The incidence relation of an L-Fuzzy Context as structuring element in Fuzzy Mathematical Morphology

C.Alcalde[1] A. Burusco[2] R. Fuentes-González[2]

[1]Dpt. Matemática Aplicada. Universidad del País Vasco UPV/EHU.
Plaza de Europa 1, 20018 San Sebastián. Spain.
Email: c.alcalde@ehu.es
[2]Dpt. Automática y Computación. Universidad Pública de Navarra.
Campus de Arrosadía, 31006 Pamplona. Spain.
Email: {burusco,rfuentes}@unavarra.es

Abstract

The goal of this work is to prove a link between the Fuzzy Mathematical Morphology and the L-fuzzy Concept Analysis when we are using structuring relations which represent the effect that we want to produce over an initial fuzzy image or signal. In this case, we prove that the problem of obtaining the L-fuzzy concepts of an L-fuzzy context is equivalent to the problem of finding fuzzy images or signals that remain invariant under a fuzzy morphological opening or closing.

Keywords: L-fuzzy Concept Analysis, Fuzzy Mathematical Morphology, Formal Concept Analysis, Morphological Image Processing.

1. Introduction

The L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology were developed in different contexts but in both cases the lattice theory is used as algebraic framework.

In the case of the L-fuzzy Concept Analysis, we define the L-fuzzy concepts using a fuzzy implication and a composition operator associated with it. In the Fuzzy Mathematical Morphology, a fuzzy implication is also used to define the erosion operator but a t-norm also appears to introduce the dilation operator.

On the other hand, both theories have been used in knowledge extraction processes in data bases [17, 18, 19].

Recently, an interesting relation between L-fuzzy concept lattices and fuzzy mathematical morphology has been introduced in [1]. In this paper we have extended this relation to the case of working with any structuring relation $R \in L^{X\times X}$ which is not necessarily obtained from a structuring image.

Next, we will show a brief description of these two theories.

2. Preliminary theories

2.1. L-fuzzy Concept Analysis

The Formal Concept Analysis was introduced by R. Wille [30, 20] and, based on order theory and complete lattices, tries to process knowledge and represent the conceptual structures of a data set.

A formal context is defined as a triple (X, Y, R) where X and Y are two finite sets of objects and attributes respectively and $R \subseteq X \times Y$ is a binary relation defined among them. The hidden information is obtained by means of the formal concepts that are pairs (A, B) with $A \subseteq X$, $B \subseteq Y$ verifying $A^\star = B$ and $B^\star = A$, where $\star$ is the derivation operator that assigns to every object set A the attributes related to the elements of A, and to every attribute set B the objects related to the attributes of B. These formal concepts can be interpreted as a group of objects A sharing a group of attributes B. The set A is said to be the extension of the concept and the set B is the intension.

The set of concepts derived from a formal context (X, Y, R) is a complete lattice and it is usually represented by a line diagram.

The first extension of the Formal Concept Analysis to the fuzzy field, the L-fuzzy Concept Analysis, is due to Burusco and Fuentes-González [14, 15], and its fundamental structures are fuzzy Galois connections and fuzzy closure operators [6, 7]. This theory takes as starting point an L-fuzzy context (L, X, Y, R), where L is a complete lattice, X and Y are sets of objects and attributes respectively and $R \in L^{X\times Y}$ is a fuzzy relation that represents the relationship between the elements of X and Y, taking values in the complete lattice L.

To work with these L-fuzzy contexts, the derivation operators 1 and 2 are defined by means of the following expressions:

$$\forall A \in L^X, \forall B \in L^Y \quad A_1(y) = \inf_{x\in X}\{I(A(x), R(x, y))\}$$
$$B_2(x) = \inf_{y\in Y}\{I(B(y), R(x, y))\}$$

with I a fuzzy implication operator defined in the lattice $(L, \leq)$, which is decreasing in its first argument, and where A_1 represents the set of all the attributes related to the objects of A in a fuzzy way, and B_2, the objects related to all the attributes of B.

