determinant

$$|C| = \prod_i \frac{1}{x_i - y_i} \prod_{i<j} \frac{(x_i - x_j)(y_i - y_j)}{(x_i - y_j)(y_i - x_j)}. \tag{A.2}$$

This Cauchy identity shows in particular that C is regular. In the main text we work with the matrix $C(r)$ obtained by substituting

$$x_i = e^{r_i}, \qquad y_j = e^{-r_j}, \tag{A.3}$$

with

$$\operatorname{Im} r_k \in (0, \pi) \cup (-\pi, 0), \quad k = 1, \ldots, N, \quad e^{r_m} \neq e^{\pm r_n}, \quad 1 \leq m < n \leq N. \tag{A.4}$$

Basically, we already obtained the following identities in Lemma A.7 of Ref. [11], where we employed a slightly different Cauchy matrix. For completeness we include a proof.

Lemma A.1. *One has the identities*

$$\begin{gathered} \left(C(r)^{-1}\zeta\right)_m = \left(e^{r_m} - e^{-r_m}\right) \prod_{n=1,\, n\neq m}^{N} \frac{e^{-r_m} - e^{r_n}}{e^{-r_m} - e^{-r_n}}, \\ \zeta \equiv (1, \ldots, 1)^t, \quad m = 1, \ldots, N. \end{gathered} \tag{A.5}$$

Proof. This is equivalent to the functional identities

$$\sum_{k=1}^{N} \frac{1}{e^{r_j} - e^{-r_k}} \left(e^{r_k} - e^{-r_k}\right) \prod_{l=1,\, l\neq k}^{N} \frac{e^{-r_k} - e^{r_l}}{e^{-r_k} - e^{-r_l}} = 1, \qquad j = 1, \ldots, N. \tag{A.6}$$

By permutation invariance we need only prove (A.6) for $j = 1$. In that case the lhs can be rewritten as $F(r_1)$, with

$$F(\rho) \equiv \prod_{l=2}^{N} \frac{e^{-\rho} - e^{r_l}}{e^{-\rho} - e^{-r_l}} + \sum_{k=2}^{N} \frac{e^{r_k} - e^{-r_k}}{e^{-\rho} - e^{-r_k}} \prod_{l=2,\, l\neq k}^{N} \frac{e^{-r_k} - e^{r_l}}{e^{-r_k} - e^{-r_l}}. \tag{A.7}$$

Clearly, $F(\rho)$ is $2\pi i$-periodic and has limit 1 for $\operatorname{Re}\rho \to -\infty$. Since it has a finite limit for $\operatorname{Re}\rho \to \infty$ as well, it suffices to verify that the residues at the simple poles $r_2, \ldots, r_N$ cancel, which is routine. ■

In Appendix C we have occasion to use the identity

$$\sum_{m=1}^{N} \left(C(r)^{-1}\zeta\right)_m = \sum_{m=1}^{N} \left(e^{r_m} - e^{-r_m}\right), \tag{A.8}$$

which can be derived from (A.5) and its proof. Indeed, substituting (A.5) on the lhs, and setting $r_1 = \rho$, we see that (A.8) amounts to the function

$$G(\rho) \equiv \left(e^{\rho} - e^{-\rho}\right)\left(\prod_{l=2}^{N} \frac{e^{-\rho} - e^{r_l}}{e^{-\rho} - e^{-r_l}} - 1\right) + \sum_{k=2}^{N}(e^{r_k} - e^{-r_k})\left(\frac{e^{\rho} - e^{-r_k}}{e^{-\rho} - e^{-r_k}} \prod_{l=2,\, l\neq k}^{N} \frac{e^{-r_k} - e^{r_l}}{e^{-r_k} - e^{-r_l}} - 1\right) \tag{A.9}$$

being identically zero. Now $G(\rho)$ is obviously $2\pi i$-periodic, and is readily seen to be entire. It is straightforward to check that its limit for $\mathrm{Re}\,\rho \to -\infty$ vanishes. Thus it remains to verify that its limit for $\mathrm{Re}\,\rho \to \infty$ is finite. Now the coefficient of $\exp(\rho)$ equals $F(\rho) - 1$ (cf. (A.7)), so it vanishes. Hence $G(\rho)$ vanishes for $\mathrm{Re}\,\rho \to \infty$, too, so that (A.8) is proved.

