Preservation of t-norm and t-conorm based properties of fuzzy relations during aggregation process

Urszula Dudziak

Institute of Mathematics, University of Rzeszów, Al. Rejtana 16 C, 35-959 Reszów

Abstract

In the paper aggregations of fuzzy relations using functions of n variables are considered. After recalling properties of fuzzy relations, aggregation functions which preserve: reflexivity, irreflexivity, T-asymmetry, T-antisymmetry, symmetry, S-connectedness, T-transitivity, negative S-transitivity, T-S-semitransitivity and T-S-Ferrers property of fuzzy relations, where T is a t-norm and S is a t-conorm, are examined.

Keywords: Fuzzy relations, properties of fuzzy relations, aggregation functions

1. Introduction

Fuzzy relations are often considered in the context of preservation of their properties in the aggregation process (cf. [3], [4], [9], [11], [14] [15], [17]) which is due to the possible applications, e.g. in multicriteria decision making problems. There are diverse types of fuzzy relation properties. We especially take into account the ones which are based on triangular norms and triangular conorms ([9], Chapter 2.5), i.e. T-asymmetry, T-antisymmetry, S-connectedness, T-transitivity, negative S-transitivity, T-S-semitransitivity and T-S-Ferrers property of fuzzy relations, where T is a t-norm and S is a t-conorm.

In this contribution, we concentrate on a fixed property of fuzzy relations and characterize all aggregation functions preserving this property. Moreover, the aim of this paper is to give characterizations, under the weakest assumptions on functions used for aggregation procedure. Therefore, we start with an arbitrary n-ary function (not necessarily aggregation function).

In this paper, we try to present necessary and sufficient conditions for n-argument functions to preserve fixed properties of fuzzy relations. Firstly (Section 2), we give basic definitions concerning n-ary functions in $[0, 1]$, domination between functions and useful notions concerning fuzzy relations. Next, (Section 3), we present theorems for the considered properties of fuzzy relations in the aggregation process.

2. Basic definitions

Here we present properties of aggregation functions, the concept of domination and the properties of fuzzy relations considered in this paper.

2.1. Aggregation functions

We recall some information about functions in the interval $[0, 1]$.

Definition 1 ([2]). Let $n \geqslant 2$. A function $F \colon [0, 1]^n \to [0, 1]$ is called an aggregation function if it is increasing in each of its arguments and fulfils the boundary conditions

$$F(0, ..., 0) = 0, \quad F(1, ..., 1) = 1.$$

There are many examples of aggregation functions. We present these examples, which will be used in the sequel (cf. [2], [6], [15]).

Example 1. Let $t_1, ..., t_n, w_1, ..., w_n \in [0, 1]$. Aggregation functions are:
- the weighted minimum

$$F(t_1, ..., t_n) = \min_{1 \leqslant k \leqslant n} \max(1 - w_k, t_k), \quad \max_{1 \leqslant k \leqslant n} w_k = 1,$$

- the weighted maximum

$$F(t_1, ..., t_n) = \max_{1 \leqslant k \leqslant n} \min(w_k, t_k), \quad \max_{1 \leqslant k \leqslant n} w_k = 1,$$

- the projections

$$P_k(t_1, \ldots, t_n) = t_k, \quad k \in \{1, ..., n\},$$

- the median value

$$\mathrm{med}(t_1, \ldots, t_n) = \begin{cases} \frac{s_k + s_{k+1}}{2}, & \text{for } n = 2k \\ s_{k+1}, & \text{for } n = 2k+1 \end{cases},$$

where $(s_1, \ldots, s_n)$ is the increasingly ordered sequence of the values $t_1, \ldots, t_n$, i.e. $s_1 \leqslant \ldots \leqslant s_n$.
- the quasi–linear means

$$F(t_1, ..., t_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k \varphi(t_k)\right),$$

where $\varphi \colon [0, 1] \to \mathbb{R}$ is an increasing bijection and the weights $w_k > 0$, $k = 1, ..., n$, fulfil the condition $\sum_{k=1}^{n} w_k = 1,$

• the weighted arithmetic mean

$$F(t_1, ..., t_n) = \sum_{k=1}^{n} w_k t_k,$$

which we get if $f(t) = t$ in the formula for a quasi-linear mean,

• the weighted geometric mean

$$F(t_1, ..., t_n) = \prod_{k=1}^{n} t_k^{w_k}$$

which we get if $f(t) = \log t$ in the formula for a quasi-linear mean.

In our further considerations we will need the following concepts.

Definition 2. Let $n \in \mathbb{N}$. We say that a function $F : [0, 1]^n \to [0, 1]$:

• has a zero element $z \in [0, 1]$ (cf. [2], Definition 10) if for each $k \in \{1, \ldots, n\}$ and each $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n \in [0, 1]$ one has

$$F(x_1, ..., x_{k-1}, z, x_{k+1}, ..., x_n) = z,$$

• is without zero divisors if

$$\underset{x_1, ..., x_n \in [0,1]}{\forall} \quad (F(x_1, ..., x_n) = z \Rightarrow (\underset{1 \leqslant k \leqslant n}{\exists} x_k = z)).$$

Definition 3 ([12], p. 4). A triangular norm $T \colon [0, 1]^2 \to [0, 1]$ (triangular conorm $S \colon [0, 1]^2 \to [0, 1]$) is an arbitrary associative, commutative function having a neutral element $e = 1$ ($e = 0$), which is increasing in each of its arguments.