In this work, we are going to use the following notation for these derivation operators to stand out their dependence to relation R:

$\forall A \in L^X, \forall B \in L^Y$, we define the operators $\mathcal{D}_R : L^X \to L^Y$ and $\mathcal{D}_{R^{op}} : L^Y \to L^X$ as follows:

$$\mathcal{D}_R(A)(y) = A_1(y) = \inf_{x \in X}\{I(A(x), R(x,y))\}$$
$$\mathcal{D}_{R^{op}}(B)(x) = B_2(x) = \inf_{y \in Y}\{I(B(y), R^{op}(y,x))\}$$

where we denote by R^{op} the opposite relation of R, that is, $\forall (x,y) \in X \times Y$, $R^{op}(y,x) = R(x,y)$.

The derivation operators are used to define the constructor operators $\varphi : L^X \to L^X$ with $\varphi(A) = (A_1)_2 = A_{12}$ and $\psi : L^Y \to L^Y$ such that $\psi(B) = (B_2)_1 = B_{21}$.

The information stored in the context is visualized by means of the L-fuzzy concepts that are the pairs $(A, A_1) \in (L^X, L^Y)$ with $A \in \text{fix}(\varphi)$, set of fixed points of the operator φ. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a set of objects that share a set of attributes in a fuzzy way.

The set $\mathcal{L} = \{(A, A_1)/A \in \text{fix}(\varphi)\}$ with the order relation $\leq$ defined as:

$$(A, A_1) \leq (C, C_1) \text{ if } A \leq C \quad (\text{or } A_1 \geq C_1)$$

is a complete lattice that is said to be the L-fuzzy concept lattice [14, 15].

The calculation of the L-fuzzy concept lattice is reduced, as we mentioned above, to calculate the set of fixed points of the constructor operator φ. If the implication I used in the definition of the derivation operators is a residuated implication, then the constructor operators φ and ψ are closure operators. Therefore, the use of residuated implications will greatly facilitate the calculation of the L-fuzzy concepts because $\forall A \in L^X$, the pair (A_{12}, A_1) is an L-fuzzy concept of the L-fuzzy context.

Other important results about this theory are in [2, 13, 27, 16, 26, 8].

A very interesting particular case of L-fuzzy contexts appears trying to analyze situations where the object and the attribute sets are coincident [3, 4], that is, L-fuzzy contexts (L, X, X, R) with $R \in L^{X \times X}$. In these situations, the L-fuzzy concepts are pairs (A, B) such that $A, B \in L^X$.

These are the L-fuzzy contexts that we are going to use to obtain the main results of this work. Specifically, we are going to take a complete chain $(L, \leq)$ as the valuation set. Moreover, in the case of the use of the L-fuzzy contexts $(L, \mathbb{R}^n, \mathbb{R}^n, R)$ or $(L, \mathbb{Z}^n, \mathbb{Z}^n, R)$, the L-fuzzy concepts (A, B) are interpreted as signal or image pairs or digital versions of these signals or images, respectively. These L-fuzzy contexts play an important role in our paper.

2.2. Mathematical Morphology

The Mathematical Morphology is a theory concerned with the processing and analysis of images or signals using filters and operators that modify them. The foundations of this theory (initiated by G. Matheron [24, 25] and J. Serra [28]), are in set theory, integral geometry and e lattice algebra. Actually, this methodology is used in general contexts related to activities as information extraction in digital images, noise elimination or pattern recognition.

2.2.1. Mathematical Morphology in binary images and grey level images

In this theory, binary images A from $X = \mathbb{R}^n$ or $X = \mathbb{Z}^n$ (digital images or signals when n=1) are analyzed.

The *morphological filters* are defined as operators $F : \wp(X) \to \wp(X)$ that transform, simplify, clean or extract relevant information from these images $A \subseteq X$, information that is encapsulated by the filtered image $F(A) \subseteq X$.

These morphological filters are obtained by means of two basic operators, the *dilation* δ_S and the *erosion* ε_S, that are defined in the case of binary images with the *sum* and *difference of Minkowski* [28], respectively.

$$\delta_S(A) = A \oplus S = \bigcup_{s \in S} A_s$$
$$\varepsilon_S(A) = A \ominus \breve{S} = \bigcap_{s \in \breve{S}} A_s$$

where A is an image that is treated with another $S \subseteq X$, that is said to be *structuring element*, or with its opposite $\breve{S} = \{-x/x \in S\}$ and where A_s represents a translation of A: $A_s = \{a + s/a \in A\}$.

