Distance calculation between Level-2 fuzzy regions

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Abstract

In many applications, spatial data often are prone to uncertainty and imprecision. To model this, fuzzy regions have been developed. Our initial model was a fuzzy set over a two dimensional domain, allowing for fuzzy regions and fuzzy points to be modelled. The model had some limitations: all points where treated independently, and it was not possible to group points together. Furthermore, it depended on meta-information to specify the interpretation. The model was extended to a level-2 fuzzy region to overcome these limitations; here the impact on the definition of the distance between regions will be considered.

Keywords: fuzzy region, fuzzy spatial reasoning, fuzzy distance, spatial databases

1. Introduction

Geographic data can be represented in different ways, depending on the application. Two common, distinct approaches are field based and entity-based ([1]). In this contribution, attention goes to the level-2 fuzzy regions; these are an extension of traditional entity based concepts, in which both uncertainty and imprecision can be modelled independently. The entity based approach involves representing real world entities (roads, objects, etc.) using basic geometric objects (lines, polygons, etc.). As such, the data aims to reflect a real world situation. However, data usually are prone to uncertainty and imprecision, which can have a number of causes: the uncertainty or imprecision can be inherent to the data (so it is an intrinsic part of the features that are modelled), it can be introduced due to limitations in measurements (the features that are modelled are not uncertain or imprecise, but we cannot asses them exactly) or it can result from combining data from different sources (the sources can contradict or be incompatible). In order to represent such features, different models exist, but the approaches usually involve modelling a number of candidate boundaries ([2], [3]) and are not really further developed beyond the model of representation.

In [4], we introduced the concept of fuzzy regions, which considered a region as a set of points and a fuzzy region essentially a fuzzy set over a two dimensional domain. The fuzzy region model allows for the representation of fuzzy regions (i.e. regions with partial membership), or fuzzy points (i.e. points at an imprecise or uncertain location), by considering a veristic respectively a possibilistic interpretation of the fuzzy set.

While this allows the modelling of fuzzy features, some shortcomings prevent a true modelling. First all points in a fuzzy region are considered independently, making it impossible for a user to specify that some points belong together. A solution to this was presented in [5], where points carrying the same membership can be grouped. This solution however had its limitations and did not solve the second shortcoming of the original model, which is more subtle and concerns interpretations. In a region with a veristic interpretation, it is not possible to specify e.g. candidate boundaries. As such, it is not possible to specify the outline of different possible regions: all points are independent. We can not give the points a possibilistic interpretation, as this would mean that all points are candidates. The concept of a candidate boundary is interesting, as it allows users to clearly specify possibilities. For this purpose, a new extension has been developed and presented in [6]; it defines a level-2 fuzzy region as a level-2 fuzzy set - a set defined over a fuzzy domain (not to be confused with a type-2 fuzzy set, a set which has fuzzy membership grades). In this contribution we consider calculating the distance between two level-2 fuzzy regions. After introducing and defining the fuzzy regions in section 2, the level-2 fuzzy regions will be defined in 3. Section 4 concerns the distance between level-2 fuzzy regions. Section 5 contains information on future work and applications; a conclusion summarizes the findings.

2. Preliminaries

2.1. Fuzzy regions

The concept of the original fuzzy regions requires a different view on regions. Traditionally, a region is defined by means of its outline (usually represented by means of a polygon), but it is also possible to

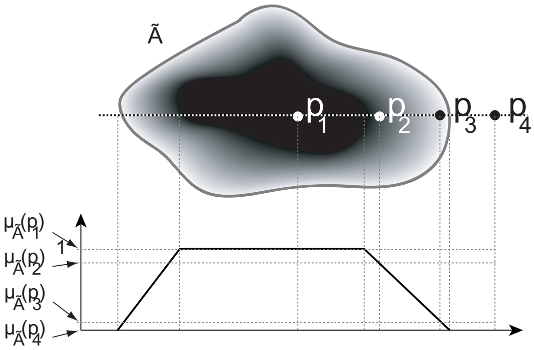

Figure 1: The concept of a fuzzy region $\tilde{A}$; a fuzzy set over a two dimensional domain. All points that have a non-zero membership value belong to some extent to the region. This membership grade is indicated on by means of grey scale values: the higher the value, the darker the shade (there is a dark outline to indicate the maximal extent of the region). In the lower half, a cross section is displayed.

