

Compatible operations in some subvarieties of the variety of weak Heyting algebras

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Abstract

Weak Heyting algebras are a natural generalization of Heyting algebras (see [2], [5]). In this work we study certain subvarieties of the variety of weak Heyting algebras in order to extend some known results about compatible functions in Heyting algebras.

Keywords: weak Heyting algebras, compatible functions, locally affine completeness.

1. Introduction

The algebraic similarity type of the algebras we consider in this paper is the similarity type $\{\wedge, \vee, 0, 1\}$ of bounded distributive lattices augmented with a binary operation symbol \rightarrow , i.e., it is the standard similarity type of Heyting algebras.

A *weak Heyting algebra*, or *WH-algebra* ([2], [5]), is an ordered algebraic structure $\langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$, where the reduct algebra $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice and $\rightarrow: A \times A \rightarrow A$ is a map such that for all $a, b, c \in A$ satisfies the following conditions:

1. $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$,
2. $(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$,
3. $(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c$,
4. $a \rightarrow a = 1$.

We write WH to indicate the variety of WH-algebras.

Other examples of WH-algebras that appear in the literature are the Basic algebras introduced by M. Ardesir and W. Ruitenburg in [1] and the subresiduated lattices of G. Epstein and A. Horn in [7]; these last structures were introduced as a generalization of Heyting algebras.

A Basic algebra is a WH-algebra that in addition satisfies the inequality

$$(I) \quad a \leq 1 \rightarrow a.$$

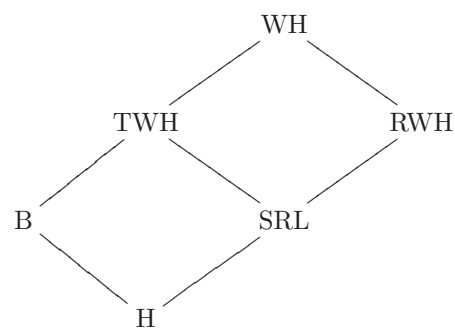
A subresiduated lattice ^a is a WH-algebra that in addition satisfies the inequalities

^aIn Theorem 1 of [7] was proved that a subresiduated lattice is an algebra $\langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$, where $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice and the following four conditions hold for every $a, b, c \in A$: $(a \wedge b) \rightarrow b = 1$, $c \rightarrow (a \wedge b) = (c \rightarrow a) \wedge (c \rightarrow b)$, (R) and (T). From this theorem and some

$$(T) \quad a \rightarrow b \leq c \rightarrow (a \rightarrow b),$$

$$(R) \quad a \wedge (a \rightarrow b) \leq b.$$

Besides Basic algebras and subresiduated lattices, it can be considered other varieties of WH-algebras that are obtained by considering arbitrary combinations of the three inequalities (R), (T) and (I) above. These varieties are the varieties of WH-algebras that correspond to certain subintuitionistic logics. In every WH-algebra the inequality (I) implies (T). There are at most five subvarieties obtainable in that way, and in fact there are exactly five. They are the variety of subresiduated lattices, denoted SRL, the variety of Basic algebras, denoted B, the variety of the WH-algebras that satisfy (R), whose elements will be called RWH-algebras, the variety of the WH-algebras that satisfy (T), whose elements will be called TWH-algebras, and finally the variety of Heyting algebras (which are the WH-algebras that satisfy the three inequalities (R), (T) and (I)). The variety of RWH-algebras will be denoted by RWH and the variety of TWH-algebras by TWH. If in addition we denote by H to the variety of Heyting algebras, the relation between all these varieties is depicted in the following figure:



The variety WH is very well behaved from the point of view of the properties which for the varieties that are the equivalent algebraic semantics ([3]) of some algebraizable logic correspond to logical properties. In particular, the five varieties abovementioned inherit some of the properties of their subvariety of Heyting algebras. Recall that a variety is locally finite if its finitely generated elements are

properties given in [7] about subresiduated lattices, we have that an algebra $\langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a subresiduated lattice iff it is a WH-algebra satisfying the inequalities (R) and (T). This fact justifies our definition.

finite. Since H is not locally finite, the other five varieties (WH, RWH, TWH, SRL and B) are not locally finite either. The variety WH is an arithmetical variety and has the congruence extension property, therefore all of its subvarieties, and in particular the five considered, have these properties too. Moreover, the varieties TWH and SRL have equationally definable principal congruences, but WH and RWH do not (see [5]).