The structuring image S represents the effect that we want to produce over the initial image A.

These operators are not independent since they are dual transformations with respect to the complementation [29], that is, if A^c represents the complementary set of A, then:

$$\varepsilon_S(A) = (\delta_{\breve{S}}(A^c))^c, \forall A, S \in \wp(X)$$

We can compose these dilation and erosion operators associated with the structuring element S and obtain the basic filters *morphological opening* $\gamma_S : \wp(X) \to \wp(X)$ and *morphological closing* $\phi_S : \wp(X) \to \wp(X)$ defined by:

$$\gamma_S = \delta_S \circ \varepsilon_S \quad \phi_S = \varepsilon_S \circ \delta_S$$

The opening γ_S and the closing ϕ_S over these binary images verify the two conditions that characterize the morphological filters: They are isotone and idempotent operators, and moreover it is verified, $\forall A, S \in \wp(X)$:

i) $\gamma_S(A) \subseteq A \subseteq \phi_S(A)$

ii) $\gamma_S(A) = (\phi_S(A^c))^c$

These operators will characterize some special images (the S-open and the S-closed ones) that will play an important role in this work.

This theory is generalized introducing some tools to treat images with grey levels [28]. The images A and the structuring elements S are now maps defined in $X = \mathbb{R}^n$ and with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ or defined in $X = \mathbb{Z}^n$ and with values in finite chains as, for instance, $\{0, 1, \dots, 255\}$. Now, the erosion and dilation can be defined as follows:

$$\varepsilon_S(A)(x) = \inf\{A(y) - S(y - x)/y \in X\}$$
$$\delta_S(A)(x) = \sup\{A(y) + S(x - y)/y \in X\}$$

The previous definitions can be embedded in a more general framework that considers each image as a point $x \in L$ of a partially ordered structure $(L, \leq)$ (complete lattice), and the filters as operators $F : L \to L$ with properties related to the order in these lattices [28, 22].

Now, the erosions $\varepsilon : L \to L$ are operators that preserve the infimum $\varepsilon(\inf M) = \inf \varepsilon(M), \forall M \subseteq L$ and the dilations $\delta : L \to L$, the supremum: $\delta(\sup M) = \sup \delta(M), \forall M \subseteq L$. The openings $\gamma : L \longrightarrow L$ and the closings $\phi : L \longrightarrow L$ are isotone and idempotent operators verifying $\gamma(x) \leq x \leq \phi(x), \forall x \in L$.

2.2.2. Fuzzy Mathematical Morphology

In this new framework and associated with lattices, a new *fuzzy morphological image processing* has been developed [9, 10, 5, 11, 12, 23] using L-fuzzy sets A and S as images and structuring elements.

In this interpretation, the filters are operators $F_S : L^X \to L^X$, where L is the chain $L = [0, 1]$ or a finite chain $L = L_k = \{0 = \alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k = 1\}$ with $0 < \alpha_1 < ... < \alpha_{k-1} < 1$, and the set $X = \mathbb{R}^2$ or $X = \mathbb{Z}^2$.

In all these cases, *fuzzy morphological dilations* $\delta_S : L^X \to L^X$ and *fuzzy morphological erosions* $\varepsilon_S : L^X \to L^X$ are defined using some operators of the fuzzy logic [5, 9, 12].

In general, there are two types of relevant operators in the Fuzzy Mathematical Morphology. One of them is formed by those obtained by using some pairs $(*, I)$ of adjunct operators related by:

$$(\alpha * \beta \leq \psi) \Longleftrightarrow (\beta \leq I(\alpha, \psi))$$

The other type are the morphological operators obtained by pairs $(*, I)$ related by a strong negation $' : L \to L$:

$$\alpha * \beta = (I(\alpha, \beta'))', \forall(\alpha, \beta) \in L \times L$$

An example of one of these pairs that belongs to both types is the formed by the t-norm and the implication of Lukasiewicz.

In this paper, we will work taking the commutative group $(\mathbb{R}^n, +)$ or $(\mathbb{Z}^n, +)$ as $(X, +)$, and the complete chain $L = [0, 1]$ or a finite chain as $L = L_k = \{0 = \alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k = 1\}$ as $(L, \leq, ', I, *)$, with the Zadeh negation and $(*, I)$ the Lukasiewicz t-norm and implication:

$$a * b = \max(0, a + b - 1), \forall a, b \in L$$
$$I(a, b) = \min(1, 1 - a + b), \forall a, b \in L$$

We interpret the L-fuzzy sets $A : X \to L$ and $S : X \to L$ as n-dimensional images in the space $X = \mathbb{R}^n$ (or n-dimensional digital images in the case of $X = \mathbb{Z}^n$).