consider the region to be a set of points in the two dimensional space. It is then a small step to consider it a fuzzy subset of this two dimensional space ([7]). This concept of a fuzzy regions was first introduced in [4], using the domain $\mathbb{R}^2$. With each element (point); a membership grade was associated.

2.1.1. Definitions

A fuzzy region is a fuzzy set defined over a two dimensional domain; this concept is illustrated on figure 1. The formal definition is:

$$\tilde{R} = \{(p, \mu_{\tilde{R}}(p)) | p \in \mathbb{R}^2\} \tag{1}$$

With the membership function is defined as:

$$\begin{array}{rcl} \mu_{\tilde{R}} : \mathbb{R}^2 & \mapsto & [0,1] \\ p & \rightarrow & \mu_{\tilde{R}}(p) \end{array}$$

According to [8], three different interpretations are possible for fuzzy sets: veristic, possibilistic and degrees of truth. In the context of the fuzzy region model, only the first two have been considered. The veristic interpretation expresses a partial membership. As such, it is known that all elements belong to the set and the membership degree expresses a *degree of beloning to*. In a possibilistic interpretation, there is doubt concerning which elements belong to the set. The membership grade of an element expresses the possiblitity the element belongs to the set. The veristic interpretation of a set of points implies a fuzzy region: all points belong, but not to the same extent. The possibilistic interpretation of a set of points on the other hand implies a fuzzy point: all points are candidates for a single crisp point, but it is not known which of the candidates is the crisp point. The representation in both cases is exactly the same, however the difference in interpretation impacts also the operations such as the distance between fuzzy regions.

The above definition was extended to allow for grouping of points with the same membership grade ([5]). For this purpose, the domain was altered from $\mathbb{R}^2$ to $\wp(\mathbb{R}^2)$; the powerset of $\mathbb{R}^2$. The powerset $\wp$ of a set A is defined as the set of all possible subsets of that set, including the empty set and the set itself. An example is given below.

$$\begin{array}{rcl} \wp(\{0,1,2\}) & = & \{\{\},\{0\},\{1\},\{2\},\{0,1\},\{0,2\}, \\ & & \{1,2\},\{0,1,2\}\} \end{array}$$

Using this concept, the fuzzy region can be defined with $\wp(\mathbb{R}^2)$ as the domain. This makes the basic elements of the fuzzy region subregions ([5]).

$$\begin{array}{rcl} \tilde{R} & = & \{(P, \mu_{\tilde{R}}(P)) | P \in \wp(\mathbb{R}^2) \\ & & \wedge \forall P_1, P_2 \in \tilde{R} : P_1 \cap P_2 = \emptyset\} \end{array} \tag{2}$$

With the membership function is defined as:

$$\begin{array}{rcl} \mu_{\tilde{R}} : \wp(\mathbb{R}^2) & \mapsto & [0,1] \\ P & \rightarrow & \mu_{\tilde{R}}(P) \end{array}$$

Note that in this definition the intersection between any two elements should be empty: it is required that no two elements of the fuzzy region share points. A point can only be considered to belong to the region once, even if it is to a membership grade less than 1. The reason for imposing this restriction is that intersecting subregions would yield unexpected behaviour in the different operations; the basic elements in the original regions are single points and also don't intersect. The concept is illustrated in figure 2. When a fuzzy region is defined by means of a limited number of subregions, the concept bears resemblance to the concept of plateau regions ([9]). Each of the subregions is given a membership grade, which carries a veristic interpretation. The concept allows for points to be grouped together in subregions, but it implies that all points that are in a single group will have the same membership grade.