A second type of matrix playing an important role in this paper is the Vandermonde matrix. Choosing $\rho_1, \ldots, \rho_N \in \mathbb{C}$, it can be defined as the matrix with kth row

$$\gamma_k \equiv \left(1, \rho_k, \ldots, \rho_k^{N-1}\right), \qquad k = 1, \ldots, N. \tag{A.10}$$

As is well known (and easily checked), it has determinant

$$|\mathrm{Row}\,(\gamma_1, \ldots, \gamma_N)| = \prod_{1\leq m<n\leq N} (\rho_n - \rho_m). \tag{A.11}$$

In the main text we encounter matrices of the form

$$V(\rho_1, \ldots, \rho_N) \equiv \mathrm{Row}\,(\eta_1, \ldots, \eta_N), \qquad \eta_k \equiv \left(1 - \rho_k, 1 - \rho_k^2, \ldots, 1 - \rho_k^N\right). \tag{A.12}$$

Presumably, the following result is known, too. In any case, the proof we supply is short and simple.

Lemma A.2. *One has*

$$|V(\rho)| = \prod_{k=1}^{N}(1 - \rho_k) \cdot \prod_{1\leq m<n\leq N} (\rho_n - \rho_m). \tag{A.13}$$

Proof. We may write

$$\eta_k = (1 - \rho_k)\left(1, 1 + \rho_k, \ldots, 1 + \rho_k^2 + \cdots + \rho_k^{N-1}\right) \equiv (1 - \rho_k)\zeta_k, \tag{A.14}$$

so that

$$|V(\rho)| = \prod_{k=1}^{N}(1 - \rho_k) \cdot |\mathrm{Row}\,(\zeta_1, \ldots, \zeta_N)|. \tag{A.15}$$

In the matrix $\mathrm{Row}(\zeta_1, \ldots, \zeta_N)$ we now subtract column l from column $l + 1$ for $l = N - 1, N - 2, \ldots, 1$, to obtain the Vandermonde matrix $\mathrm{Row}(\gamma_1, \ldots, \gamma_N)$. Using (A.11), we deduce (A.13). ∎

Appendix B. Casorati determinants

We begin by summarizing some known general results concerning the solutions $F \in \mathcal{M}^*$ to the AΔEs at issue in this paper, cf. Ref. [12]. Thus we have

$$F(x-i) + V_a(x)F(x+i) + (V_b(x) - c)F(x) = 0, \tag{B.1}$$

where $c = \exp(p) + \exp(-p)$ is assumed to be a fixed complex number for the moment. Assuming $F_1, F_2 \in \mathcal{M}^*$ satisfy (B.1), we define their Casorati determinant by

$$\mathcal{D}(F_1(x), F_2(x)) \equiv F_1(x-i)F_2(x) - F_1(x)F_2(x-i). \tag{B.2}$$

Clearly, this function vanishes identically iff F_1/F_2 belongs to $\mathcal{P}_i$ (1.9). Moreover, it satisfies the AΔE

$$\mathcal{D}(x) = V_a(x)\mathcal{D}(x+i), \tag{B.3}$$

as is easily verified.

Assuming from now on $F_1/F_2 \notin \mathcal{P}_i$, suppose $F_3 \in \mathcal{M}^*$ is a third solution to (B.1). Then it is straightforward to check that one has

$$F_3(x) = \nu_1(x)F_2(x) - \nu_2(x)F_1(x), \tag{B.4}$$

where

$$\nu_j(x) \equiv \mathcal{D}(F_j(x), F_3(x))/\mathcal{D}(F_1(x), F_2(x)), \qquad j = 1, 2. \tag{B.5}$$

Now it follows from (B.3) that quotients of non-zero Casorati determinants are i-periodic. Thus, one has $\nu_j \in \mathcal{P}_i$ whenever $\nu_j \in \mathcal{M}^*$. Conversely, for $\nu_j \in \mathcal{P}_i$ the rhs of (B.4) obviously solves (B.1). Thus, the space of meromorphic solutions to (B.1) is two-dimensional over the field of i-periodic meromorphic functions.