In the literature, (see [5, 9, 21]), fuzzy erosion and dilation operators are introduced associated with the residuated pair $(*, I)$ as follows:

If $S : X \to L$ is an image that we take as *structuring element*, then we consider the following definitions associated with (L, X, S) :

Definition 1 *[9] The fuzzy erosion of the image $A \in L^X$ by the structuring element S is the L-fuzzy set $\varepsilon_S(A) \in L^X$ defined as:*

$$\varepsilon_S(A)(x) = \inf\{I(S(y-x), A(y))/y \in X\} \qquad \forall x \in X$$

The fuzzy dilation of the image A by the structuring element S is the L-fuzzy set $\delta_S(A)$ defined as:

$$\delta_S(A)(x) = \sup\{S(x - y) * A(y)/y \in X\} \qquad \forall x \in X$$

Then, we obtain fuzzy erosion and dilation operators $\varepsilon_S, \delta_S : L^X \to L^X$.

3. Structuring relations

We are interested in the application of this theory to the more general contexts without the use of commutative group structures, laws as sums or differences, and translation operators. In this way, we are going to use structuring relations $R \in L^{X \times X}$ that represent different effects that we want to produce over an initial fuzzy image or signal $A \in L^X$.

In this new framework, we can redefine the fuzzy erosion and dilation associated with the pair $(*, I)$ as follows: $\forall x \in X$,

$$\begin{aligned}\varepsilon_R(A)(x) &= \inf\{I(R(y, x), A(y))/y \in X\} \\ &= \inf\{I(R^{op}(x, y), A(y))/y \in X\} \\ \delta_R(A)(x) &= \sup\{R(x, y) * A(y)/y \in X\}\end{aligned}$$

We remark that, in this paper, we are using the Lukasiewicz operators.

We have the following result in this case:

Proposition 1 *If A' is the negation of A defined by $A'(x) = (A(x))', \forall x \in X$ and if $R \in L^{X \times X}$ represents the structuring relation, then it is verified:*

$$\varepsilon_R(A') = (\delta_{R^{op}}(A))'$$
$$\delta_R(A') = (\varepsilon_{R^{op}}(A))'$$

Proof: Let be $x \in X$.

$$\begin{aligned}\varepsilon_S(A')(x) &= \inf\{I(R(y,x), A'(y))/y \in X\}\\ &= \inf\{I(R^{op}(x,y), A'(y))/y \in X\}\\ &= \inf\{(R^{op}(x,y) * A(y))'/y \in X\}\\ &= (\sup\{R^{op}(x,y) * A(y)/y \in X\})'\\ &= (\delta_{R^{op}}(A)(x))' = (\delta_{R^{op}}(A))'(x)\end{aligned}$$

The second equality is proved analogously.

□

Using these fuzzy erosion and dilation operators, we are going to define the basic morphological filters: the fuzzy opening and the fuzzy closing (see [5, 9, 21]) for an structuring relation.

Definition 2 *The fuzzy opening of the image $A \in L^X$ by the structuring relation $R \in L^{X \times X}$ is the fuzzy subset $\gamma_R(A)$ that results from the composition of the fuzzy erosion $\varepsilon_R(A)$ of A by R followed by its fuzzy dilation:*

$$\gamma_R(A) = \delta_R(\varepsilon_R(A)) = (\delta_R \circ \varepsilon_R)(A)$$

The fuzzy closing of the image $A \in L^X$ by the structuring element $R \in L^{X \times X}$ is the fuzzy subset $\phi_R(A)$ that results from the composition of the fuzzy dilation $\delta_R(A)$ of A by R followed by its fuzzy erosion:

$$\phi_R(A) = \varepsilon_R(\delta_R(A)) = (\varepsilon_R \circ \delta_R)(A)$$

It can be proved that the operators γ_R and ϕ_R are morphological filters, that is, they preserve the order and they are idempotent, that is,
$\forall A, A_1, A_2 \in L^X, \forall R \in L^{X \times X}$:

i) $A_1 \leq A_2 \Longrightarrow \gamma_R(A_1) \leq \gamma_R(A_2)$
ii) $A_1 \leq A_2 \Longrightarrow \phi_R(A_1) \leq \phi_R(A_2)$
iii) $\gamma_R(\gamma_R(A)) = \gamma_R(A)$
iv) $\phi_R(\phi_R(A)) = \phi_R(A)$