As the regions are fuzzy sets, the traditional fuzzy operations for intersection and union (t-norms and t-conorms) are immediately applicable.

The set operations are independent of the interpretation; but it is assumed that both arguments carry the same interpretation, and that the end result will also carry the same interpretation. The set operations to suit the powerset extension mentioned above are also straight forward and were presented in [5].

2.1.2. Distance definitions

The distance between two fuzzy regions is a fuzzy set reflecting all the possible distances between both regions. A distinction is made between fuzzy regions interpreted in a veristic way, and fuzzy regions interpreted in a possibilistic way. This distinction is

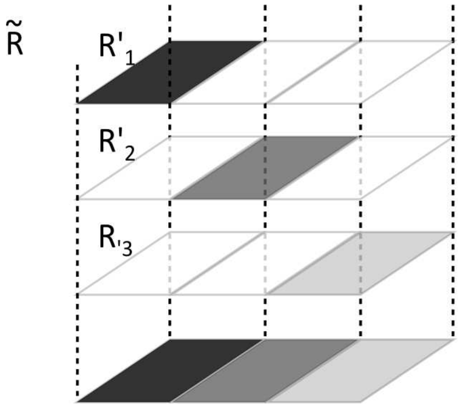

Figure 2: The concept of a fuzzy region defined over the powerset of the two dimensional domain. The region $\tilde{R}$ is comprised of three elements, the regions R'_1, R'_2 and R'_3). These are crisp regions that are given a membership grade with a veristic interpretation; they are elements or subregions.

necessary because in a possibilistic interpretation we need to consider the possible distances between all points of the region; in a veristic interpretation it was sufficient to only consider the distances between all possible α levels.

The distance between fuzzy regions in a veristic interpretation has been defined in [10] as:

$$\tilde{d}(\tilde{A},\tilde{B}) = \{(x, \mu_{\tilde{d}(\tilde{A},\tilde{B})}(x)) \mid x \in \mathbb{R}\} \tag{3}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{A},\tilde{B})} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup\{\alpha \mid d(\tilde{A}_\alpha, \tilde{B}_\alpha) \leq x \\ &\qquad \leq d(\tilde{A}_{\overline{\alpha}}, \tilde{B}_{\overline{\alpha}})\} \end{aligned}$$

The origin from the definition stems from the fact that it was desirable for the distance to carry the same properties as a fuzzy number, even though technically it carries a veristic interpretation.

The distance between fuzzy regions in a possibilistic interpretation is slightly different and has been defined in [10].

$$\tilde{d}(\tilde{p}^A, \tilde{p}^B) = \{(x, \mu_{\tilde{d}(\tilde{p}^A,\tilde{p}^B)}(x))\} \tag{4}$$

with membership function

$$\begin{aligned} \mu_{\tilde{d}(\tilde{p}^A,\tilde{p}^B)} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup_{\alpha \in]0,1]} \{\alpha \mid p_1 \in \tilde{p}^A_\alpha, p_2 \in \tilde{p}^B_\alpha \\ &\qquad \wedge d(p_1,p_2) = x\} \end{aligned}$$

2.1.3. Limitations

In the introduction, we briefly mentioned the limitations of fuzzy regions. There are basically three shortcomings to the model: the inability to group elements, the dependency of metadata to know the interpretation and the inability to combine interpretations. The first limitation concerns grouping elements: there are many situations in which a user has additional data concerning the distribution of membership grades, for instance knowledge that some points either belong to the region at the same time or don't belong to the region. The extension using the powerset compensates for this to some extent, but it still is impossible to have points with different membership grades belong in the same group.