In this paper are given two characterizations for a function to be compatible in an algebra of RWH and in a subresiduated lattice, one for unary functions and from this one for functions of arbitrary arity is derived. These characterizations are used to show that these two varieties are locally affine complete, that is, on a finite subset of an algebra in these varieties any compatible function is a polynomial. Then it is proved that if an unary function preserves finite infimum and it is expansive, then it is compatible. Finally are given generalizations of the notions of frontal operator, successor function and gamma function in weak Heyting algebras.

2. Basic results

The following properties follows from [5].

Proposition 1. (Prop. 3.2) *Let A be a WH-algebra. Then for every $a, b, c \in A$ we have the following conditions:*

1. *If $a \leq b$, then $c \rightarrow a \leq c \rightarrow b$ and $b \rightarrow c \leq a \rightarrow c$.*
2. *If $a \leq b$, then $a \rightarrow b = 1$.*
3. *$(a \rightarrow b) \wedge (a \rightarrow c) \leq a \rightarrow (c \vee b)$.*

Proposition 2. (Prop. 4.22) *Let $A \in \text{WH}$. The following conditions are equivalent:*

- (a) *A is a RWH-algebra.*
- (b) *For every $a, b, c \in A$, if $a \leq b \rightarrow c$, then $a \wedge b \leq c$.*

Given a WH-algebra A , a filter F of A is said to be an *open filter* if for every $a \in F$, $1 \rightarrow a \in F$. We will abbreviate $1 \rightarrow a$ by $\Box(a)$, and then the iterated operator \Box^n is defined in the usual way.

Remark 1. *Let A be a WH-algebra and $a, b \in A$.*

- (a) *Straightforward computations show that $\Box(1) = 1$ and $\Box(a \wedge b) = \Box(a) \wedge \Box(b)$. Thus, \Box is monotonic.*
- (b) *If in addition A is a RWH algebra, then $\Box(a) \leq a$. It follows from that $1 \rightarrow a \leq 1 \rightarrow a$ and Proposition 2. Then $\Box^n(a) \leq \Box^m(a)$ when $n \geq m$.*

Let \mathbb{N} be the set of natural numbers. A direct computation proves that for every $a \in A$, the open filter generated by $\{a\}$ is the filter

$$\{x \in A : a \wedge \Box(a) \wedge \dots \wedge \Box^n(a) \leq x, \text{ for some } n \in \mathbb{N}\}.$$

The previous filter will be denoted by $F^o(a)$.

As usual, for $a, b \in A$ we define

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

Theorem 1. (Th. 6.12) *For every RWH-algebra A there is an isomorphism between the lattice of open filters of A and the lattice $\text{Con}(A)$ of congruences of A . Moreover, the isomorphism is given by the function H defined on the set of open filters of A by declaring*

$$H(F) = \{(a, b) \in A \times A : a \leftrightarrow b \in F\},$$

and the inverse of H is such that for every congruence θ of A , $H^{-1}(\theta) = \{a \in A : (a, 1) \in \theta\}$.

The previous theorem is a generalization of the Theorem 2 given in [7].

3. Compatible functions

Definition 1. *Let A be an algebra and let $f : A^k \rightarrow A$ be a function (not necessarily an homomorphism).*

1. *We say that f is compatible with a congruence θ of A if $(a_i, b_i) \in \theta$ for $i = 1, \dots, k$ implies $(f(a_1, \dots, a_k), f(b_1, \dots, b_k)) \in \theta$.*
2. *We say that f is a compatible function of A provided it is compatible with all the congruences of A .*

Note that if A is an algebra and $f : A^n \rightarrow A$ is a function, then f is compatible iff the algebras A and $\langle A, f \rangle$ have the same congruences.

For $\theta \in \text{Con}(A)$ and $a \in A$, we write a/θ for the equivalence class of a . For $a, b \in A$, the subset $\theta(a, b)$ of $A \times A$ denotes the smallest congruence that contains the element (a, b) .

Let $f : A \rightarrow A$ be a function. Recall the following convenient remark: f is compatible iff $(f(a), f(b)) \in \theta(a, b)$ for every $a, b \in A$.

The simplest examples of compatible functions on an algebra are the polynomial functions; note that in particular, all constant functions are compatible.