It should be repeated that the latter conclusion involves the assumption that F_1, F_2 are solutions with a *non-zero* Casorati determinant. To our knowledge, the *existence* of such a basis is not known to follow from our assumptions $V_a \in \mathcal{M}^*$, $V_b \in \mathcal{M}$, even when one also assumes the asymptotics (1.3). For the class of potentials obtained above Theorem 2.3, however, we have solutions $\mathcal{W}(x, \pm p)$ available. (Here and from now on we take the p-dependence into account.) We proceed by deriving more information pertaining to the basis properties of the latter solutions.

Theorem B.1. *Assuming*

$$e^p \neq e^{\pm r_1}, \ldots, e^{\pm r_N}, \tag{B.6}$$

we have

$$\mathcal{D}(\mathcal{W}(r, \mu; x, p), \mathcal{W}(r, \mu; x, -p)) = \left(e^p - e^{-p}\right) \lambda(r, \mu; x), \tag{B.7}$$

where $\lambda(r, \mu; x)$ is given by (2.47).

Proof. It is clear from the definition (2.54) of $V_a(x)$ and the AΔE (B.3) that $\mathcal{D}(x)/\lambda(x)$ is i-periodic. The problem is, therefore, to show that this i-periodic function is the constant $\exp(p)-\exp(-p)$. But we are going to prove this by focusing on the p-dependence of the Casorati determinant. Indeed, using (2.57), it can be written

$$\begin{aligned} e^p & \left(1-\sum_{m=1}^N \frac{R_m(x-i)}{e^p-e^{-r_m}}\right)\left(1-\sum_{n=1}^N \frac{R_n(x)}{e^{-p}-e^{-r_n}}\right) \\ & -e^{-p}\left(1-\sum_{m=1}^N \frac{R_m(x)}{e^p-e^{-r_m}}\right)\left(1-\sum_{n=1}^N \frac{R_n(x-i)}{e^{-p}-e^{-r_n}}\right). \end{aligned} \tag{B.8}$$

This shows that we are dealing with a $2i\pi$-periodic meromorphic function of p, with simple poles in the period strip $\operatorname{Im} p \in (-\pi,\pi]$ that can be located only at $p=\pm r_n$, $n=1,\ldots,N$, and with zeros for $p=0, i\pi$.

Consider now the residues at the pole $p=-r_k$. The first factor in brackets has residue $-\exp(r_k)R_k(x-i)$ and the third one $-\exp(r_k)R_k(x)$. The second factor in brackets has value $d(r_k,\mu_k;x)R_k(x)$ (by virtue of the system (2.37)), and, likewise, the fourth one has value $d(r_k,\mu_k;x-i)R_k(x-i)$. Thus we obtain a total residue

$$\begin{aligned} & e^{-r_k}\left(-e^{r_k}R_k(x-i)\right)d(r_k,\mu_k;x)R_k(x) \\ & \quad -e^{r_k}\left(-e^{r_k}R_k(x)\right)d(r_k,\mu_k;x-i)R_k(x-i)=0, \end{aligned} \tag{B.9}$$

where we used (2.36).

We continue by observing that (B.8) is odd in p. Hence the total residue for $p=r_k$ vanishes, too. We are now in the position to conclude that the function obtained upon dividing (B.8) by $\exp(p)-\exp(-p)$ is $2i\pi$-periodic, entire and even in p. Its limit for $\operatorname{Re} p \to \pm\infty$ equals $1+\sum_{n=1}^N \exp(r_n)R_n(x)=\lambda(x)$, so by Liouville's theorem it equals $\lambda(x)$ for all p. ■

From Theorem B.1 we infer in particular that with the restriction (B.6) in force, $\mathcal{W}(x,p)$ and $\mathcal{W}(x,-p)$ yield a basis in the above sense, as long as $\exp(p)$ is not equal to ± 1. It is indeed evident from (2.57) that the quotient

$$\mathcal{W}(x,ik\pi)/\mathcal{W}(x,-ik\pi)=e^{-2k\pi x}, \qquad k\in\mathbb{Z}, \tag{B.10}$$

is i-periodic. However, for $p=ik\pi$ one can obtain a further solution

$$\lim_{\epsilon\to 0}(2\epsilon)^{-1}\left[\mathcal{W}(x,ik\pi+\epsilon)-e^{-2k\pi x}\mathcal{W}(x,-ik\pi-\epsilon)\right]=\partial_p\mathcal{W}(x,ik\pi), \quad k\in\mathbb{Z}. \tag{B.11}$$

