

Type-2 aggregation operators

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Abstract

The paper deals with an extension of aggregation operators from the set of real numbers (or interval $[0, 1]$) to the set of fuzzy truth values (fuzzy sets in $[0, 1]$). We define so-called type-2 aggregation operator and show that an extension of ordinary aggregation operator by convolution is a type-2 aggregation operator. Finally we show that ordinary aggregation operator, as well as aggregation operator for intervals and for n -dimensional intervals are special cases of our type-2 aggregation operator.

Keywords: Aggregation operator, Fuzzy truth values, Type-2 fuzzy sets, Type-2 aggregation operator, n -Dimensional fuzzy set, Fuzzy multiset

1. Introduction

At some point, aggregation plays a fundamental role in all kinds of knowledge based systems [1], [2]. The theory of aggregation of real numbers is well established (see e.g. [3], [4], [5]) and is applied in fuzzy logic systems based on (type-1) fuzzy sets.

Aggregation operators for real numbers were extended to the aggregation operators for intervals ([6], [7]). These generalized aggregation operators are applicable in systems based on the interval-valued fuzzy sets [8], Atanassov's intuitionistic fuzzy sets [9] and interval type-2 fuzzy sets [10]. Recently, Shang et al. [11] generalized the concept of interval-valued fuzzy sets to n -dimensional fuzzy sets (also called fuzzy multisets) and Bedregal et al. [12] proposed aggregation operator for n -dimensional intervals. Note that the n -dimensional intervals are the membership grades of n -dimensional fuzzy sets.

Zadeh [13] introduced the concept of type-2 fuzzy sets as an extension of type-1 fuzzy sets. The membership grades of type-2 fuzzy sets are type-1 fuzzy sets in $[0, 1]$ (we will refer to as fuzzy truth values). The type-2 fuzzy sets are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set [14]. This makes them to be an attractive tool in many real problems. However, there is no theory allowing us to aggregate fuzzy truth values. This is one of several obstacles for applicability of the systems based on type-2 fuzzy sets. Our goal is to overcome this lack of knowledge.

This article is a first announcement of our research activity concerning with aggregation of fuzzy

truth values. The aim of this paper is to propose an aggregation operator for fuzzy truth values (type-2 aggregation operator), and to provide a theoretical basis for the concept of type-2 aggregation operator. Moreover, we show that usual definition of (ordinary) aggregation operator, the definition of aggregation operator for intervals [6], and also the definition of aggregation operator for n -dimensional intervals [12] are special cases of our definition of type-2 aggregation operator.

Finally, we show that an n -dimensional fuzzy set can be interpreted as a class of α -cuts of some fuzzy truth value. Thus, membership grades of type-2 fuzzy sets can be approximated via n -dimensional fuzzy sets. Note that this approach corresponds to α -plane representation [15] and zSlice representation [16] of type-2 fuzzy sets.

The paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. Section 3 presents the extension of (ordinary) aggregation operator via convolution. In Section 4, we provide the axiomatic basis for type-2 aggregation operator and show that the proposed extension satisfies stated axioms. In Section 5 we show that ordinary aggregation operator, as well as aggregation operator for intervals and for n -dimensional intervals are special cases of our type-2 aggregation operator. The conclusions are discussed in Section 6.

2. Preliminaries

In this section we present some basic concepts and terminology that will be used throughout the paper.

A mapping $f : X \rightarrow [0, 1]$ is called a fuzzy set (or type-1 fuzzy set) in a set X , the value $f(x)$ is called a membership grade of x . A fuzzy set f in X is convex if for all $\lambda \in [0, 1]$ holds $f(\lambda x_1 + (1 - \lambda)x_2) \geq \min(f(x_1), f(x_2))$ where x_1, x_2 are arbitrary elements of X . A fuzzy set f in X is normal if there exists $x \in X$ such that $f(x) = 1$. A crisp set $Ker(f) = \{x \in X \mid f(x) = 1\}$ is called a kernel of f . A crisp set $f_\alpha = \{x \in X \mid f(x) \geq \alpha\}$, where $\alpha \in]0, 1]$ is called an α -cut of f .