The following lemma is useful in order to give a description for compatible functions:

Lemma 2. *Let A be a RWH-algebra and $a, b \in A$. Then*

- (a) *$F^o(a) = \{x \in A : \Box^n(a) \leq x \text{ for some } n \in \mathbb{N}\}$, where we define $\Box^0(a) = a$ for every $a \in A$.*
- (b) *Let $\theta \in \text{Con}(A)$, Then we have that $(a, b) \in \theta$ iff $a \leftrightarrow b \in 1/\theta$.*
- (c) *$1/\theta(a, b) = F^o(a \leftrightarrow b) = \{x \in A : \Box^n(a \leftrightarrow b) \leq x, \text{ for some } n \in \mathbb{N}\}$.*

Proof. (a) It follows from Remark 1.

(b) By Theorem 1 we have that $H(H^{-1}(\theta)) = \theta$, i.e., $(a, b) \in \theta$ iff $(a \leftrightarrow b, 1) \in \theta$.

(c) It is consequence from (a), (b) and Theorem 1. \square

As immediate consequence of previous lemma we have the following

Proposition 3. *Let A be a RWH-algebra and $f : A \rightarrow A$ a function.*

The following conditions are equivalent:

1. f is compatible.
2. For every $a, b \in A$ there exists $n \in \mathbb{N}$ such that $\Box^n(a \leftrightarrow b) \leq f(a) \leftrightarrow f(b)$.

Let A be a subresiduated lattice and $a, b \in A$, then

$$a \leftrightarrow b \leq \Box(a \rightarrow b) \wedge \Box(b \rightarrow a) = \Box(a \leftrightarrow b).$$

The previous remark shows that for every $n \in \mathbb{N}$ it holds that $a \leftrightarrow b \leq \Box^n(a \leftrightarrow b)$. Then by item (a) of Lemma 2 we have that

$$a \leftrightarrow b = \Box^n(a \leftrightarrow b).$$

Thus we have the following

Corollary 3. *Let A be a SRL-algebra and let $f : A \rightarrow A$ be a function.*

The following conditions are equivalent:

1. f is compatible.
2. For every $a, b \in A$, $a \leftrightarrow b \leq f(a) \leftrightarrow f(b)$.

Let A be an algebra and let $f : A^k \rightarrow A$ be a function. For every $b_i \in A$ ($i = 1, \dots, k$) we define $\bar{b} = (b_1, \dots, b_k)$ and $\bar{b}(i) = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_k)$. Then we define the functions $f_{\bar{b}(i)} : A \rightarrow A$ by $f_{\bar{b}(i)}(a) = f(b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_k)$.

The following remark is a consequence of the definition of k -ary compatible function on an algebra.

Remark 2. *Let A be an algebra and let $f : A^k \rightarrow A$ be a function. The function f is compatible iff for every $\bar{b} \in A^k$ the functions $f_{\bar{b}(i)}$ are compatible.*

Then we obtain the following

Corollary 4. *Let $A \in \text{RWH}$ and let $f : A^k \rightarrow A$ be a function.*

The following conditions are equivalent:

1. f is compatible.
2. For every $\bar{a}, \bar{b} \in A^k$ there exists $n \in \mathbb{N}$ such that

$$\Box^n(a_1 \leftrightarrow b_1) \wedge \dots \wedge \Box^n(a_k \leftrightarrow b_k) \leq f(\bar{a}) \leftrightarrow f(\bar{b}). \quad (1)$$

Proof. Suppose that f is a compatible function, and let $\bar{a}, \bar{b} \in A^k$. By Proposition 3 there are $n_1, \dots, n_k \in \mathbb{N}$ such that

$$\begin{aligned} \Box^{n_1}(a_1 \leftrightarrow b_1) &\leq f(\hat{a}) \leftrightarrow f(b_1, a_2, \dots, a_k) \\ \Box^{n_2}(a_2 \leftrightarrow b_2) &\leq f(b_1, a_2, \dots, a_k) \leftrightarrow f(b_1, b_2, a_3, \dots, a_k) \\ &\dots \\ \Box^{n_k}(a_k \leftrightarrow b_k) &\leq f(b_1, b_2, \dots, b_{k-1}, a_k) \leftrightarrow f(\hat{b}) \end{aligned}$$

Let $n = \max \{n_i : i = 1, \dots, k\}$. Then we obtain that

$$\Box^n(a_i \leftrightarrow b_i) \leq f(\bar{a}) \leftrightarrow f(\bar{b}),$$

for every $i = 1, \dots, k$. Hence,

$$\Box^n(a_1 \leftrightarrow b_1) \wedge \dots \wedge \Box^n(a_k \leftrightarrow b_k) \leq f(\bar{a}) \leftrightarrow f(\bar{b}).$$

Therefore, we deduce the condition (1).