Let us observe that triangular norms (conorms) are aggregation functions for $n = 2$. Directly from the definition of a triangular norm (conorm) we obtain a useful property.

Corollary 1. *A triangular norm (conorm) has a zero element $z = 0$ ($z = 1$).*

In our further considerations we will use the abbreviation 't-norm' ('t-conorm') when speaking of a triangular norm (a triangular conorm).

Example 2 ([12], pp. 4, 11). The four well-known examples of t-norms T and corresponding t-conorms S are:

$$T_M(s, t) = \min(s, t), \qquad T_P(s, t) = st,$$
$$S_M(s, t) = \max(s, t), \qquad S_P(s, t) = s + t - st,$$

$$T_L(s, t) = \max(s + t - 1, 0),$$
$$S_L(s, t) = \min(s + t, 1),$$

$$T_D(s, t) = \begin{cases} s, & t = 1 \\ t, & s = 1 \\ 0, & otherwise \end{cases},$$

$$S_D(s, t) = \begin{cases} s, & t = 0 \\ t, & s = 0 \\ 1, & otherwise \end{cases}$$

for $s, t \in [0, 1]$.

Lemma 1 (cf. [12], pp. 6–7). *For arbitrary t-norm T and t-conorm S one has*

$$T_D \leqslant T \leqslant T_M, \quad T_D \leqslant T_L \leqslant T_P \leqslant T_M.$$

Moreover

$$S_M \leqslant S \leqslant S_D, \quad S_D \geqslant S_L \geqslant S_P \geqslant S_M.$$

Definition 4 ([12], p. 28). A strict t-norm $T \colon [0, 1]^2 \to [0, 1]$ is a t-norm which is continuous and strictly increasing in $(0, 1]^2$.

Theorem 1 ([9], p. 11). *Any strict t-norm T is a function isomorphic to the product t-norm T_P, i.e.*

$$T(x_1, x_2) = \varphi^{-1}(T_P(\varphi(x_1), \varphi(x_2))), \quad x_1, x_2 \in [0, 1],$$

where $\varphi \colon [0, 1] \to [0, 1]$ is an increasing bijection.

Corollary 2. *Triangular norms: $\min$, T_P, strict t-norms are functions without zero divisors.*

In our further considerations we will need the concept of a dual function and a self-dual function.

Definition 5 (cf. [2], p.31). Let $F \colon [0, 1]^n \to [0, 1]$. A function F^d is called a dual function to F, if for all $x_1, \ldots, x_n \in [0, 1]$

$$F^d(x_1, \ldots, x_n) = 1 - F(1 - x_1, \ldots, 1 - x_n).$$

F is called a self-dual function, if it holds $F = F^d$.

Example 3 ([2], p.31, [18]). Pairs of dual functions are for example: T_M and S_M, T_L and S_L, T_P and S_P, T_D and S_D. In general, t-norms are dual functions to t-conorms and vice versa. The concept of self-duality is especially developed for aggregation functions. Interesting properties and characterizations of self-dual aggregation functions one can find in [18]. The weighted arithmetic mean, the median and all quasi-linear means for which $f \colon [0, 1] \to [0, 1]$ fulfils $f(1 - x) = 1 - f(x)$, are self-dual aggregation functions. Any self-dual and commutative binary aggregation function F satisfies $F(x, 1 - x) = \frac{1}{2}$ for all $x \in [0, 1]$, so neither t-norms nor t-conorms are self-dual.

2.2. Domination

Definition 6 (cf. [17], Definition 2.5). Let $m, n \in \mathbb{N}$. A function $F \colon [0, 1]^m \to [0, 1]$ dominates function $G \colon [0, 1]^n \to [0, 1]$ ($F \gg G$) if for an arbitrary matrix $[a_{ik}] = A \in [0, 1]^{m \times n}$ the following inequality holds

$$F(G(a_{11}, ..., a_{1n}), ..., G(a_{m1}, ..., a_{mn})) \geqslant$$

$$G(F(a_{11}, ..., a_{m1}), ..., F(a_{1n}, ..., a_{mn})).$$

There exist a characterization theorem of all functions which dominate minimum (similarly for a drastic t-norm [17]).

Theorem 2 (cf. [17], Proposition 5.1)**.** *A function* $F \colon [0,1]^n \to [0,1]$, *which is increasing in each of its arguments dominates minimum if and only if for each* $t_1, ..., t_n \in [0,1]$

$$F(t_1, ..., t_n) = \min(f_1(t_1), ..., f_n(t_n)),$$

where $f_k \colon [0,1] \to [0,1]$ *is increasing with* $k = 1, ..., n$.