Definition 1 A function $A : [0, 1]^n \rightarrow [0, 1]$ is called a type-1 aggregation operator on $[0, 1]$ if and only if it satisfies the conditions:

- (A1) $A(0, \dots, 0) = 0$;
- (A2) $A(1, \dots, 1) = 1$;

(A3) $x_1 \leq y_1, \dots, x_n \leq y_n$ implies $A(x_1, \dots, x_n) \leq A(y_1, \dots, y_n)$.

for all $x_1, y_1, \dots, x_n, y_n \in [0, 1]$.

A type-2 fuzzy set in X is a fuzzy set whose membership grades are type-1 fuzzy sets in $[0, 1]$. Let \mathcal{F} denotes a class of all type-1 fuzzy sets in $[0, 1]$. Then type-2 fuzzy set in X is a mapping $\tilde{f} : X \rightarrow \mathcal{F}$ and elements of \mathcal{F} are called fuzzy truth values. We denote by $\mathcal{F}_N / \mathcal{F}_C / \mathcal{F}_{NC}$ a class of all normal / convex / normal convex fuzzy truth values, respectively. The algebra of fuzzy truth values $\mathbf{F} = (\mathcal{F}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$ is closely describe in [17] and [18], whereas

$$(f \sqcup g)(z) = \sup_{x \vee y = z} (f(x) \wedge g(y)),$$

$$(f \sqcap g)(z) = \sup_{x \wedge y = z} (f(x) \wedge g(y)),$$

$$f \sqsubseteq g \quad \text{iff} \quad f \sqcap g = f,$$

$$f \preceq g \quad \text{iff} \quad f \sqcup g = g,$$

$$\mathbf{0}(x) = \begin{cases} 1 & , \text{if } x = 0, \\ 0 & , \text{otherwise,} \end{cases}$$

$$\mathbf{1}(x) = \begin{cases} 1 & , \text{if } x = 1, \\ 0 & , \text{otherwise.} \end{cases}$$

Let $f \in \mathcal{F}$. Then unary operations

$$f^L(x) = \sup_{y \leq x} f(y) \quad \text{and} \quad f^R(x) = \sup_{y \geq x} f(y).$$

enable us to express the operations \sqcup, \sqcap and relations \sqsubseteq, \preceq pointwise (see [19], [20], [18]):

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L),$$

$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R),$$

$$f \sqsubseteq g \quad \text{iff} \quad f^R \wedge g \leq f \leq g^R,$$

$$f \preceq g \quad \text{iff} \quad f \wedge g^L \leq g \leq f^L.$$

Moreover, a fuzzy truth value f is convex if and only if $f = f^L \wedge f^R$ (Proposition 33 in [18]).

3. Extension of type-1 aggregation operators

We extend a type-1 aggregation operator. Afterwards we show that under some conditions the extended type-1 aggregation operator preserves normality and convexity of fuzzy truth values.

According to Zadeh's extension principle [13] n -ary type-1 aggregation operator $A : [0, 1]^n \rightarrow [0, 1]$ can be extended by the convolution with respect to minimum \wedge and maximum \vee to n -ary operator $\tilde{A} : \mathcal{F}^n \rightarrow \mathcal{F}$ as follows:

$$\begin{aligned} \tilde{A}(f_1, \dots, f_n)(y) &= \\ &= \sup_{A(x_1, \dots, x_n) = y} (f_1(x_1) \wedge \dots \wedge f_n(x_n)), \end{aligned} \quad (1)$$

where $y, x_1, \dots, x_n \in [0, 1]$ and $f_1, \dots, f_n \in \mathcal{F}$.

Our approach is a generalization of some extensions of t-norms and t-conorms proposed in [18], [21] and [22]. Different approach to the subject used Zhou et al. [23] and proposed so-called type-1 OWA operators (although in our opinion the more fitting name should be 'type-2 OWA operator' and the term 'type-1 OWA operator' should stand for ordinary OWA operator).