Together with $\mathcal{W}(x,-ik\pi)$ this yields again a basis, since one clearly gets

$$\mathcal{D}(\partial_p\mathcal{W}(x,ik\pi),\mathcal{W}(x,-ik\pi))=(-)^k\lambda(x). \tag{B.12}$$

The auxiliary wave functions $\mathcal{A}(x,\pm p)$ (2.2) are well defined for all $p\in\mathbb{C}$, but of course their Casorati determinant,

$$\mathcal{D}(\mathcal{A}(x,p),\mathcal{A}(x,-p))=\left(e^p-e^{-p}\right)\lambda(x)\prod_{n=1}^N\left(e^p-e^{-r_n}\right)\left(e^{-p}-e^{-r_n}\right), \tag{B.13}$$

vanishes when $\exp(p)$ equals $\exp(\pm r_m)$, cf. also (2.66) and (2.67). Just as for $\exp(p) = \pm 1$ one can obtain solutions

$$\lim_{\epsilon\to 0}(2\epsilon)^{-1}[\mathcal{A}(x, r_m + 2ik\pi + \epsilon) - \alpha_m(x)\mathcal{A}(x, -r_m - 2ik\pi - \epsilon)] \equiv \mathcal{B}_{r_m,k}(x), \qquad \text{(B.14)}$$

with $m = 1, \ldots, N, k \in \mathbb{Z}$, yielding a basis together with $\mathcal{A}(x, -r_m - 2ik\pi)$. Indeed, their Casorati determinants

$$\mathcal{D}(\mathcal{B}_{r_m,k}(x), \mathcal{A}(x, -r_m - 2ik\pi)) = 2^{-1}\partial_p\mathcal{D}(\mathcal{A}(x,p), \mathcal{A}(x,-p))|_{p=r_m} \qquad \text{(B.15)}$$

do not vanish identically.

Appendix C. Alternative representations

Thus far, we have worked with formulas expressing $\lambda(x)$, $V_a(x)$, $V_b(x)$ and $\mathcal{W}(x,p)$ in terms of the solution $R(x)$ to the system (2.37), cf. (2.47), (2.54), (2.55) and (2.57), resp. In this appendix we derive alternative formulas involving various matrices and determinants. They will be used in Appendix D, as well as in Parts II and III.

We begin by deriving a second formula for the sum function

$$\Sigma(r, \mu; x) \equiv \sum_{n=1}^{N} R_n(r, \mu; x), \qquad \text{(C.1)}$$

and hence for

$$V_b(r, \mu; x) = \Sigma(r, \mu; x - i) - \Sigma(r, \mu; x), \qquad \text{(C.2)}$$

cf. (2.55).

Theorem C.1. *The function (C.1) can be rewritten as*

$$\Sigma(r, \mu; x) = \sum_{m,n=1}^{N} \left([D(r, \mu; x) + C(r)]^{-1}\right)_{mn}. \qquad \text{(C.3)}$$

Proof. Denoting the canonical scalar product on $\mathbb{C}^N$ by $(\cdot, \cdot)$, we clearly have

$$\Sigma(x) = (\zeta, R(x)). \qquad \text{(C.4)}$$

Invoking (2.37), this can be written as

$$\Sigma(x) = \left(\zeta, [D(x) + C]^{-1}\zeta\right), \qquad \text{(C.5)}$$

which amounts to (C.3). ■

The formula (C.3) is particularly useful in Appendix D. We now derive a third formula for $\Sigma(x)$ that is crucial in Part II. To state the latter formula, we introduce the vectors

$$\omega_n(r) \equiv \left(e^{r_n} - e^{-r_n}\right)\left(C(r)_{1n}, \ldots, C(r)_{Nn}\right)^t, \qquad n = 1, \ldots, N. \qquad \text{(C.6)}$$

Now we define the matrix $\Omega_n(r, \mu; x)$ as the matrix obtained from $D(r, \mu; x) + C(r)$ when the nth column is replaced by $\omega_n(r)$.