Moreover, these filters verify that:

$$\gamma_R(A) \leq A \leq \phi_R(A) \quad \forall A \in L^X, \forall R \in L^{X \times X}$$

Analogous results to those obtained for the fuzzy erosion and dilation operators can be proved for the fuzzy opening and fuzzy closing:

Proposition 2 *If A' is the negation of A defined by $A'(x) = (A(x))' \quad \forall x \in X$, then, $\forall A \in L^X, \forall R \in L^{X \times X}$,*

$$\gamma_R(A') = (\phi_{R^{op}}(A))'$$
$$\phi_R(A') = (\gamma_{R^{op}}(A))'$$

Proof: By proposition 1,

$$\begin{aligned}\gamma_R(A') &= \delta_R(\varepsilon_R(A')) = \delta_R((\delta_{R^{op}}(A))')\\ &= (\varepsilon_{R^{op}}(\delta_{R^{op}}(A)))' = (\phi_{R^{op}}(A))'\end{aligned}$$

The other equality can be proved in an analogous way.

□

Since the operators γ_R and ϕ_R are increasing in the complete lattice $(L^X, \leq)$, the respective fixed points sets are complete lattices (by Tarski's theorem) and therefore, non empty sets. These fixed points will be used in the following definition:

Definition 3 *An image $A \in L^X$ is said to be R-open if $\gamma_R(A) = A$ and it is said to be R-closed if $\phi_R(A) = A$.*

These R-open and R-closed sets provide a connection between the Fuzzy Mathematical Morphology and the L-fuzzy Concept Analysis, as we will see next.

4. Relation between both theories

Given the structuring relation $R \in L^{X \times X}$ and its strong negation $R' \in L^{X \times X}$, we can associate with the triple (L, X, R) an L-fuzzy context (L, X, X, R') where the sets of objects and attributes are coincident.

We will use this representation to prove the most important results that connect both theories:

Theorem 1 *Let (L, X, R) be the triple associated with the structuring relation $R \in L^{X \times X}$. Let (L, X, X, R') be the L-fuzzy context whose incidence relation $R' \in L^{X \times X}$ is the strong negation of the structuring relation R. Then the fuzzy erosion ε_R and fuzzy dilation δ_R operators en (L, X, R) are related to the derivation operators $\mathcal{D}_{R'}$ and $\mathcal{D}_{R'^{op}}$ in the L-Fuzzy context (L, X, X, R') by:*

$$\varepsilon_R(A) = \mathcal{D}_{R'}(A') \quad \forall A \in L^X$$
$$\delta_R(A) = (\mathcal{D}_{R'^{op}}(A))' \quad \forall A \in L^X$$

Proof: Taking into account the properties of the Lukasiewicz implication operator, for any $x \in X$, it is verified that:

$$\begin{aligned}\varepsilon_R(A)(x) &= \inf\{I(R(y,x), A(y))/y \in X\}\\ &= \inf\{I(A'(y), R'(y,x))/y \in X\} = \mathcal{D}_{R'}(A')(x)\end{aligned}$$

Analogously,

$$\begin{aligned}\delta_R(A)(x) &= \sup\{R(x,y) * A(y)/y \in X\}\\ &= \sup\{(I(R(x,y), A'(y)))'/y \in X\}\\ &= (\inf\{I(R(x,y), A'(y))/y \in X\})'\\ &= (\inf\{I(A(y), R'(x,y))/y \in X\})'\\ &= (\inf\{I(A(y), R'^{op}(y,x))/y \in X\})'\\ &= ((\mathcal{D}_{R'^{op}}(A))(x))' = (\mathcal{D}_{R'^{op}}(A))'(x)\end{aligned}$$

□

As a consequence, we obtain the following result which proves the connection between the outstanding morphological elements and the L-fuzzy concepts:

Theorem 2 *Let us consider the structuring relation $R \in L^{X \times X}$. The following propositions are equivalent:*

1. *The pair $(A, B) \in L^X \times L^X$ is an L-fuzzy concept of the context (L, X, X, R') where R' is the negation of the structuring relation R.*
2. *The pair $(A, B) \in L^X \times L^X$ is such that the negation A' of A is R-open ($\gamma_R(A') = A'$) and B is the fuzzy erosion of A' (that is, $B = \varepsilon_R(A')$).*
3. *The pair $(A, B) \in L^X \times L^X$ is such that B is R-closed ($\phi_R(B) = B$) and A is the negation of the fuzzy dilation of B (that is, $A = (\delta_R(B))'$).*

Proof:

1 $\Longrightarrow$ 2) Let be $R \in L^{X \times X}$ the structuring relation. Let us consider an L-fuzzy concept (A, B) of the L-fuzzy context (L, X, X, R'). By the definition of L-fuzzy concept, it is verified that $B = \mathcal{D}_{R'}(A)$ and $A = \mathcal{D}_{R'^{op}}(B)$, and, applying the previous theorem,

$$\varepsilon_R(A') = \mathcal{D}_{R'}(A) = B.$$

Moreover, it is fulfilled that

$$\gamma_R(A') = \delta_R(\varepsilon_R(A')) = \delta_R(B) = (\mathcal{D}_{R'^{op}}(B))' = \\ = A'$$

which proves that A' is R-open.

2 $\Longrightarrow$ 3) Let us suppose that the pair $(A, B) \in L^X \times L^X$ is such that $\gamma_R(A') = A'$ and $B = \varepsilon_R(A')$). Then,

$$\phi_R(B) = \varepsilon_R(\delta_R(B)) = \varepsilon_R(\delta_R(\varepsilon_R(A'))) \\ = \varepsilon_R(\gamma_R(A')) = \varepsilon_R(A') = B$$

which proves that B is R-closed.
On the other hand, from the hypothesis $B = \varepsilon_R(A')$ can be deduced that

$$\delta_R(B) = \delta_R(\varepsilon_R(A')) = \gamma_R(A')$$

and consequently, taking into account that A' is R-open, that

$$\delta_R(B) = A'$$

and finally,

$$A = (\delta_R(B))'$$

3 $\Longrightarrow$ 1) Let (A, B) be a pair fulfilling that $\phi_R(B) = B$ and $A = (\delta_R(B))'$. Let us consider the L-fuzzy context (L, X, X, R'). Then, by the previous theorem we can deduce that

$$(\mathcal{D}_{R'^{op}}(B)) = (\delta_R(B))' = A$$

On the other hand, applying the previous theorem and the hypothesis,

$$\mathcal{D}_{R'}(A) = \varepsilon_R(A') = \varepsilon_R(\delta_R(B)) = \phi_R(B) = B$$

therefore, as B is the derived set of A, the pair (A, B) is an L-fuzzy concept of the L-fuzzy context (L, X, X, R').

□

Let us see now some examples.

Example 1 Interpretation of some binary images as formal concepts.

Let us consider the binary image represented in Figure 1.

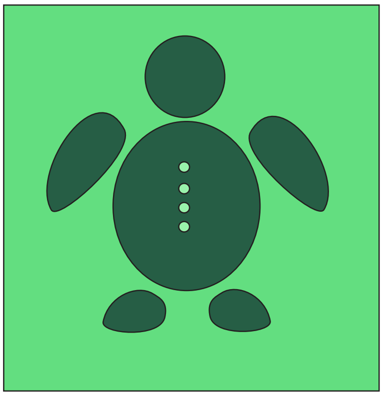

Figure 1: Binary image

Let us suppose the referential set $X = \mathbb{R}^2$ and w a fixed positive number. We will define the structuring relation $R \subset \mathbb{R}^2 \times \mathbb{R}^2$ as:

$$\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$$
$$(x_1, x_2) R (y_1, y_2) \Longleftrightarrow ((x_1 - y_1)^2 + (x_2 - y_2)^2 \leq w^2)$$

By the results of Theorem 2, the pair (A, B) showed in Figure 2 and related to Figure 1 is a formal concept of the formal context $(\mathbb{R}^2, \mathbb{R}^2, R')$, because $\gamma_R(A') = A'$ and $B = \varepsilon_R(A')$ is fulfilled.