Example 1. An extension of arithmetic mean $A(x_1, \dots, x_n) = \sum_{i=1}^n x_i / n$ is the following aggregation operator for fuzzy truth values:

$$\tilde{A}(f_1, \dots, f_n)(y) = \sup_{\sum_{i=1}^n x_i = y} (f_1(x_1) \wedge \dots \wedge f_n(x_n)),$$

Let $n = 2$ and let f_1, f_2 be fuzzy truth values with trapezoidal shapes given by $f_1 = (0.1, 0.3, 0.6, 0.7)$ and $f_2 = (0.4, 0.5, 0.7, 0.9)$. Then $\tilde{A}(f_1, f_2)$ is a fuzzy truth value with trapezoidal shape given by $\tilde{A}(f_1, f_2) = (0.25, 0.4, 0.65, 0.8)$. See Figure 1. We can see that in this case it is sufficient to compute the arithmetic means of four parameters of trapezoids. We will closely discuss when it is possible to consider just a few parameters instead of continuous domain in an upcoming comprehensive article on this subject. There we also give more examples of various extended aggregation operators.

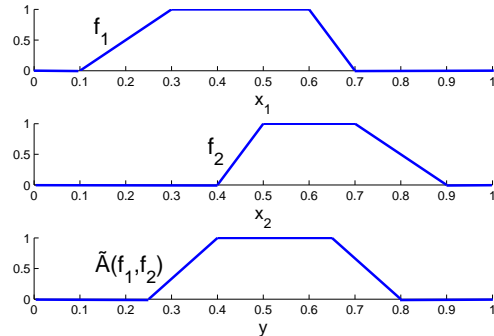


Figure 1: Aggregation of trapezoidal fuzzy truth values f_1, f_2 by extended arithmetic mean $\tilde{A}(f_1, f_2)(y) = \sup_{\frac{x_1+x_2}{2}=y} (f_1(x_1) \wedge f_2(x_2))$.

For applications of type-2 fuzzy sets, the normal and convex fuzzy truth values are of special significance. So, it is important that normal convex fuzzy truth values f_1, \dots, f_n give a normal convex fuzzy truth value $\tilde{A}(f_1, \dots, f_n)$ whenever \tilde{A} is an extension of some continuous n -ary type-1 aggregation operator A .

Theorem 1 Let A be an n -ary type-1 aggregation operator, let f_1, \dots, f_n be normal fuzzy truth values. Then $\tilde{A}(f_1, \dots, f_n)$ given by (1) is normal fuzzy truth value.

Proof. Let $x_i^p \in \text{Ker}(f_i)$, for all $i = 1, \dots, n$. Let

$A(x_1^p, \dots, x_n^p) = x^p$. Then

$$\begin{aligned} \tilde{A}(f_1, \dots, f_n)(x^p) &= \\ &= \sup_{A(x_1, \dots, x_n) = x^p} (f_1(x_1) \wedge \dots \wedge f_n(x_n)) = \\ &= f_1(x_1^p) \wedge \dots \wedge f_n(x_n^p) = 1 \wedge \dots \wedge 1 = 1. \end{aligned}$$

Thus, $x^p \in \text{Ker}(\tilde{A}(f_1, \dots, f_n))$, consequently $\tilde{A}(f_1, \dots, f_n)$ is normal. q.e.d.

Theorem 2 Let A be a continuous n -ary type-1 aggregation operator, let f_1, \dots, f_n be normal convex fuzzy truth values. Then $\tilde{A}(f_1, \dots, f_n)$ given by (1) is normal convex fuzzy truth value.