Theorem C.2. *The function (C.1) can be rewritten as*

$$\Sigma(r,\mu;x)=\sum_{n=1}^{N}|\Omega_n(r,\mu;x)|/|D(r,\mu;x)+C(r)|. \tag{C.7}$$

Proof. In view of (C.1) and (2.38), we need only show

$$\sum_{n=1}^{N}|Z_n(x)|=\sum_{n=1}^{N}|\Omega_n(x)|. \tag{C.8}$$

To this end we first replace the quantities $d(r_n,\mu_n;x)$ in the diagonals of the $2N$ matrices occurring here by $\lambda_n\in\mathbb{C}$, and denote the resulting matrices by $\hat{Z}_n(\lambda_1,\ldots,\lambda_N)$ and $\hat{\Omega}_n(\lambda_1,\ldots,\lambda_N)$. Then it clearly suffices to show

$$\sum_{n=1}^{N}|\hat{Z}_n(\lambda)|=\sum_{n=1}^{N}|\hat{\Omega}_n(\lambda)|, \tag{C.9}$$

for arbitrary $\lambda\in\mathbb{C}^N$.

To this end we compare the coefficients of the monomials $\lambda_{i_1}\cdots\lambda_{i_k}$ with $1\le i_1<\cdots<i_k\le N$. For $k=0$ they are obtained by taking $\lambda_1,\ldots,\lambda_N=0$. Expanding $|\hat{Z}_n(0,\ldots,0)|$ with respect to its nth column ζ, we see that we get on the lhs the sum of all the cofactors of the Cauchy matric $C(r_1,\ldots,r_N)$. Similarly, expanding $|\hat{\Omega}_n(0,\ldots,0)|$ with respect to its nth column ω_n, we obtain on the rhs $\sum_{n=1}^{N}(\exp(r_n)-\exp(-r_n))|C(r_1,\ldots,r_N)|$. By virtue of (A.8), these two sums are equal.

Next, consider the case $k=N$. Since λ_n does not occur in $\hat{Z}_n(\lambda)$ and $\hat{\Omega}_n(\lambda)$, this case gives rise to zero coefficients on the lhs and rhs. Thus we are left with the case $k\in\{1,\ldots,N-1\}$. Denoting the indices complementary to $i_1,\ldots,i_k$ by $j_1,\ldots,j_l$, $l=N-k$, we should show that (C.9) holds true when $\lambda_{j_1},\ldots,\lambda_{j_l}$ vanish. We may restrict the summation to $n=j_1,\ldots,j_l$. Doing so, we should choose the diagonal elements $\lambda_{i_1},\ldots,\lambda_{i_k}$ in the expansion of the determinants so as to get the pertinent coefficients. When we then expand again with respect to the special columns, we obtain on the lhs the sum of all cofactors of the $(N-k)\times(N-k)$ matrix $C(r_{j_1},,\ldots,r_{j_l})$, and on the rhs $|C(r_{j_1},\ldots,r_{j_l})|$ times the sum of $\exp(r_n)-\exp(-r_n)$ with $n\in\{j_1,\ldots,j_l\}$. Thus the required coefficient equality follows once again from (A.8). ■

Consider next the function

$$Q(r,\mu;x,p)\equiv 1-\sum_{k=1}^{N}\frac{R_k(r,\mu;x)}{e^p-e^{-r_k}}. \tag{C.10}$$

Evidently, it equals the wave function $\mathcal{W}(r,\mu;x,p)$ (2.57) up to multiplication by the plane wave $\exp(ixp)$. We proceed by obtaining a second representation for Q as a determinant quotient. As a preparation we introduce the matrix

$$\Delta(r;p)\equiv \mathrm{diag}\,(\delta(r_1;p),\ldots,\delta(r_N;p)), \tag{C.11}$$

where

$$\delta(\rho;p)\equiv 1-\frac{e^\rho-e^{-\rho}}{e^p-e^{-\rho}}, \tag{C.12}$$

and the quotient function

$$\tilde{Q}(r,\mu;x,p) \equiv |D(r,\mu;x) + C(r)\Delta(r;p)|/|D(r,\mu;x) + C(r)|. \tag{C.13}$$

For $N = 1$ we have using (2.33)

$$(D(x) + C)\tilde{Q}(x,p) = D(x) + C - C\frac{e^{r_1} - e^{-r_1}}{e^p - e^{-r_1}} = D(x) + C - \frac{1}{e^p - e^{-r_1}}, \tag{C.14}$$

whereas (C.10) and (2.38) yield

$$(D(x) + C)Q(x,p) = D(x) + C - \frac{1}{e^p - e^{-r_1}}. \tag{C.15}$$

Therefore $Q(x,p)$ and $\tilde{Q}(x,p)$ are equal for $N = 1$.