Example 4. There are a few useful examples of functions which fulfil conditions of Theorem 2:

if $f_k(t) = t$, $k = 1, ..., n$, then $F = \min$,

if for a certain $k \in \{1, ..., n\}$, function $f_k(t) = t$ and $f_i(t) = 1$ for $i \neq k$, then $F = P_k$ - projections,

if $f_k(t) = \max(1 - v_k, t)$, $v_k \in [0,1]$, $k = 1, ..., n$, $\max\limits_{1 \leqslant k \leqslant n} v_k = 1$, then F is the weighted minimum.

Here we present some examples of domination between functions.

Example 5 (cf. [8], [17])**.** The weighted geometric mean dominates t-norm T_P. The weighted arithmetic mean dominates t-norm T_L. The function

$$F(t_1, ..., t_n) = \frac{p}{n} \sum_{k=1}^{n} t_k + (1-p) \min_{1 \leqslant k \leqslant n} t_k \qquad (1)$$

dominates T_L, where $p \in (0,1)$. The weighted minimum dominates every t-norm T. Let us consider projections P_k. Then $F \gg P_k$ and $P_k \gg F$ for any function $F : [0,1]^n \to [0,1]$ (see [5]).

We recall here some results on domination in the family of t-norms and t-conorms.

Theorem 3 (cf. [17], p.16)**.** *Let* T *be a t-norm, S a t-conorm. Then*

$$T \gg T, \qquad T_M \gg T \gg T_D,$$

$$T_D \gg T \Leftrightarrow T = T_D, \qquad T \gg T_M \Leftrightarrow T = T_M.$$

Dually we obtain

$$S \gg S, \qquad S_D \gg S \gg S_M,$$

$$S \gg S_D \Leftrightarrow S = S_D, \qquad S_M \gg S \Leftrightarrow S = S_M.$$

Lemma 2 ([7])**.** *Let* $F : [0,1]^n \to [0,1]$, $F : [0,1]^n \to [0,1]$. *If* $F \gg G$, *then* $G^d \gg F^d$.

Corollary 3. *Let* $F : [0,1]^n \to [0,1]$, *and* T *be a t-norm and S be a t-conorm. If* $F \gg T$, *then* $S \gg F^d$.

Example 6. By Example 5 and Corollary 3 we see that the weighted arithmetic mean are dominated by S_L and the weighted maximum is dominated by any t-conorm S. Moreover, it is easy to verify that the function

$$F(t_1, ..., t_n) = \frac{p}{n} \sum_{k=1}^{n} t_k + (1-p) \max_{1 \leqslant k \leqslant n} t_k \qquad (2)$$

is dominated by S_L, where $p \in (0,1)$. By Theorem 2 and Corollary 3 a function $F \colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments is dominated by maximum if and only if for each $t_1, ..., t_n \in [0,1]$

$$F(t_1, ..., t_n) = \max(f_1(t_1), ..., f_n(t_n)),$$

where $f_k \colon [0,1] \to [0,1]$ is increasing with $k = 1, ..., n$. Examples of such functions are:
if $f_k(t) = t$, $k = 1, ..., n$, then $F = \max$,
if for a certain $k \in \{1, ..., n\}$, function $f_k(t) = t$ and $f_i(t) = 1$ for $i \neq k$, then $F = P_k$ - projections, if $f_k(t) = \min(v_k, t)$, $v_k \in [0,1]$, $k = 1, ..., n$, $\max\limits_{1 \leqslant k \leqslant n} v_k = 1$, then F is the weighted maximum.

For further results on domination, specially domination between t-norms, we recommend [12] (pp.152–156), [13], [19], we recall only a few of them which will be useful later on.

2.3. Fuzzy relations

We recall basic properties of fuzzy relations.

Definition 7 ([20])**.** A fuzzy relation on a set $X \neq \emptyset$ is an arbitrary function $R : X \times X \to [0,1]$. The family of all fuzzy relations on X is denoted by $FR(X)$.

Remark 1. *If* card $X = n$, $X = \{x_1, ..., x_n\}$, *then* $R \in FR(X)$ *may be presented by a matrix* $R = [r_{ik}]$, *where* $r_{ik} = R(x_i, x_k)$, $i, k = 1, ..., n$.

Definition 8 (cf. [20])**.** Let $* \colon [0,1]^2 \to [0,1]$. A sup–$*$–composition of relations $R, W \in FR(X)$ is the relation $(R \circledast W) \in FR(X)$ such that for any $(x, z) \in X \times X$

$$(R \circledast W)(x, z) = \sup_{y \in X} R(x, y) * W(y, z). \qquad (3)$$

An inf–$*$–composition of relations $R, W \in FR(X)$ is the relation $(R \circledast' W) \in FR(X)$ such that for any $(x, z) \in X \times X$

$$(R \circledast' W)(x, z) = \inf_{y \in X} R(x, y) * W(y, z). \qquad (4)$$

Next we present properties of fuzzy relations, which will be discussed in our further considerations ([1], p. 38, [9], Chapter 2.5, cf. [10], p. 445).