Proof. We prove the proposition for $n = 2$, the generalization for arbitrary n is straightforward. From Theorem 1 it follows that $\tilde{A}(f_1, f_2)$ is normal. It remains to show that it is also convex, i.e., $\tilde{A}(f_1, f_2) = (\tilde{A}(f_1, f_2))^L \wedge (\tilde{A}(f_1, f_2))^R$. The inequality \leq follows from $f \leq f^L$, $f \leq f^R$ for all $f \in \mathcal{F}$ (Proposition 3 in [18]), so we are going to show for all $y \in [0, 1]$:

$$\tilde{A}(f_1, f_2)(y) \geq \left((\tilde{A}(f_1, f_2))^L \wedge (\tilde{A}(f_1, f_2))^R \right)(y). \quad (2)$$

From (2) we get:

$$\begin{aligned} & \left((\tilde{A}(f_1, f_2))^L \wedge (\tilde{A}(f_1, f_2))^R \right)(y) = \\ &= \left(\sup_{A(s_1, s_2) = y} (f_1(s_1) \wedge f_2(s_2)) \right)^L \wedge \\ & \wedge \left(\sup_{A(t_1, t_2) = y} (f_1(t_1) \wedge f_2(t_2)) \right)^R = \\ &= \left(\sup_{u \leq y} \sup_{A(s_1, s_2) = u} (f_1(s_1) \wedge f_2(s_2)) \right) \wedge \\ & \wedge \left(\sup_{v \geq y} \sup_{A(t_1, t_2) = v} (f_1(t_1) \wedge f_2(t_2)) \right) = \\ &= \left(\sup_{A(s_1, s_2) \leq y} (f_1(s_1) \wedge f_2(s_2)) \right) \wedge \\ & \wedge \left(\sup_{A(t_1, t_2) \geq y} (f_1(t_1) \wedge f_2(t_2)) \right), \end{aligned}$$

where $s_1, s_2, t_1, t_2 \in [0, 1]$. We denote the very last term by *Term*. Then (2) can be expressed:

$$\sup_{A(x_1, x_2) = y} (f_1(x_1) \wedge f_2(x_2)) \geq \text{Term}, \quad (3)$$

which we are going to prove. We will consider the following three cases:

1. Let $y \in \text{Ker}(\tilde{A}(f_1, f_2))$. This means that $\sup_{A(x_1, x_2) = y} (f_1(x_1) \wedge f_2(x_2)) = 1$, hence, (3) holds.

2. Let $y \leq \inf(\text{Ker}(\tilde{A}(f_1, f_2)))$. Now let $A(s_1, s_2) \leq y$. Then (see Figure 2):

(i) Let $s_1 \leq \sup(\text{Ker}(f_1))$, $s_2 \leq \sup(\text{Ker}(f_2))$. Then there exists numbers $x_1^0, x_2^0 \in [0, 1]$ such that $A(x_1^0, x_2^0) = y$ and $s_1 \leq x_1^0 \leq \sup(\text{Ker}(f_1))$, $s_2 \leq x_2^0 \leq \sup(\text{Ker}(f_2))$. Recall that f_1, f_2 are convex, so they are increasing on intervals $[0, \sup(\text{Ker}(f_1))]$, $[0, \sup(\text{Ker}(f_2))]$, respectively. Thus, $f_1(x_1^0) \wedge f_2(x_2^0) \geq f_1(s_1) \wedge f_2(s_2)$ and consequently (3) holds.

(ii) Let $s_1 \leq \sup(\text{Ker}(f_1))$, $s_2 > \sup(\text{Ker}(f_2))$. Then there exists $x_1^0, x_2^0 \in [0, 1]$ such that $A(x_1^0, x_2^0) = y$ and $s_1 \leq x_1^0 \leq \sup(\text{Ker}(f_1))$, $s_2 = x_2^0$. Recall that f_1 is increasing on interval $[0, \sup(\text{Ker}(f_1))]$. Thus, $f_1(x_1^0) \wedge f_2(x_2^0) \geq f_1(s_1) \wedge f_2(s_2)$ and consequently (3) holds.

(iii) Let $s_1 > \sup(\text{Ker}(f_1))$, $s_2 \leq \sup(\text{Ker}(f_2))$. Then there exists $x_1^0, x_2^0 \in [0, 1]$ such that $A(x_1^0, x_2^0) = y$ and $s_1 = x_1^0$, $s_2 \leq x_2^0 \leq \sup(\text{Ker}(f_2))$. Recall that f_2 is increasing on interval $[0, \sup(\text{Ker}(f_2))]$. Thus, $f_1(x_1^0) \wedge f_2(x_2^0) \geq f_1(s_1) \wedge f_2(s_2)$ and consequently (3) holds.