We are now going to prove by induction on N that Q and $\tilde{Q}$ are equal for arbitrary (r,μ). For brevity we use the notation

$$F^{(M)}(\cdot) \equiv F(r_1,\ldots,r_M,\mu_1,\ldots,\mu_M;\cdot), \qquad M \in \mathbb{N}, \tag{C.16}$$

in the proof. As a further preparation, we recall that $Z_n(x)$ denotes the matrix obtained from $D(x) + C$ when the nth column is replaced by $\zeta = (1,\ldots,1)^t$. The identity

$$\left|Z_M^{(M)}(x)\right| = -\sum_{k=1}^{M-1} C_{Mk}^{(M)} \left|Z_k^{(M-1)}(x)\right| + \left|D^{(M-1)}(x) + C^{(M-1)}\right| \tag{C.17}$$

is a cruci al ingredient of our proof. It can be checked by developing the determinant on the lhs w.r.t. the last row.

Theorem C.3. *One has*

$$Q(r,\mu;x,p) = \tilde{Q}(r,\mu;x,p), \tag{C.18}$$

for all (r,μ) *satisfying (2.30), (2.32) and (2.34).*

Proof. We have already shown equality for $N = 1$. Assuming equality for $N = 1,\ldots,$ $M - 1$, we now prove equality for $N = M$.

We begin by noting that the system (2.37) entails

$$Q^{(M)}(x,r_n) = d(r_n,\mu_n;x)R_n^{(M)}(x), \qquad n = 1,\ldots,M. \tag{C.19}$$

Now we assert that we have

$$\tilde{Q}^{(M)}(x,r_M) = Q^{(M)}(x,r_M). \tag{C.20}$$

To substantiate this assertion, we introduce

$$G(x) \equiv d(r_M,\mu_M;x)^{-1}\left|D^{(M)}(x) + C^{(M)}\right|\tilde{Q}^{(M)}(x,r_M), \tag{C.21}$$

and use (C.13) to get

$$G(x) = d(r_M,\mu_M;x)^{-1}\left|D^{(M)}(x) + C^{(M)}\Delta^{(M)}(r_M)\right|. \tag{C.22}$$

We now note the key property $\delta(\rho;\rho)=0$, cf. (C.12). It implies that the Mth column of the matrix on the rhs reads $(0,\dots,0,d(r_M,\mu_M;x))^t$. Developing its determinant w.r.t. the Mth column, we then obtain

$$\begin{aligned} G(x) &= \left| D^{(M-1)}(x) + C^{(M-1)}\Delta^{(M-1)}(r_M) \right| \\ &= \tilde{Q}^{(M-1)}(x,r_M) \left| D^{(M-1)}(x) + C^{(M-1)} \right|. \end{aligned} \tag{C.23}$$

We are now in the position to use the induction assumption. It entails that (C.23) can be rewritten as

$$\begin{aligned} G(x) &= Q^{(M-1)}(x,r_M) \left| D^{(M-1)}(x) + C^{(M-1)} \right| \\ &= \left| D^{(M-1)}(x) + C^{(M-1)} \right| - \sum_{k=1}^{M-1} \frac{\left| Z_k^{(M-1)}(x) \right|}{e^{r_M} - e^{-r_k}}, \end{aligned} \tag{C.24}$$

where we used (C.10) and (2.38) in the second step. Invoking the identity (C.17), and using then (C.19) and (2.38), we infer

$$G(x) = \left| Z_M^{(M)}(x) \right| = d(r_M,\mu_M;x)^{-1} \left| D^{(M)}(x) + C^{(M)} \right| Q^{(M)}(x,r_M). \tag{C.25}$$

Comparing (C.21) and (C.25), we obtain the asserted equality (C.20).

More generally, it now follows that we have

$$\tilde{Q}^{(M)}(x,r_n) = Q^{(M)}(x,r_n), \qquad n=1,\dots,M. \tag{C.26}$$

Indeed, this can be reduced to the case $n=M$ (already proved) by relabeling r_n,μ_n.