Definition 9. Let T be a t-norm and S be a t-conorm. Relation $R \in FR(X)$ is:
- reflexive, if $\forall\limits_{x \in X} R(x, x) = 1$,
- irreflexive, if $\forall\limits_{x \in X} R(x, x) = 0$,
- symmetric, if $\forall\limits_{x, y \in X} R(x, y) = R(y, x)$,
- T-asymmetric, if $\forall\limits_{x, y \in X} T(R(x, y), R(y, x)) = 0$,
- T-antisymmetric, if $\forall\limits_{x, y, x \neq y \in X} T(R(x, y), R(y, x)) = 0$,
- totally S-connected, if $\forall\limits_{x, y \in X} S(R(x, y), R(y, x)) = 1$,

- S-connected, if $\underset{x,y,x\neq y\in X}{\forall} S(R(x,y),R(y,x))=1$,
- T-transitive, if
  $\underset{x,y,z\in X}{\forall} T(R(x,y),R(y,z))\leqslant R(x,z)$,
- negatively S-transitive, if
  $\underset{x,y,z\in X}{\forall} S(R(x,y),R(y,z))\geqslant R(x,z)$,
- T-S-Ferrers, if
  $\underset{x,y,z,w\in X}{\forall} T(R(x,y),R(z,w))\leqslant S(R(x,w),R(z,y))$,
- T-S-semitransitive, if
  $\underset{x,y,z,w\in X}{\forall} T(R(x,w),R(w,y))\leqslant S(R(x,z),R(z,y))$.

In the above definition a t-norm takes place of a minimum and a t-conorm takes place of a maximum in the standard properties introduced for fuzzy relations by L.A. Zadeh [21] (for the adequate properties of crisp relations see [16]).

With the use of n–argument function F, $n\in\mathbb{N}$, $n\geqslant 2$, we aggregate given fuzzy relations $R_1,\dots,R_n$.

Definition 10 ([17], Definition 2.4)**.** Let $F : [0,1]^n \to [0,1]$, $R_1,\dots,R_n\in FR(X)$. An aggregated fuzzy relation $R_F\in FR(X)$ is described by the formula

$$R_F(x,y)=F(R_1(x,y),\dots,R_n(x,y)), \quad x,y\in X.$$

A function F preserves a property of fuzzy relations if for every $R_1,\dots,R_n\in FR(X)$ having this property, R_F also has this property.

We recall here the concept of the complement R' of a fuzzy relation R, i.e.

$$R'(x,y)=1-R(x,y),\ x,y\in[0,1].$$

Lemma 3. *A fuzzy relation R is T-asymmetric (T-antisymmetric, T-transitive), if and only if its complement R' is totally S-connected (S-connected, negatively S-transitive).*

In virtue of this lemma conditions for aggregated totally S-connected, S-connected and negatively S-transitive fuzzy relations can be obtained by negation of conditions considered above for aggregated T-asymmetric, T-antisymmetric and T-transitive fuzzy relations, respectively.

3. Preservation of fuzzy relation properties

In this section we present the weakest assumptions on functions to preserve fuzzy relation properties. Aggregation functions (cf. Definition 1) are given as examples. We recall here the results from [8] concerning preservation of reflexivity and symmetry.

Theorem 4. *Let $R_1,\dots,R_n\in FR(X)$ be reflexive (respectively irreflexive). The relation R_F is reflexive (respectively irreflexive), if and only if the function F satisfies the condition $F(1,\dots,1)=1$ (respectively $F(0,\dots,0)=0$).*

Any aggregation function and any idempotent function F, i.e. function which fulfils $F(t,\dots,t)=t$, for $t\in[0,1]$ preserves reflexivity and irreflexivity of fuzzy relations.

Theorem 5. *Let $R_1,\dots,R_n\in FR(X)$ be symmetric. For every function F the fuzzy relation R_F is also symmetric.*

Here theorems concerning preservation of t-norm and t-conorm based fuzzy relation properties will be presented.

Theorem 6. *Let card $X\geqslant 2$, T be a t-norm without zero divisors. A function F preserves T-asymmetry (T-antisymmetry) if and only if it satisfies the following condition for all $s,t\in[0,1]^n$*

$$\underset{1\leqslant k\leqslant n}{\forall} \min(s_k,t_k)=0 \Rightarrow \min(F(s),F(t))=0. \tag{5}$$

Proof. Let F fulfil (5) for all $s,t\in[0,1]^n$, $x,y\in X$. If $R_1,\dots,R_n\in FR(X)$ are T-asymmetric, then using substitution $s_k=R_k(x,y)$, $t_k=R_k(y,x)$ for $k=1,\dots,n$ we see that for a t-norm T without zero divisors we get $s_k=0$ or $t_k=0$, $k=1,\dots,n$, as a result

$$\underset{1\leqslant k\leqslant n}{\forall} \min(s_k,t_k)=0 \tag{6}$$

and the relation R_F is T-asymmetric by (5) and the assumption on a t-norm T.