Conversely, suppose that it holds the condition (1). Let $\theta \in \text{Con}(A)$ and $a_i \theta b_i$ for $i = 1, \dots, k$. By Lemma 2 we obtain $(a_i \leftrightarrow b_i)\theta 1$, so

$$\Box^n(a_1 \leftrightarrow b_1) \wedge \Box^n(a_2 \leftrightarrow b_2) \wedge \dots \wedge \Box^n(a_k \leftrightarrow b_k) \in 1/\theta.$$

By condition (1) we have that $f(\bar{a}) \leftrightarrow f(\bar{b}) \in 1/\theta$. Taking into account Lemma 2 we deduce that $(f(\bar{a}), f(\bar{b})) \in \theta$, i.e., f is compatible. \square

Then we have the following

Corollary 5. *Let $A \in \text{SRL}$ and let $f : A^k \rightarrow A$ be a function.*

The following conditions are equivalent:

1. f is compatible.
2. For every $\bar{a}, \bar{b} \in A^k$, $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge \dots \wedge (a_k \leftrightarrow b_k) \leq f(\bar{a}) \leftrightarrow f(\bar{b})$.

The characterization for compatible functions on Heyting algebras given in Lemma 2.1 of [4] is exactly the same given in Corollary 5.

4. Local affine completeness

A function obtained by composition of basic operations of the algebra and parameters (polynomial function) is compatible in every algebra. It naturally rise the question of whether there are compatible functions different from polynomials. In the variety of boolean algebras the answer is no (see [11]), thus we say that it is an *affine complete variety*. On the other hand, the variety of Heyting algebras is not an affine complete variety (see Example 2.1 of [4], and [8]). However, it is *locally* affine complete in the sense that any restriction of a compatible function to a finite subset is a polynomial (see [10], [11], [13]).

Remark 3. *Let $A \in \text{RWH}$, let $f : A^k \rightarrow A$ be a compatible function and let B be a finite subset of A^k . Let n be the maximum of the natural number associated in Corollary 4 to all pairs (\bar{b}, \bar{x}) where \bar{x} and \bar{b} range over all points of B . The monotony of \square implies that*

$$\Box^n(b_1 \leftrightarrow x_1) \wedge \dots \wedge \Box^n(b_k \leftrightarrow x_k) \leq f(\bar{b}) \leftrightarrow f(\bar{x}). \quad (2)$$

In the next we show the locally affine completeness of the variety RWH.

Theorem 6. Let $A \in \text{RWH}$, let $f : A^k \rightarrow A$ be a compatible function, let B be a finite subset of A^k and let $\bar{x} \in B$. Let

$$T_{\bar{x}} = \{\Box^n(b_1 \leftrightarrow x_1) \wedge \dots \wedge \Box^n(b_k \leftrightarrow x_k) \wedge f(\bar{b}) : \bar{b} \in B\},$$

where n is the natural number given in Remark 3. Then, $f(\bar{x}) = \bigvee T_{\bar{x}}$.

Proof. Let $\bar{x} \in B$. For every $\bar{b} \in B$, by (2) of Remark 3 we obtain that

$$\Box^n(b_1 \leftrightarrow x_1) \wedge \dots \wedge \Box^n(b_k \leftrightarrow x_k) \wedge f(\bar{b}) \leq f(\bar{x}).$$

It proves that $f(\bar{x})$ is an upper bound of $T_{\bar{x}}$.

On the other hand, since $\Box^n(x_i \leftrightarrow x_i) = 1$ for every $i = 1, \dots, k$, we have that

$$\Box^n(x_1 \leftrightarrow x_1) \wedge \dots \wedge \Box^n(x_k \leftrightarrow x_k) \wedge f(\bar{x}) = f(\bar{x}).$$

Therefore, $f(\bar{x}) = \bigvee T_{\bar{x}}$. \square

Corollary 7. The varieties RWH and SRL are locally affine complete.