We continue by noting that (C.11)–(C.13) entail $\tilde{Q}^{(M)}(x,p)$ is of the form

$$\sum_{i_1,\dots,i_M=0}^{1} c_{i_1\cdots i_M}(x) \prod_{k=1}^{M} \left(e^p - e^{-r_k}\right)^{-i_k}, \tag{C.27}$$

with $c_{0\cdots 0}(x)=1$. Thus the function

$$P(x,p) \equiv \prod_{k=1}^{M} \left(e^p - e^{-r_k}\right) \cdot \left(Q^{(M)}(x,p) - \tilde{Q}^{(M)}(x,p) \right) \tag{C.28}$$

is a polynomial in $z=\exp(p)$ of degree at most $M-1$. Since $P(x,p)$ vanishes in M distinct points $z=\exp(r_1),\dots,\exp(r_M)$ (due to (C.26)), it must be zero, entailing $Q^{(M)}(x,p) = \tilde{Q}^{(M)}(x,p)$. ■

Finally, let us observe that (C.10) and (2.47) entail

$$\lim_{\mathrm{Re}\, p\to-\infty} Q(r,\mu;x,p) = \lambda(r,\mu;x). \tag{C.29}$$

We can therefore use Theorem C.3 to obtain a second representation for $\lambda(x)$, and hence for $V_a(x)$ as well.

Theorem C.4. *The functions (2.47) and (2.54) can be rewritten as*

$$\lambda(r,\mu;x) = \prod_{k=1}^{N} e^{2r_k} \cdot |D(r,\mu;x-i) + C(r)|/|D(r,\mu;x) + C(r)|, \tag{C.30}$$

$$V_a(r,\mu;x) = |D(r,\mu;x-i) + C(r)||D(r,\mu;x+i) + C(r)|/|D(r,\mu;x) + C(r)|^2. \tag{C.31}$$

Proof. From (C.12) we obtain

$$\lim_{\mathrm{Re}\, p \to -\infty} \delta(\rho;p) = e^{2\rho}, \tag{C.32}$$

so that (C.13) yields

$$\begin{aligned} \lim_{\mathrm{Re}\, p \to -\infty} \tilde{Q}(x,p) &= \left| D(x) + C \operatorname{diag}\left(e^{2r_1},\dots,e^{2r_N}\right)\right| / |D(x) + C| \\ &= \prod_{k=1}^{N} e^{2r_k} \cdot \left| D(x) \operatorname{diag}\left(e^{-2r_1},\dots,e^{-2r_N}\right) + C\right| / |D(x) + C|. \end{aligned} \tag{C.33}$$

Recalling the definition (2.35)–(2.36) of $D(x)$, we deduce that $D(x)\operatorname{diag}\left(e^{-2r_1},\dots,e^{-2r_N}\right)$ equals $D(x-i)$. Using Theorem C.3, we then obtain

$$\lim_{\mathrm{Re}\, p \to -\infty} Q(x,p) = \prod_{k=1}^{N} e^{2r_k} \cdot |D(x-i) + C|/|D(x) + C|. \tag{C.34}$$

Comparing this to (C.29), one reads off (C.30). Now (C.31) is plain from (2.54). ■

Appendix D. Formal self-adjointness

Consider the complex translations $T_{\pm i}$ in the AΔO A (1.1). They have a quite natural interpretation as self-adjoint operators on $L^2(\mathbb{R}, dx)$. Specifically, they can be viewed as the Fourier transforms of multiplication by $\exp(\pm p)$ on $L^2(\mathbb{R}, dp)$.

Next, letting $f \in \mathcal{M}$, we introduce the conjugate meromorphic function

$$f^*(x) \equiv \overline{f(\overline{x})}. \tag{D.1}$$

Now suppose that V_a and V_b satisfy

$$V_a^*(x) = V_a(x-i), \tag{D.2}$$

$$V_b^*(x) = V_b(x). \tag{D.3}$$

Then one readily checks that A is formally self-adjoint on $L^2(\mathbb{R}, dx)$.

In this appendix we obtain conditions on (r,μ) entailing (D.2) and (D.3). Let us first note that we have

$$V_a^*(x) = \lambda^*(x)/\lambda^*(x-i), \tag{D.4}$$

$$V_b^*(x) = \Sigma^*(x+i) - \Sigma^*(x). \tag{D.5}$$

Therefore, (D.2) and (D.3) will follow whenever

$$\lambda^*(x) = 1/\lambda(x), \tag{D.6}$$

$$\Sigma^*(x) = -\Sigma(x-i), \tag{D.7}$$

resp.