Conversely, let $s,t\in[0,1]^n$ fulfil (6). Since card $X\geqslant 2$, then there exist $a,b\in X$, $a\neq b$. Fuzzy relations

$$R_k(x,y)=\begin{cases} s_k, & \text{if } (x,y)=(a,b) \\ t_k, & \text{if } (x,y)=(b,a)\,, \quad k=1,\dots,n \\ 0, & \text{otherwise} \end{cases}$$

are T-asymmetric. Thus, the relation R_F is also T-asymmetric and we get

$$T(F(s),F(t))=$$

$$T(F(R_1(a,b),\dots,R_n(a,b)),$$

$$F(R_1(b,a),\dots,R_n(b,a)))=0,$$

so $\min(F(s),F(t))=0$ which proves (5). In the case of T-antisymmetric fuzzy relations the proof is similar. $\qquad\square$

Now we give examples of functions preserving T-asymmetry and T-antisymmetry. We apply here Corollary 2.

Example 7 ([8])**.** Let T be a strict t-norm. The function $F = \min$ preserves T-asymmetry (T-antisymmetry). Functions F which has the zero element $z=0$ with respect to certain coordinate, i.e.

$$\underset{1\leqslant k\leqslant n}{\exists}\ \underset{i\neq k}{\forall}\ \underset{t_i\in[0,1]}{\forall} F(t_1,\dots,t_{k-1},0,t_{k+1},\dots,t_n)=0$$

fulfil (5), so they preserve T-asymmetry (T-antisymmetry). In particular, the weighted geometric mean fulfil (5). As another example we may consider the median function. If a function F fulfils the condition

$$\underset{t \in [0,1]^n}{\forall} \quad \text{card}\{k : t_k = 0\} > \frac{n}{2} \Rightarrow F(t) = 0, \quad (7)$$

then we also get (5) (e.g. the median fulfils this condition). However, the above condition is not necessary for (5), because it does not cover the projections which preserve T-asymmetry (T-antisymmetry).

Similarly as in Theorem 6 we get

Theorem 7. *Let card $X \geqslant 2$, S be a t-conorm without zero divisors. A function F preserves total S-connectedness (S-connectedness) if and only if it satisfies the following condition for all $s, t \in [0,1]^n$*

$$\underset{1 \leqslant k \leqslant n}{\forall} \quad \max(s_k, t_k) = 1 \Rightarrow \max(F(s), F(t)) = 1. \quad (8)$$

Proof. Let F fulfil (8) for all $s, t \in [0,1]^n$, $x, y \in X$. If $R_1, \dots, R_n \in FR(X)$ are totally S-connected, then using substitution $s_k = R_k(x,y)$, $t_k = R_k(y,x)$ for $k = 1, \dots, n$ we see that for a t-conorm S without zero divisors we get $s_k = 1$ or $t_k = 1$, $k = 1, \dots, n$, as a result

$$\underset{1 \leqslant k \leqslant n}{\forall} \quad \max(s_k, t_k) = 1 \quad (9)$$

and the relation R_F is totally S-connected by (8) and the assumption on a t-conorm S.

Conversely, let $s, t \in [0,1]^n$ fulfil (9). Since card $X \geqslant 2$, then there exist $a, b \in X$ such that $a \neq b$. Fuzzy relations

$$R_k(x,y) = \begin{cases} s_k, & \text{if } (x,y) = (a,b) \\ t_k, & \text{if } (x,y) = (b,a) \\ 1, & \text{otherwise} \end{cases}, \quad k = 1, \dots, n$$

are totally S-connected. Thus, the relation R_F is also totally S-connected and we get

$$S(F(s), F(t)) =$$
$$S(F(R_1(a,b), \dots, R_n(a,b)),$$
$$F(R_1(b,a), \dots, R_n(b,a))) = 1,$$

so $\max(F(s), F(t)) = 1$ which proves (8). In the case of S-connected fuzzy relations the proof is analogous. $\qquad\square$

Example 8. Examples of functions fulfilling (8) for all $s, t \in [0,1]^n$ are $F = \max$, $F = \text{med}$ or functions F with the zero element $z = 1$ with respect to a certain coordinate, i.e.

$$\underset{1 \leqslant k \leqslant n}{\exists} \underset{i \neq k}{\forall} \underset{t_i \in [0,1]}{\forall} F(t_1, \dots, t_{k-1}, 1, t_{k+1}, \dots, t_n) = 1.$$

The dual property for (7) have the form

$$\underset{t \in [0,1]^n}{\forall} \quad \text{card}\{k : t_k = 1\} > \frac{n}{2} \Rightarrow F(t) = 1.$$

We will give theorems (Theorems 8, 9) providing a necessary and a sufficient condition for the preservation of T-transitivity (cf. also [17]). This are modified versions of theorems obtained in [8] for more general class of $*$-transitive fuzzy relations, where $* : [0,1]^2 \to [0,1]$.