From the previous three cases (i)-(iii) it follows:

$$\begin{aligned} & \sup_{A(x_1, x_2) = y} (f_1(x_1) \wedge f_2(x_2)) \geq \\ & \geq \sup_{A(s_1, s_2) \leq y} (f_1(s_1) \wedge f_2(s_2)), \end{aligned} \quad (4)$$

for all $y \leq \inf(\text{Ker}(\tilde{A}(f_1, f_2)))$.

3. Let $y \geq \sup(\text{Ker}(\tilde{A}(f_1, f_2)))$. The proof of

$$\begin{aligned} & \sup_{A(x_1, x_2) = y} (f_1(x_1) \wedge f_2(x_2)) \geq \\ & \geq \sup_{A(t_1, t_2) \geq y} (f_1(t_1) \wedge f_2(t_2)), \end{aligned} \quad (5)$$

is similar to item 2 with the exception that functions f_1, f_2 are decreasing on intervals $[\inf(\text{Ker}(f_1)), 1]$, $[\inf(\text{Ker}(f_2)), 1]$, respectively.

Finally, (4) and (5) implies (3). q.e.d.

4. Type-2 aggregation operators

In this section we propose a definition of so-called type-2 aggregation operator, which is a generalization of the standard definition of (type-1) aggregation operator on the set $[0, 1]$ to aggregation operator on the set of fuzzy truth values.

Definition 2 Let $(\mathcal{U}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$ be a subalgebra of $\mathbf{F} = (\mathcal{F}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$. A function $\tilde{A} : \mathcal{U}^n \rightarrow \mathcal{U}$ is called a type-2 aggregation operator on \mathcal{U} if and only if it satisfies the conditions $(\tilde{A}1)$, $(\tilde{A}2)$ and, for all $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{U}$, at least one of the conditions $(\tilde{A}3)$, $(\tilde{A}3')$:

- $(\tilde{A}1)$ $\tilde{A}(\mathbf{0}, \dots, \mathbf{0}) = \mathbf{0}$;
- $(\tilde{A}2)$ $\tilde{A}(\mathbf{1}, \dots, \mathbf{1}) = \mathbf{1}$;
- $(\tilde{A}3)$ $f_1 \sqsubseteq g_1, \dots, f_n \sqsubseteq g_n$ implies $\tilde{A}(f_1, \dots, f_n) \sqsubseteq \tilde{A}(g_1, \dots, g_n)$,

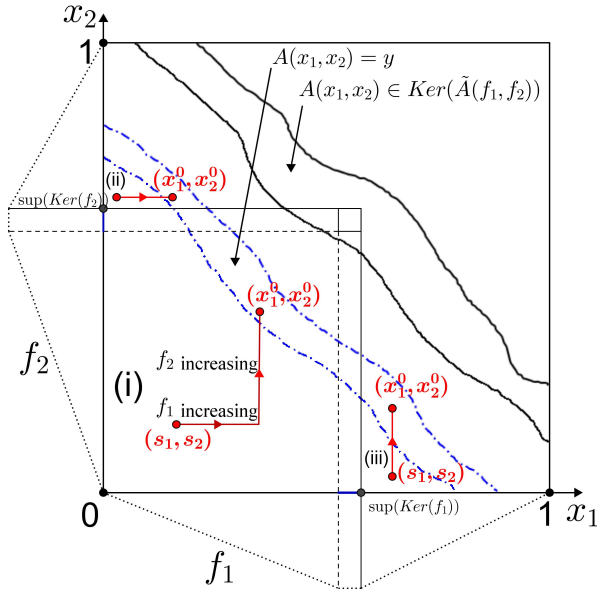


Figure 2: Figure to the proof of Theorem 2.

(A3') $f_1 \preceq g_1, \dots, f_n \preceq g_n$ implies $\tilde{A}(f_1, \dots, f_n) \preceq \tilde{A}(g_1, \dots, g_n)$.