The second limitation tells us we always need to supply metadata to know which interpretation the fuzzy region is given. As this impacts the operators, the meaning and the querying of the fuzzy region, it is an important aspect that should not be overlooked. Especially for the distance operation, the interpretation of the fuzzy set determines which definition of the distance is applicable. This dependency on meta data (the interpretation) makes the model less transparent and may give rise to confusion. Even though fuzzy regions with different interpretations are very similar, it is also necessary to define additional operators on the possible combinations: the distance between a fuzzy region with a possibilistic interpretation and one with a veristic interpretation is a concept that makes sense, but cannot be deduced from the current definitions that require the same interpretation for both arguments. Combining both interpretations in a single unified model would not only remove the dependency of the metadata, but would also allow for unified definitions for various operators.

The third limitation is more subtle and requires an example; the classic example we use for this is the representation of a lake with varying water level: points at the same altitude around the lake will be flooded at the same time. The current model only allows for the lake to be modelled, with points that belong to some extent to it. While we can say that points that are not flooded all the time belong to a lesser extent to the lake (and we can use the definition 2 to group the points), it still leaves a problem. At a given point in time the lake will have a fairly crisp boundary, but if we don't know the water level we just don't know what this boundary is. To model this possibility, an additional level of uncertainty is needed. Adding a second level of uncertainty to the model would allow the fuzzy region in different circumstances to be modelled. Combining all three limitations led us to developing a unified representation of both fuzzy points and fuzzy regions which was first introduced in [6]. For fuzzy regions, a number of operations have been defined in the past, including distance and surface area ([11]).

3. Level-2 fuzzy regions

3.1. Concept

The level-2 fuzzy region is an extension to the previous fuzzy regions. Previously, a fuzzy region was de-

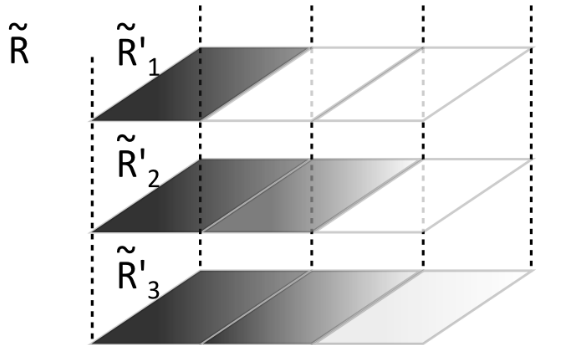

Figure 3: The concept of the level-2 fuzzy regions. The region $\tilde{\tilde{R}}$ has three possible candidates: the regions $\tilde{R}'_1$, $\tilde{R}'_2$ and $\tilde{R}'_3$). Each of these is a fuzzy region (as defined in 1) with a veristic interpretation which is given a membership grade. As they are candidates or possibilities, their assigned membership grade is interpreted possibilistically.

fined as a fuzzy set of non-overlapping crisp regions. While it solved some initial issues, it had limitations among which the fact that there still was the need for additional metadata to carry the interpretation. The level-2 fuzzy region overcomes this by allowing candidate fuzzy regions to be represented. For this, the concept of the fuzzy powerset is used. The fuzzy powerset, denoted $\tilde{\wp}$, of a set A is the set of all possible fuzzy subsets of A. Using a similar representation as before, a level-2 fuzzy region can be represented as is shown on figure 3.

3.2. Definition

Using the fuzzy powerset, it is possible to define a fuzzy region similarly as has been done with the powerset.

$$\tilde{\tilde{R}} = \{(\tilde{R}', \mu_{\tilde{\tilde{R}}}(\tilde{R}'))|\tilde{R}' \in \tilde{\wp}(\mathbb{R}^2)\} \qquad (5)$$

with the membership function is defined as:

$$\begin{aligned} \mu_{\tilde{\tilde{R}}} : \tilde{\wp}(\mathbb{R}^2) &\mapsto [0,1] \\ \tilde{R}' &\to \mu_{\tilde{\tilde{R}}}(\tilde{R}') \end{aligned}$$

The elements of the fuzzy region are fuzzy regions as in definition 1. It is also possible to define them as fuzzy regions according to definition (2). Important to note is that unlike the previous extension, now it possible for different candidate regions to share elements: regions are no longer considered as subregions, but rather as candidate regions.