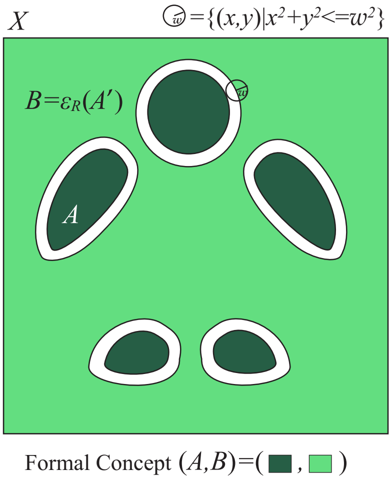

Figure 2: A formal concept of the context $(\mathbb{R}^2, \mathbb{R}^2, R')$

However, if we take now the set D represented in Figure 3,

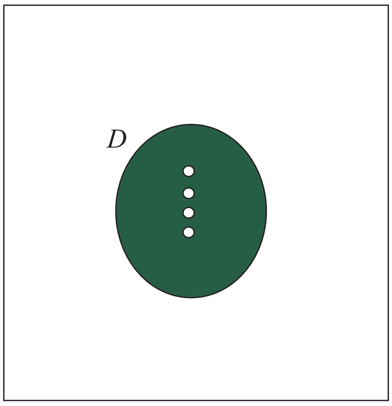

Figure 3: Image defined by set D.

calculating the fuzzy opening, we can verify that $\gamma_R(D') \neq D'$. That is, D' is not an R-open set (see Figure 4).

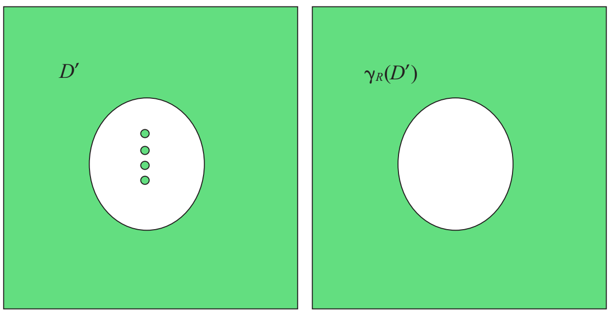

Figure 4: D' is not an R-open set.

Therefore, there is no formal concept in the formal context $(\mathbb{R}^2, \mathbb{R}^2, R')$ which extension is the set D.

Example 2 Interpretation of some open digital signals as L-fuzzy concepts.

If $X \subseteq \mathbb{Z}$ and $L = \{0, 0.1, 0.2, ..., 0.9, 1\}$ then, the maps $A : X \to L$ can be interpreted as 1-D discrete signals.

Let us consider the referential set $X = \{0, 1, 2, \ldots, 20\}$ and the discrete signal represented by the L-fuzzy set A (see Figure 5):

$$A = \{0/0.3, 1/0.3, 2/0.3, 3/0.6, 4/0.6, 5/0.5, 6/0, 7/0, 8/0, 9/0, 10/0, 11/0, 12/0.5, 13/0.8, 14/1, 15/1, 16/0.5, 17/0.5, 18/0.5, 19/0.5, 20/0.9\}$$

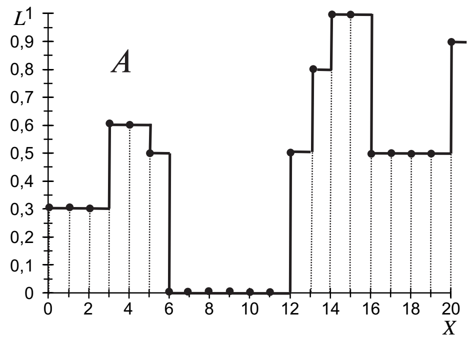

Figure 5: Discrete signal A as an L-Fuzzy set

The structuring relation $R \in L^{X \times X}$ is defined as:

$$R(x, y) = \begin{cases} 0 & \text{if } |x - y| > 2 \\ 0.5 & \text{if } 1 < |x - y| \leq 2 \\ 1 & \text{if } |x - y| \leq 1 \end{cases}$$