Recall that for normal convex fuzzy truth values the two orderings coincide, i.e. $\sqsubseteq = \preceq$ (Proposition 19 and Proposition 37 in [18]).

The main result of this section shows that, on the set of normal convex fuzzy truth values, the extended n -ary continuous aggregation operator given by (1) is a type-2 aggregation operator.

Theorem 3 Let A be a continuous n -ary type-1 aggregation operator, let \mathcal{F}_{NC} be the set of normal convex fuzzy truth values. Then the extended aggregation operator $\tilde{A} : \mathcal{F}_{NC}^n \rightarrow \mathcal{F}_{NC}$ with

$$\tilde{A}(f_1, \dots, f_n)(y) = \sup_{A(x_1, \dots, x_n) = y} (f_1(x_1) \wedge \dots \wedge f_n(x_n)), \quad (6)$$

where $y, x_1, \dots, x_n \in [0, 1]$ and $f_1, \dots, f_n \in \mathcal{F}_{NC}$, is a type-2 aggregation operator.

Proof. Omitted for the reason of space. The proof will be given in an upcoming comprehensive article on this subject.

Example 2. In Example 1 we showed extended arithmetic mean $A(x_1, \dots, x_n) = \sum_{i=1}^n x_i / n$, for $n = 2$, which is a type-2 aggregation operator for each $n = 1, 2, \dots$. Arithmetic mean is a continuous aggregation operator, so, it preserves convexity and normality of fuzzy truth values (see Figure 1).

Now we give an example of discontinuous aggregation operator whose extension is not a type-2 aggregation operator and does not preserve the convexity. Let A be a binary aggregation operator with:

$$A(x_1, x_2) = \begin{cases} 0 & , \text{if } (x_1, x_2) \in [0, 0.5] \times [0, 0.5], \\ 0.3 & , \text{if } (x_1, x_2) \in [0, 0.5] \times [0.5, 1], \\ 0.7 & , \text{if } (x_1, x_2) \in [0.5, 1] \times [0, 0.5], \\ 1 & , \text{if } (x_1, x_2) \in [0.5, 1] \times [0.5, 1] \end{cases}$$

and f_1, f_2, g_1, g_2 be trapezoidal fuzzy truth values with the following parameters $f_1 = (0.3, 0.4, 0.6, 0.7)$, $f_2 = (0.1, 0.3, 0.4, 0.6)$, $g_1 = (0.35, 0.45, 0.65, 0.75)$, $g_2 = (0.6, 0.7, 0.8, 0.9)$. Clearly $f_1 \sqsubseteq g_1$ and $f_2 \sqsubseteq g_2$. Values of $\tilde{A}(f_1, f_2)$ and $\tilde{A}(g_1, g_2)$ are in the second and third row of the following table. Values of $\tilde{A}(f_1, f_2) \sqcap \tilde{A}(g_1, g_2)$ are in the fourth row:

y	0	0.3	0.7	1
$\tilde{A}(f_1, f_2)(y)$	1	0.5	1	0.5
$\tilde{A}(g_1, g_2)(y)$	0	1	0	1
$(\tilde{A}(f_1, f_2) \sqcap \tilde{A}(g_1, g_2))(y)$	1	1	1	0.5

We can see that neither $\tilde{A}(f_1, f_2)$ nor $\tilde{A}(g_1, g_2)$ is convex. Moreover, \tilde{A} is not type-2 aggregation operator, because $(\tilde{A}(f_1, f_2) \sqcap \tilde{A}(g_1, g_2)) \neq \tilde{A}(f_1, f_2)$, i.e. $\tilde{A}(f_1, f_2) \sqsubseteq \tilde{A}(g_1, g_2)$ does not hold.

5. Type-2 aggregation operators on various subalgebras of \mathcal{F}

In this section we show that the ordinary aggregation operator, as well as the aggregation operator for intervals and for n -dimensional intervals are special cases of our type-2 aggregation operator. We discuss the relation between our generalized aggregation operator given by Definition 2 and some known aggregation operators on the sets isomorphic to some subalgebras of \mathcal{F} .