In literature, a fuzzy set defined over a fuzzy domain is referred to as a level-2 fuzzy set ([12], [13]). Most commonly, and also for the level-2 fuzzy regions, the interpretation at the second level is possibilistic. This concept is not to be confused with a type-2 fuzzy set ([14]), which is a fuzzy set defined over a crisp domain but with fuzzy membership grades. In type-2 fuzzy sets, uncertainty concerning the membership grades is expressed, in level-2 fuzzy sets uncertainty concerning fuzzy possibilities is expressed. To some extent, the same could be achieved using type-2 fuzzy sets, but the concept of a candidate region would be lost (everything would be at the level of points). The concept of the candidate regions is useful, as it allows us to use spatial operations at this lower level. These have been defined for fuzzy regions in the past: [4], [15], [11]. In [6], it is shown that any statements regarding individual points of the level-2 fuzzy region will result in a type-2 fuzzy set, which is an interesting correlation between level-2 and type-2 sets.

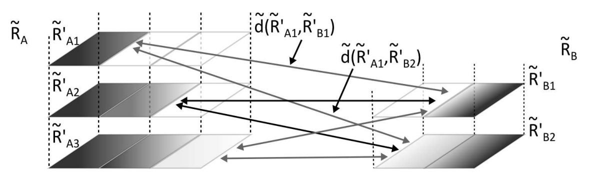

Figure 4: The distance between level-2 fuzzy region; every possibility needs to be considered.

In this contribution, the operations of distance between level-2 fuzzy regions. These have been defined in the past for normal fuzzy regions ([11])and fuzzy regions defined using the powerset ([5]). The surface area of a level-2 fuzzy regions has been considered in [16].

4. Distance between level-2 fuzzy regions

4.1. Concept

The concept of the distance between level-2 fuzzy regions is defined by means of the distances between fuzzy regions: the distances between all possible regions are determined and combined to form a single result. This is illustrated on figure 4. From the definition, a level-2 fuzzy region consists of fuzzy regions in a veristic interpretation that has an additional membership grade which is interpreted in a possibilistic way. As such, it is necessary to use the definition 3 to obtain the different distances between any two fuzzy regions. All the different candidate distances then have a possibility, depending on the possibility of the regions.

The result of the distance can be represented in different ways. The first way is obtained when combining distances, only keeping the distances with the lowest possibility. This results in a level-2 fuzzy set: a fuzzy set of fuzzy distances, each with a possibility. The second interpretation is obtained by considering the crisp distances (i.e. the distances contained inside the fuzzy distance between two fuzzy regions), and then combining the numbers at this level. This results in a fuzzy set of crisp distances, where each distance has a fuzzy membership grade.

The reason for the two different definitions is mainly semantic: the first definition is very intuitive when looking at the concept of candidate regions, but makes it more difficult to quickly view the possible distances. Level-2 fuzzy sets are also usually more difficult to work with and to perform computations on. The second definition is a type-2 fuzzy set over a real axis, which is closer to the notion that the distance should be represented as a real number (or an extension of that).

4.2. Distance between level-2 fuzzy regions with level-2 result

4.2.1. Definition

In this definition, the concept of the distance is such that it maintains the idea of the candidate regions. The result is a fuzzy set that indicates the different possibilities of all the possible distances. The distances themselves are however also fuzzy distances as the different candidate regions are fuzzy regions. First all possible (fuzzy) distances, as in definition 3 are considered. As each of these fuzzy distances is possible to the extent that the regions that yielded the distance are possible, a possibility distribution over the fuzzy distances can be defined. The distance therefore is a function of the form:

$$\begin{aligned} \tilde{\tilde{d}}^{l2} : \tilde{\tilde{\mathbb{R}}}^2 \times \tilde{\tilde{\mathbb{R}}}^2 &\to \tilde{\tilde{\mathbb{R}}} \\ (\tilde{\tilde{R}}_1, \tilde{\tilde{R}}_2) &\mapsto \tilde{\tilde{d}}^{l2}(\tilde{\tilde{R}}_1, \tilde{\tilde{R}}_2) \end{aligned}$$