Theorem 8. *Let card $X \geqslant 3$. If a function $F : [0,1]^n \to [0,1]$ preserves T-transitivity, then it dominates T ($F \gg T$), it means that for all $(s_1, \dots, s_n), (t_1, \dots, t_n) \in [0,1]^n$*

$$F(T(s_1, t_1), \dots, T(s_n, t_n)) \geqslant$$

$$T(F(s_1, \dots, s_n), F(t_1, \dots, t_n)).$$

Theorem 9. *If a function $F : [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F \gg T$, then it preserves T-transitivity.*

Example 9 (cf. [17], p. 30)**.** Each quasi–linear mean dominates T_D. Moreover, for $n = 2$ arbitrary t-norm $F = T$ dominates T_D. By Theorem 2 and Examples 4–5 we see that the minimum, the weighted minimum and the projections dominate min. The weighted geometric mean preserves T_P-transitivity, the weighted arithmetic mean preserves T_L-transitivity, the minimum preserves T-transitivity with arbitrary t-norm T. The function F described by the formula (1) preserves T_L-transitivity.

In Theorems 8 and 9 we present respectively necessary and sufficient conditions for the preservation of T-transitivity. We give separate results to point out the adequate assumptions for each theorem. Dually to the result for T-transitivity (see [8]) we may obtain the statement for negative S-transitivity. To prove it, we will need the property connected with the extension of a relation $R_0 \in FR(A)$ to the relation $R \in FR(X)$, where $A \subset X$, $A \neq \emptyset$.

Lemma 4. *Let $R_0, W_0 \in FR(A)$, $\emptyset \neq A \subset X$, $R, W \in FR(X)$ be of the form*

$$R(x,y) = \begin{cases} R_0(x,y), & x, y \in A \\ 1, & \text{otherwise} \end{cases}, \quad (10)$$

$$W(x,y) = \begin{cases} W_0(x,y), & x, y \in A \\ 1, & \text{otherwise} \end{cases}, \quad (11)$$

respectively. If an operation $* : [0,1]^2 \to [0,1]$ has the zero element $z = 1$, then

$$(R \circledast' W)(x,y) = \begin{cases} (R_0 \circledast' W_0)(x,y), & x, y \in A \\ 1, & \text{otherwise} \end{cases}. \quad (12)$$

Proof. Let $x, y \in X$, $B = X \setminus A$. By the associativity of the infimum on $[0,1]$ we may consider the following cases:
if $(x, y) \in A \times A$, then

$$(R \circledast' W)(x,y) = \underset{w \in X}{\inf} R(x,w) * W(w,y) =$$

$$\min(\inf_{w\in A} R(x,w)*W(w,y), \inf_{w\in B} R(x,w)*W(w,y)) =$$

$$\min(\inf_{w\in A} R_0(x,w)*W_0(w,y), \inf_{w\in B} 1*1) =$$

$$\min((R_0 \circledast' W_0)(x,y), 1) = (R_0 \circledast' W_0)(x,y),$$

if $(x,y) \in A \times B$, then

$$(R \circledast' W)(x,y) = \inf_{w\in X} R(x,w)*W(w,y) =$$

$$\min(\inf_{w\in A} R(x,w)*W(w,y), \inf_{w\in B} R(x,w)*W(w,y)) =$$

$$\min(\inf_{w\in A} R_0(x,w)*1, \inf_{w\in B} 1*1) = \min(1,1) = 1,$$

if $(x,y) \in B \times A$, then

$$(R \circledast' W)(x,y) = \inf_{w\in X} R(x,w)*W(w,y) =$$

$$\min(\inf_{w\in A} R(x,w)*W(w,y), \inf_{w\in B} R(x,w)*W(w,y)) =$$

$$\min(\inf_{w\in A} 1*W_0(w,y), \inf_{w\in B} 1*1) = \min(1,1) = 1,$$

if $(x,y) \in B \times B$, then

$$(R \circledast' W)(x,y) = \inf_{w\in X} R(x,w)*W(w,y) =$$

$$\inf_{w\in X}(1*1) = 1.$$

As a result $R \circledast' W$ is of the form (12). $\qquad\square$

Lemma 5. *Let $*$ be a t-conorm. A relation $R \in FR(X)$ is negatively $*$–transitive if and only if*

$$R \circledast' R \geqslant R. \qquad (13)$$

Proof. Let $R \in FR(X)$. Directly by (4) and definition of negative S-transitivity we get

$$R \circledast' R \geqslant R \Leftrightarrow \underset{x,z\in X}{\forall}\ \inf_{y\in X} R(x,y)*R(y,z) \geqslant R(x,z)$$

$$\Leftrightarrow \underset{x,y,z\in X}{\forall}\ R(x,y)*R(y,z) \geqslant R(x,z),$$

which means that R is negatively $*$-transitive for a t-conorm $*$. $\qquad\square$

Lemma 6. *Let S be a t-conorm, $A \subset X$, $A \neq \emptyset$. If relation $R_0 \in FR(A)$ is negatively S-transitive in A, then relation $R \in FR(X)$ described by (10) is negatively S-transitive in X.*

Proof. Let $(x,y) \in A \times A$. By Lemma 4, Lemma 5 and by negative S-transitivity of relation R_0 in A we get for any $x,y \in A$

$$(R\circledast' R)(x,y) = (R_0\circledast' R_0)(x,y) \geqslant R_0(x,y) = R(x,y).$$

If $(x,y) \in X \times X \setminus A \times A$, then $(R \circledast' R)(x,y) = 1 \geqslant R(x,y)$. As a result relation R is negatively S-transitive in X. $\qquad\square$