5.1. Fuzzy grades of (type-1) fuzzy sets

Let \mathcal{S} be a set of all the singletons from \mathcal{F} , i.e.,

$$f \in \mathcal{S} \quad \text{iff} \quad f(x) = \begin{cases} 1 & , \text{if } x = a, \\ 0 & , \text{otherwise,} \end{cases}$$

for some $a \in [0, 1]$.

Then $\mathbf{S} = (\mathcal{S}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq)$ is a subalgebra of \mathbf{F} isomorphic with the algebra $([0, 1], \vee, \wedge, 0, 1, \leq)$ (see [18]), i.e. isomorphic with the fuzzy grades of type-1 fuzzy sets. If $\mathcal{U} = \mathcal{S}$, Definition 2 coincides with the usual notion of n -ary aggregation operator (e.g. [3], [2]).

5.2. Fuzzy grades of interval-valued fuzzy sets

Let \mathcal{I} be a set of all the characteristic functions of closed subintervals of $[0, 1]$, i.e.,

$$f \in \mathcal{I} \quad \text{iff} \quad f(x) = \begin{cases} 1 & , \text{if } x \in [a, b], \\ 0 & , \text{otherwise,} \end{cases}$$

for some $[a, b] \subseteq [0, 1]$.

Then $\mathbf{I} = (\mathcal{I}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq)$ is a subalgebra of \mathbf{F} isomorphic with the algebra

$([0, 1]^2, \vee, \wedge, [0, 0], [1, 1], \leq)$ (see [18]), i.e. isomorphic with the fuzzy grades of interval-valued fuzzy sets under standard maximum \vee , minimum \wedge and ordering \leq for intervals. If $\mathcal{U} = \mathcal{I}$, Definition 2 coincides with the definition of n -ary aggregation operator for intervals (e.g. [6], [7]).

5.3. Fuzzy grades of n -dimensional fuzzy sets

Shang et al. [11] introduced an n -dimensional fuzzy set A on Z as a mapping $A : Z \rightarrow [0, 1]^n$, where $A(z) = (A_1(z), \dots, A_n(z))$ with $A_1(z) \leq A_2(z) \leq \dots \leq A_n(z)$ is called an n -dimensional interval. Recall that type-1 fuzzy sets and interval-valued fuzzy sets are special cases of n -dimensional fuzzy sets for $n = 1$ and $n = 2$, respectively. One of the possible interpretations is: an n -dimensional interval can be seen as a chain of nested intervals of length $k = \frac{n}{2}$ (for even n) or $k = \frac{n+1}{2}$ (for odd n) representing different uncertainty levels on the membership degree [12]. Now, let us interpret these nested intervals as α -cuts of some fuzzy truth value f (fuzzy grade of type-2 fuzzy set for some $z \in Z$) for $\alpha = \frac{1}{k}, \frac{2}{k}, \dots, 1$:

$$\begin{aligned} f_{\frac{1}{k}} &= [A_1(z), A_n(z)], f_{\frac{2}{k}} = [A_2(z), A_{n-1}(z)], \dots \\ &\dots, f_1 = [A_k(z), A_{n-k+1}(z)], \end{aligned}$$

for even n , and

$$\begin{aligned} f_{\frac{1}{k}} &= [A_1(z), A_n(z)], f_{\frac{2}{k}} = [A_2(z), A_{n-1}(z)], \dots \\ &\dots, f_1 = [A_k(z), A_k(z)], \end{aligned}$$

for odd n . Now, let \mathcal{V} be, for some fixed n , a subset of \mathcal{F} given by:

$$g \in \mathcal{V} \quad \text{iff} \quad g(x) = \begin{cases} \frac{1}{k} & , \text{if } x \in f_{\frac{1}{k}} - f_{\frac{2}{k}}, \\ \frac{2}{k} & , \text{if } x \in f_{\frac{2}{k}} - f_{\frac{3}{k}}, \\ \vdots & \\ \frac{k-1}{k} & , \text{if } x \in f_{\frac{k-1}{k}} - f_1, \\ 1 & , \text{if } x \in f_1, \end{cases}$$

for some $f \in \mathcal{F}$.