Here, $\tilde{\tilde{d}}^{l2}$ is a level-2 fuzzy set defined over $\mathbb{R}$ as follows:

$$\tilde{\tilde{d}}^{l2}(\tilde{R}_1, \tilde{R}_2) = \bigcup_{\tilde{R}'_1 \in \tilde{R}_1, \tilde{R}'_2 \in \tilde{R}_2} \{(\tilde{d}(\tilde{R}'_1, \tilde{R}'_2), \min(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2)))\} \tag{6}$$

With the distance $\tilde{d}$ defined as for veristic fuzzy regions:

$$\tilde{d}(\tilde{R}'_1, \tilde{R}'_2) = \{(x, \mu_{\tilde{d}(\tilde{R}'_1, \tilde{R}'_2)}(x)) \mid x \in \mathbb{R}\} \tag{7}$$

with membership function

$$\begin{aligned} \mu_{\tilde{d}(\tilde{R}'_1, \tilde{R}'_2)} : \mathbb{R} &\to [0,1] \\ x &\mapsto \sup\{\alpha \mid d(\tilde{R}'_{1\alpha}, \tilde{R}'_{2\alpha}) \leq x \\ &\qquad \leq d(\tilde{R}'_{1\overline{\alpha}}, \tilde{R}'_{2\overline{\alpha}})\} \end{aligned}$$

This level-2 fuzzy set represents all the possible distances between the different possibilities for both fuzzy regions.

4.2.2. Compatibility

Veristic fuzzy regions To represent a fuzzy region with a veristic interpretation as a level-2 fuzzy region, it suffices to only allow a single candidate region and assign it possibility 1. The compatibility with the distance between fuzzy regions in a veristic interpretation is obvious from the construction, as the definition reverts back to the original definition.

Possibilistic fuzzy regions For fuzzy regions in a possibilistic interpretation, it is necessary to consider first how they would be represented. To represent a fuzzy region with a possibilistic interpretation as a level-2 fuzzy region, the different candidate regions are reduced to singleton sets in which the only element has membership grade 1; it also is required that no singleton set occurs more than once. On the second level, the singleton set itself is given the membership grade of the point in the original possibilistic fuzzy set. The definition for the distance between level-2 fuzzy regions can then be rewritten by representing the points rather than the singleton sets; this also allows for the distance to be simplified to the standard Euclidean distance between points:

$$\bigcup_{p_1 \in \tilde{R}1, p_2 \in \tilde{R}_2} \{(d(p_1, p_2), \min(\mu_{\tilde{R}_1}(p_1), \mu_{\tilde{R}_2}(p_2)))\} \tag{8}$$

This definition is equivalent to the original definition for the distance between fuzzy regions in a possibilistic interpretation: the original definition looks for the highest alpha level at which both points p_1 and p_2 still are present; it is obvious that

$$\begin{aligned} &\sup_{\alpha \in]0,1]} \{\alpha \mid p_1 \in \tilde{R}_{1\alpha}, p_2 \in \tilde{R}_{2\alpha}\} \\ &\qquad = \min(\mu_{\tilde{R}_1}(p_1), \mu_{\tilde{R}_2}(p_2)) \end{aligned}$$

which allows us to conclude that the definition also is compatible with fuzzy regions in a possibilistic interpretation.

4.3. Distance between level-2 fuzzy regions with type-2 result

4.3.1. Definition

As the use of level-2 fuzzy sets is less common, and makes it more difficult to defuzzify, a second definition will be introduced.