Theorem 10. *Let card $X \geqslant 3$. An increasing in each of its arguments function $F: [0,1]^n \to [0,1]$ preserves negative S-transitivity if and only if $S \gg F$, it means that for any $(s_1,...,s_n)$, $(t_1,...,t_n) \in [0,1]^n$ the following inequality holds*

$$S(F(s_1,\ldots,s_n), F(t_1,\ldots,t_n)) \geqslant$$

$$F(S(s_1,t_1),\ldots,S(s_n,t_n)). \qquad (14)$$

Proof. Let us fix $s_k, t_k \in [0,1]$, $k = 1,...,n$ and take $u,v,w \in X$. By assumption that *card* $X \geqslant 3$, we may write $u \neq v$, $v \neq w$, $u \neq w$. We also denote $A = \{u,v,w\}$. Firstly, we will create fuzzy relations W_k, $k = 1,...,n$ in the set A described by the matrices

$$W_k = \begin{bmatrix} 1 & s_k & S(s_k,t_k) \\ 1 & 1 & t_k \\ 1 & 1 & 1 \end{bmatrix}, k = 1,...,n.$$

In other words

$$W_k(u,v) = s_k,\ W_k(v,w) = t_k,\ W_k(u,w) = S(s_k,t_k). \qquad (15)$$

By Lemma 5 relations W_k are negatively S-transitive in A because

$$W_k\circledast' W_k = \begin{bmatrix} 1 & 1 & S(s_k,t_k) \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \geqslant W_k,\ \ k = 1,...,n.$$

In virtue of Lemma 6 relations $R_k \in FR(X)$, $k = 1,...,n$,

$$R_k(x,y) = \begin{cases} W_k(x,y), & x,y \in A \\ 1, & otherwise \end{cases} \qquad (16)$$

are negatively S-transitive in X. By assumption F preserves negative S-transitivity of fuzzy relations so relation $R_F = F(R_1,...,R_n)$ is also negatively S-transitive for $x,y,z \in X$, i.e.

$$S(F(R_1(x,y),...,R_n(x,y)), F(R_1(y,z),...,R_n(y,z)))$$

$$\geqslant F(R_1(x,z),...,R_n(x,z)).$$

In particular, for elements $u,v,w \in X$ the above inequality is also fulfilled. Thus applying the notations (15) and formula (16) we have

$$S(F(s_1,...,s_n), F(t_1,...,t_n)) =$$

$$S(F(R_1(u,v),...,R_n(u,v)), F(R_1(v,w),...,R_n(v,w)))$$

$$\geqslant F(R_1(u,w),...,R_n(u,w)) =$$

$$F(S(s_1,t_1),...,S(s_n,t_n)),$$

which means that operation F fulfils (14).

Now, let an increasing operation F fulfil (14), $x,y,z \in X$. If relations R_k are negatively S-transitive for $k = 1,...,n$, then $S(R_k(x,y), R_k(y,z)) \geqslant R_k(x,z)$. We will prove

that the relation R_F is negatively S-transitive. Applying the notations $R_k(x,y) = s_k$, $R_k(y,z) = t_k$, for $k = 1, ..., n$ we obtain

$$S(R_F(x,y), R_F(y,z)) =$$

$$S(F(R_1(x,y), ..., R_n(x,y)), F(R_1(y,z), ..., R_n(y,z))) =$$

$$S(F(s_1, ..., s_n), F(t_1, ..., t_n)) \geqslant$$

$$F(S(s_1, t_1), ..., S(s_n, t_n)) =$$

$$F(S(R_1(x,y), R_1(y,z)), ..., S(R_n(x,y), R_n(y,z))) \geqslant$$

$$F(R_1(x,z), ..., R_n(x,z)) = R_F(x,z).$$

As a result operation F preserves negative S-transitivity. $\square$

Example 10. By Example 6 we see that the weighted maximum preserves negative S-transitivity for any t-conorm S. The weighted arithmetic mean and the functions of the form (2) preserve negative S_L-transitivity. Moreover, negative S_M-transitivity (i.e. negative transitivity) is preserved if and only if a function F is of the form

$$F(t_1, ..., t_n) = \max(f_1(t_1), ..., f_n(t_n)),$$

for each $t_1, ..., t_n \in [0,1]$ and $f_k \colon [0,1] \to [0,1]$ being increasing with $k = 1, ..., n$.

Now will consider T-S-Ferrers property. The other result for preservation of this property (and also T-transitivity) for quasi-arithmetic means one can find in [9], p. 140–141.