Let us consider now the negation of the signal represented by the L-fuzzy set A (see Figure 6):

$$A' = \{0/0.7, 1/0.7, 2/0.7, 3/0.4, 4/0.4, 5/0.5, 6/1, 7/1, 8/1, 9/1, 10/1, 11/1, 12/0.5, 13/0.2, 14/0, 15/0, 16/0.5, 17/0.5, 18/0.5, 19/0.5, 20/0.1\}$$

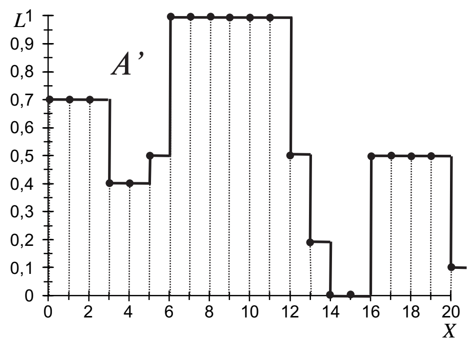

Figure 6: Negation of the discrete signal A

and let us calculate the fuzzy erosion of the negation of the signal. The obtained signal B is showed in Figure 7.

$$B = \varepsilon_R(A') = \{0/0.7, 1/0.7, 2/0.4, 3/0.4, 4/0.4, 5/0.4, 6/0.5, 7/1, 8/1, 9/1, 10/1, 11/0.5, 12/0.2, 13/0, 14/0, 15/0, 16/0, 17/0.5, 18/0.5, 19/0.1, 20/0.1\}$$

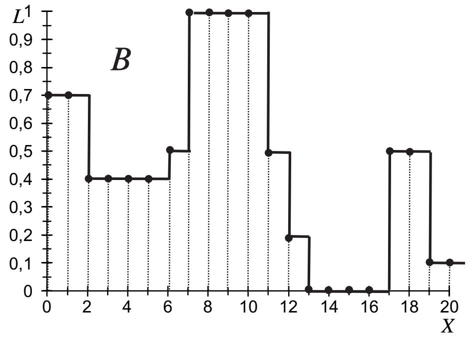

Figure 7: Discrete signal $B = \varepsilon_R(A')$

If we take now this last signal and calculate its fuzzy dilation, we can see that

$$\gamma_R(A') = \delta_R(\varepsilon_R(A')) = A'.$$

Then, the set A' is an R-open set.

Therefore, by the Theorem 2, the pair $(A, \varepsilon_R(A'))$ represented in Figure 8 is an L-fuzzy concept of the associated L-fuzzy context (L, X, X, R').

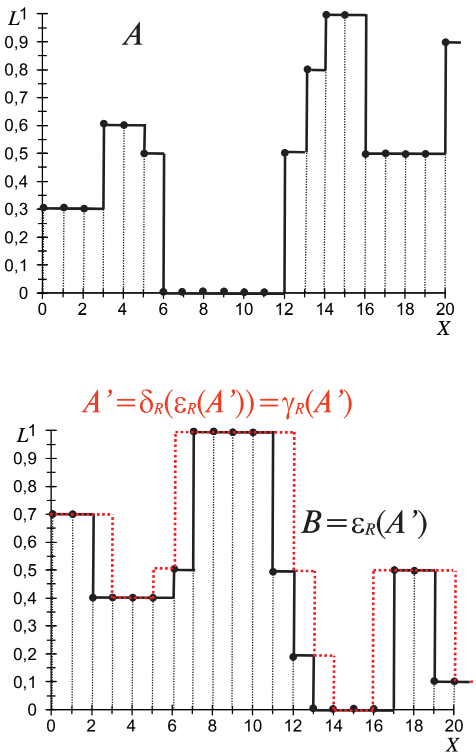

Figure 8: The pair (A, B) is an L-Fuzzy concept of the L-Fuzzy context (L, X, X, R')

5. Conclusions and Future work

The main results of this paper show an interesting relation between the L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology. So, we can apply the algorithms for the calculus of L-fuzzy concepts in Fuzzy Mathematical Morphology and vice versa.

In future works, we want to extend these results to other type of operators as other implications, t-norms, conjunctive uninorms etc... and to some L-fuzzy contexts where the objects and the attributes are not related to signal or images. Moreover, we will study the use of complete lattices different from the chains considered in this paper.

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