In this definition, the distance is defined as a fuzzy set over the real numeric axis. The reason for this is that — while aggregating the concept of candidate regions — the distance concept is more compatible and easier to compare to other distances, other fuzzy sets or even crisp numbers. The interpretation is that, with each distance, every occuring extent and its possibility are associated. This yields a type-2 fuzzy set; the definition is:

$$\begin{aligned} \tilde{\tilde{d}}^{t2} : \tilde{\tilde{\mathbb{R}}}^2 \times \tilde{\tilde{\mathbb{R}}}^2 &\to \tilde{\mathbb{R}} \\ (\tilde{\tilde{R}}_1, \tilde{\tilde{R}}_2) &\mapsto \tilde{\tilde{d}}^{t2}(\tilde{\tilde{R}}_1, \tilde{\tilde{R}}_2) \end{aligned}$$

$$\tilde{\tilde{d}}^{t2}(\tilde{R}_1, \tilde{R}_2) = \{(x, \tilde{\mu}_{\tilde{\tilde{d}}(\tilde{R}_1, \tilde{R}_2)}(x)) \mid x \in \mathbb{R}\} \tag{9}$$

where

$$\tilde{\mu}_{\tilde{\tilde{d}}(\tilde{R}_1, \tilde{R}_2)}(d) = \bigcup_{\tilde{R}'_1 \in \tilde{R}1, \tilde{R}'_2 \in \tilde{R}_2} \{(\mu_{\tilde{d}(\tilde{R}'_1, \tilde{R}'_2)}(d), \min(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_2)))\} \tag{10}$$

With the distance $\tilde{d}$ defined as for veristic fuzzy regions in 3.

4.3.2. Compatibility

Veristic fuzzy regions The compatibility with veristic regions in the traditional definition is verified by first considering how the result of the distance between veristic regions looks. Both regions in a level-2 representation would only have one possibility, and as such every distance would have only one possibility for every extent.

$$\begin{aligned}
&\tilde{\mu}_{\tilde{d}(\tilde{R}_1,\tilde{R}_2)}(x) \\
&= \bigcup_{\tilde{R}'_1 \in \tilde{R}1, \tilde{R}'_2 \in \tilde{R}_2} \{(\mu_{\tilde{d}(\tilde{R}'_1,\tilde{R}'_2)}(x), \\
&\qquad \min(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_2)))\} \\
&= \bigcup_{\tilde{R}'_1 \in \tilde{R}1, \tilde{R}'_2 \in \tilde{R}_2} \{(\mu_{\tilde{d}(\tilde{R}'_1,\tilde{R}'_2)}(x), 1)\}
\end{aligned}$$

As a type-2 fuzzy set in which all membership grades have only one possibility and all of those have only one membership grade, is equivalent with the normal fuzzy set, this result implies

$$\begin{aligned}
\tilde{\tilde{d}}^{t2}(\tilde{\tilde{R}}_1, \tilde{\tilde{R}}_2) &= \tilde{d}(\tilde{R}'_1, \tilde{R}'_2) \\
&= \{(x, \mu_{\tilde{d}(\tilde{R}_1,\tilde{R}_2)}(x)) \mid x \in \mathbb{R}\}
\end{aligned}$$

which carries a veristic interpretation. Applying this definition on traditional veristic fuzzy regions with level-2 representation yields a comparable result as the traditional definition for distance.

Possibilistic fuzzy regions Possibilistic fuzzy regions can be represented as a level-2 fuzzy region by considering all candidate regions as singleton sets, with the only element having membership grade 1. The singleton sets can be rewritten as points; the distance between two such regions then becomes

$$\tilde{\mu}_{\tilde{d}(\tilde{R}_1,\tilde{R}_2)}(d) = \bigcup_{p_1 \in \tilde{R}_1, p_2 \in \tilde{R}_2} \{(1, \min(\mu_{\tilde{R}_1}(p_1), \mu_{\tilde{R}_2}(p_2)))\}$$

We can convert this type-2 fuzzy set to a fuzzy set, by removing the membership grade 1 in the first instance: it adds no information as it is 1 for all elements of $\mathbb{R}$. The resulting fuzzy set in a possibilistic interpretation is the same as the initial definition for the distance between possibilistic fuzzy regions.