Theorem 11. *If a function $F \colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F \gg T$ and $S \gg F$, then it preserves T-S-Ferrers property.*

Proof. Let an increasing function F fulfil $F \gg T$ and $S \gg F$, $x, y, u, v \in X$. If relations R_k have T-S-Ferrers property for $k = 1, ..., n$, then $T(R_k(x,y), R_k(u,v)) \leqslant S(R_k(x,v), R_k(u,y))$. We will prove that the relation R_F has T-S-Ferrers property. We obtain

$$T(R_F(x,y), R_F(u,v)) =$$

$$T(F(R_1(x,y), ..., R_n(x,y)), F(R_1(u,v), ..., R_n(u,v)))$$

$$\leqslant$$

$$F(T(R_1(x,y), R_1(u,v)), ..., T(R_n(x,y), R_n(u,v))) \leqslant$$

$$F(S(R_1(x,v), R_1(u,y)), ..., S(R_n(x,v), R_n(u,y))) \leqslant$$

$$S(F(R_1(x,v), ..., R_n(x,v)), F(R_1(u,y), ..., R_n(u,y)))$$

$$= S(R_F(x,v), R_F(u,y)).$$

As a result function F preserves T-S-Ferrers property. $\square$

The next statement gives a sufficient condition for a function F to preserve T-S-Ferrers property.

Lemma 7. *Let T be a t-norm and S be a corresponding dual t-conorm. If $F \colon [0,1]^n \to [0,1]$ is a self-dual function, then $F \gg T$ implies $S \gg F$.*

Proof. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$. If $F \gg T$ we obtain

$$F(T(x_1, y_1), \ldots, T(x_n, y_n)) \geqslant$$

$$T(F(x_1, \ldots, x_n), F(y_1, \ldots, y_n)).$$

Since $T(x,y) = 1 - S(1-x, 1-y)$ for $x, y \in [0,1]$, it follows

$$F(1 - S(1-x_1, 1-y_1), \ldots, 1 - S(1-x_n, 1-y_n)) \geqslant$$

$$1 - S(1 - F(x_1, \ldots, x_n), 1 - F(y_1, \ldots, y_n)).$$

As a result

$$1 - F(1 - S(1-x_1, 1-y_1), \ldots, 1 - S(1-x_n, 1-y_n)) \leqslant$$

$$S(1 - F(x_1, \ldots, x_n), 1 - F(y_1, \ldots, y_n)),$$

and by $F = F^d$ we get

$$F(S(1-x_1, 1-y_1), \ldots, S(1-x_n, 1-y_n)) \leqslant$$

$$S(1 - F(x_1, \ldots, x_n), 1 - F(y_1, \ldots, y_n)).$$

By substitution $1 - x_i = a_i$ and $1 - y_i = b_i$ for $i = 1, \ldots, n$, where $a_i, b_i \in [0,1]$ and applying again the assumption $F = F^d$ we obtain

$$F(S(a_1, b_1), \ldots, S(a_n, b_n)) \leqslant$$

$$S(F(a_1, \ldots, a_n), F(b_1, \ldots, b_n)),$$

which proves that $S \gg F$. $\square$

Example 11. By Example 5 we know that the weighted arithmetic mean dominate T_L and the weighted arithmetic mean are self-dual aggregation functions (Example 3), so by Lemma 7 we see that the weighted arithmetic mean preserve T-S-Ferrers property for T_L and S_L.

Now T-S-semitransitivity will be discussed.

Theorem 12. *If a function $F \colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F \gg T$ and $S \gg F$, then it preserves T-S-semitransitivity.*

Proof. Let an increasing function F fulfil $F \gg T$ and $S \gg F$, $x, y, u, v \in X$. If relations R_k are T-S-semitransitive for $k = 1, ..., n$, then $T(R_k(x,y), R_k(y,v)) \leqslant S(R_k(x,u), R_k(u,v))$. We will prove that the relation R_F is T-S-semitransitive. We obtain

$$T(R_F(x,y), R_F(y,v)) =$$

$$T(F(R_1(x,y), ..., R_n(x,y)), F(R_1(y,v), ..., R_n(y,v)))$$

$$\leqslant$$

$$F(T(R_1(x,y), R_1(y,v)), ..., T(R_n(x,y), R_n(y,v))) \leqslant$$

$$F(S(R_1(x,u), R_1(u,v)), ..., S(R_n(x,u), R_n(u,v))) \leqslant$$

$$S(F(R_1(x,u), ..., R_n(x,u)), F(R_1(u,v), ..., R_n(u,v)))$$

$$= S(R_F(x,u), R_F(u,v)).$$

As a result function F preserves T-S-semitran–sitivity. $\square$

Examples of functions which preserve T-S-semi–transitivity are the same as for T-S-Ferrers property.

Example 12. Conditions given in Theorems 11 and 12 are only the sufficient ones. Let us consider function $F(s,t) = st$ (so $F = T_P$) and fuzzy relations presented by the matrices

$$R_1 = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right], \quad R_2 = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right].$$

Relations R_1, R_2 are min-max-Ferrers ([9], p. 142) and min-max-semitransitive, and $R = F(R_1, R_2)$ is both min-max-Ferrers and min-max-semitransitive, where $R \equiv 0$. However, it is not true that $F \gg$ min (the only t-norm that dominates minimum is minimum itself, see Theorem 3).

4. Conclusion

In this contribution we presented the necessary and sufficient conditions for the preservation of fuzzy relation properties. The considered properties involve triangular norms T and triangular conorms S. In the case of T-S-Ferrers property and T-S-semitransitivity only the sufficient conditions were obtained, but suitable counter-examples showing that the necessity does not hold were provided.

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