For example of 6-dimensional fuzzy set and corresponding $g \in \mathcal{V}$ see Figure 3. Clearly, $\mathbf{V} = (\mathcal{V}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq)$ is a subalgebra of \mathbf{F} isomorphic with the algebra $([0, 1]^n, \vee, \wedge, [0, \dots, 0], [1, \dots, 1], \leq)$, i.e. isomorphic with the fuzzy grades of n -dimensional fuzzy sets under standard maximum \vee , minimum \wedge and ordering \leq for n -dimensional intervals. If $\mathcal{U} = \mathcal{V}$, Definition 2 coincides with the definition of n -ary aggregation operator for n -dimensional intervals (Definition 3.1 in [12]).

Thus, membership grades of type-2 fuzzy sets can be approximated via n -dimensional fuzzy sets, i.e. as classes of α -cuts of fuzzy truth values. Consequently, various applications of n -dimensional fuzzy sets can be adapted as an approximate applications for type-2 fuzzy sets. For instance, the example

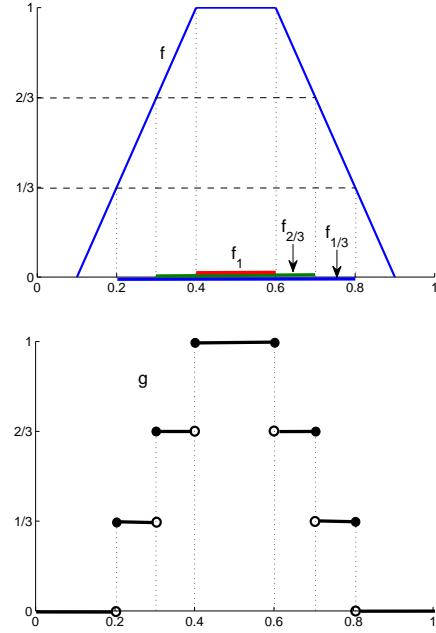


Figure 3: A trapezoidal fuzzy truth value f with parameters $(0.1, 0.4, 0.6, 0.9)$. Its α -cuts $f_{\frac{1}{3}} = [0.2, 0.8]$, $f_{\frac{2}{3}} = [0.3, 0.7]$ and $f_1 = [0.4, 0.6]$ produce 6-dimensional fuzzy set $A(z) = (0.2, 0.3, 0.4, 0.6, 0.7, 0.8)$. Corresponding fuzzy truth value $g \in \mathcal{V}$.

of fuzzy multicriteria decision making based on n -dimensional fuzzy sets in Section 5 of [12] can be used as an approximate procedure for multicriteria decision making based on type-2 fuzzy sets.

Note that this approach corresponds to α -plane representation [15] and zSlice representation [16] of type-2 fuzzy sets.

6. Conclusions

We proposed a new concept of type-2 aggregation operator. These operators aggregate fuzzy truth values, so it is a step to developing fuzzy logic systems based on type-2 fuzzy sets. Type-2 aggregation operator is an extension of known aggregation operators for real numbers, intervals and n -dimensional intervals. We also showed that an n -dimensional fuzzy set can be interpreted as a class of α -cuts of fuzzy truth values. Thus, the membership grades of type-2 fuzzy sets can be approximated via n -dimensional fuzzy sets. Consequently, various applications of n -dimensional fuzzy sets can be adapted as an approximate applications for type-2 fuzzy sets.

This paper is a first announcement of our research activity concerning with aggregation of fuzzy truth values. Our emphasis was on the theoretical side. In an upcoming comprehensive article on this subject we will develop the theoretical aspects in more details, we will also give more examples of various extended aggregation operators, and mainly we will

study a techniques for computation of type-2 aggregation operators for some specific kinds of fuzzy truth values, e.g. triangular, trapezoidal, gaussian shapes.

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