To study (D.6) and (D.7), we exploit the alternative formulas for $\Sigma(x)$ and $\lambda(x)$ obtained in Appendix C (cf. (C.3) and (C.30)). We begin by comparing

$$\Sigma^*(x) = \sum_{m,n=1}^{N} \left(\left[D^*(x) + \overline{C} \right]^{-1} \right)_{mn}, \tag{D.8}$$

and

$$-\Sigma(x-i) = \sum_{m,n=1}^{N} \left(\left[-D(x-i) - C \right]^{-1} \right)_{mn}. \tag{D.9}$$

Since $D(x)$ is a diagonal matrix, the formula (D.8) still holds true when we replace $\overline{C}$ by its transpose $\overline{C}^t$. Now from (2.33) we have

$$\left(\overline{C}^t \right)_{kl} = \left[\exp(\overline{r}_l) - \exp(-\overline{r}_k) \right]^{-1}, \tag{D.10}$$

$$(-C)_{kl} = \left[\exp(-r_l) - \exp(r_k) \right]^{-1}. \tag{D.11}$$

Thus we can ensure equality of $\overline{C}^t$ and $-C$ by choosing $r_1, \ldots, r_N$ purely imaginary.

Doing so from now on, we conclude that (D.7) will follow when the multipliers are restricted in such a way that we have

$$D^*(x) = -D(x-i). \tag{D.12}$$

From (2.35) and (2.36) we deduce that (D.12) amounts to

$$\mu_n^*(x) = -\mu_n(x) e^{-2r_n}, \qquad n = 1, \ldots, N. \tag{D.13}$$

Since r_n is purely imaginary, we need only introduce multipliers $\hat{\mu}_n(x)$ by

$$\hat{\mu}_n(x) \equiv i e^{-r_n} \mu_n(x), \qquad n = 1, \ldots, N, \tag{D.14}$$

and require

$$\hat{\mu}_n^*(x) = \hat{\mu}_n(x), \quad n = 1, \ldots, N, \tag{D.15}$$

to obtain (D.13).

The restrictions just obtained are also sufficient for (D.6) to hold true. Indeed, from (C.30) we have

$$\lambda^*(x) = \prod_{n=1}^{N} \exp(2\overline{r}_n) \cdot \left| D^*(x+i) + \overline{C} \right| / \left| D^*(x) + \overline{C} \right|. \tag{D.16}$$

Now we use the restriction $\overline{r}_n = -r_n$, (D.12) and $\overline{C} = -C^t$ to obtain

$$\lambda^*(x) = \prod_{n=1}^{N} \exp(2\overline{r}_n) \cdot \left|-D(x) - C^t\right| / \left|-D(x-i) - C^t\right| = 1/\lambda(x). \tag{D.17}$$

(We used $|-M^t| = (-)^N |M|$ in the second step.)

We summarize the main result of this appendix in the following theorem.

Theorem D.1. *Assume that the numbers r_n, $n = 1, \ldots, N$, are purely imaginary, and that the functions $ie^{-r_n}\mu_n(x)$, $n = 1, \ldots, N$, are real-valued for real x. Then the potentials $V_a(x)$ and $V_b(x)$ satisfy the formal self-adjointness relations (D.2) and (D.3).*

Proof. This follows from the above reasoning. (Note (D.15) is equivalent to real-valuedness of $\hat{\mu}_n(x)$ for real x.) ■

It is an open question whether the above restictions on (r, μ) entailing (D.2) and (D.3) are necessary. In particular, our conditions imply that $\lambda(x)$ is a phase factor for real x (cf. (D.6)), a feature that is conceivably not necessary for (D.2) to be valid. At any rate, this feature ensures that the formal operator

$$V_a(x)T_{-i} = \lambda(x)T_{-i}\lambda(x)^{-1} \tag{D.18}$$

may be viewed as a unitary transform of the self-adjoint operator $\exp(i\partial_x)$ on $L^2(\mathbb{R}, dx)$.

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Note added in proof

The uniqueness argument in the proof of Theorem 2.3 was inspired by a similar argument in the context of reflectionless Schrödinger operators, which we learned from Newell's monograph Ref. [6]. Recently, E. Date informed us that he has used the same type of arguments to handle soliton equations that admit a Zakharov-Shabat formulation (in particular, the infinite Toda lattice), cf. his papers [13] and [14]. This reasoning appears to have been used for the first time by Krichever in his theory of finite-gap solutions [15].

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