4.4. Relation between both definitions.

The level-2 distance representation contains all the data: all possible fuzzy distances with their possibility. The type-2 representation groups possible (crisp) distance together, even if they originate from different candidate regions. As such, the type-2 representation looses the notion of the candidate regions, and thus also of the candidate distances. The type-2 representation can be derived from the level-2 representation.

$$\begin{gathered}
\tilde{\tilde{d}}^{l2}(\tilde{R}_1, \tilde{R}_2) \\
\Updownarrow \text{(definition } \tilde{\tilde{d}}^{l2}(\tilde{R}_1, \tilde{R}_2)) \\
\bigcup_{\tilde{R}'_1 \in \tilde{R}_1, \tilde{R}'_2 \in \tilde{R}_2} (\tilde{d}(\tilde{R}'_1, \tilde{R}'_2), \min(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2))) \\
\Updownarrow \text{(definition } \tilde{d}(\tilde{R}'_1, \tilde{R}'_2)) \\
\bigcup_{\tilde{R}'_1 \in \tilde{R}_1, \tilde{R}'_2 \in \tilde{R}_2} (\{(x, \mu_{\tilde{d}(\tilde{R}'_1,\tilde{R}'_2)}(x)) \mid x \in \mathbb{R}\}, \\
\min(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2))) \\
\Downarrow \\
\{(x, \bigcup_{\tilde{R}'_1 \in \tilde{R}1, \tilde{R}'_2 \in \tilde{R}_2} \{(\mu_{\tilde{d}(\tilde{R}'_1,\tilde{R}'_2)}(d), \\
\min(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_2)))\}) \mid x \in \mathbb{R}\} \\
\Updownarrow \text{(definition } \tilde{d}(\tilde{R}'_1, \tilde{R}'_2)) \\
\{(x, \tilde{\mu}_{\tilde{d}(\tilde{R}_1,\tilde{R}_2)}(x)) \mid x \in \mathbb{R}\} \\
\Updownarrow \text{(definition } \tilde{\tilde{d}}^{t2}(\tilde{R}_1, \tilde{R}_2)) \\
\tilde{\tilde{d}}^{t2}(\tilde{R}_1, \tilde{R}_2)
\end{gathered}$$

5. Applications

The distance measurement between level-2 fuzzy regions fits in the level-2 fuzzy region model. The model is derived from the fuzzy region model, for which several operations have been defined and implemented in a prototype. The level-2 fuzzy region model so far has not been implemented, as the theoretical foundations needed to be made first. The model is quite complex, and practical representation models have already been considered in [17]. Operations can be build on top of the operations for the fuzzy region model, as most can be brought back to multiple applications of corresponding operations on the candidate fuzzy regions.

The applications of the model are in representing uncertain and imprecise spatial data. By allowing candidate fuzzy regions, the models can be used to represent multiple situations at once. One example would be the water levels of rivers and different flood regions. The different water levels can be represented by different candidate regions, each with a possibility, and each water level itself is a fuzzy region. This model can then be used to assess the areas at risk of flooding, given a change in the water level. The operations presented here can be used to provide additional measurements to provide e.g. low and high distances to the flooded region.

6. Conclusion

In this contribution, the concept of level-2 fuzzy sets was considered; this concept was first defined in [6]. It provides for a unified representation of fuzzy regions in both a veristic and a possibilistic approach. The model was developed to overcome the dependency on the metadata regarding the interpretation, to properly represent regions for which there is knowledge on how the boundaries can change and to allow for more elegant unified definitions. Both approaches are combined and the model thus allows candidate regions to be considered. In this contribution, the impact of this to the calculation of distances between fuzzy regions and the surface area of fuzzy regions was considered. The definitions for the surface area and the distance are such that they are still compatible with the original models.

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