It follows from the previous corollary that every finite algebra in RWH or in SRL is affine complete.

5. Some examples of compatible functions

In the following we will write A to indicate a RWH -algebra.

We say that a function $f : A \rightarrow A$ is a C -function if it satisfies the following two equations:

$$(C1) \quad f(a \wedge b) = f(a) \wedge f(b),$$

$$(C2) \quad a \leq f(a).$$

Proposition 4. Let $f : A \rightarrow A$ be a C -function. Then f is compatible.

Proof. Let $a, b \in A$. Using the equation $a \wedge (a \rightarrow b) \leq b$ and (C1), we have that

$$f(a) \wedge f(a \rightarrow b) \leq f(b).$$

Thus,

$$f(a) \rightarrow (f(a) \wedge f(a \rightarrow b)) \leq f(a) \rightarrow f(b),$$

i.e.,

$$f(a) \rightarrow f(a \rightarrow b) \leq f(a) \rightarrow f(b). \quad (3)$$

On the other hand, it follows from (C2) the fact that $a \rightarrow b \leq f(a \rightarrow b)$, so

$$f(a) \rightarrow (a \rightarrow b) \leq f(a) \rightarrow f(a \rightarrow b).$$

Hence, by (3) we obtain that

$$f(a) \rightarrow (a \rightarrow b) \leq f(a) \rightarrow f(b). \quad (4)$$

As $f(a) \leq 1$ we have that

$$f(a) \rightarrow (a \rightarrow b) \geq 1 \rightarrow (a \rightarrow b) = \Box(a \rightarrow b).$$

Then,

$$\Box(a \rightarrow b) \leq f(a) \rightarrow (a \rightarrow b),$$

so by (4) we have that $\Box(a \rightarrow b) \leq f(a) \rightarrow f(b)$. In consequence we have that

$$\Box(a \leftrightarrow b) \leq f(a) \leftrightarrow f(b).$$

Therefore, by Proposition 3 we conclude that f is a compatible function. \square

Definition 2. Let $n \in \mathbb{N}$ and let $\tau_n : A \rightarrow A$ be a C -function. We say that τ_n is a n -frontal operator if it satisfies the additional equation

$$(Fn) \quad \tau_n(a) \leq b \vee (\Box^n(b) \rightarrow a).$$

If $n = 0$ we have that τ_0 is a frontal operator in the sense of [6].

Definition 3. Let $n \in \mathbb{N}$. We define the function $s_n : A \rightarrow A$ through equations (C2), (Fn) and the additional equation

$$(sn) \quad \Box^n(s_n(a)) \rightarrow a \leq a.$$

The function s_0 is the successor function in the sense of [6] (see also [4], [9], [12] for the case of Heyting algebras).

Proposition 5. Let $n \in \mathbb{N}$ and suppose that there exists s_n on A . Then s_n is a n -frontal operator and $s_n(a) = \min \{b \in A : \Box^n(b) \rightarrow a \leq b\}$.

Proof. Similar to the proof of Lemma 3.3 of [6]. \square

Definition 4. Let $n \in \mathbb{N}$. We define the function $t_n : A \rightarrow A$ through equation (Fn) and the following two additional equations:

$$(t1)_n \quad \Box^n(t_n(0)) \rightarrow 0 = 0,$$

$$(t2)_n \quad t_n(a) = a \vee t_n(0).$$

The function t_0 is the gamma function in the sense of [6]. A direct computation based in the previous definition shows that if there exists t_n , then it is a n -frontal operator and $t_n(0) = \min \{b \in A : \Box^n(b) \rightarrow 0 \leq b\}$.

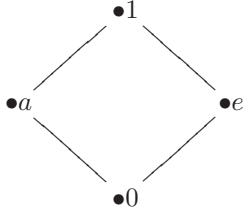
We end this work with some concrete examples of compatible functions on finite algebras, for which the considered functions will be polynomials.

Example 1. Consider the chain of three elements $H_3 = \{0, a, 1\}$ with the following binary operation:

\rightarrow	0	a	1
0	1	1	1
a	0	1	1
1	0	0	1

Then we have that $\langle H_3, \rightarrow \rangle \in \text{SRL}$, it is not a Heyting algebra, and $0 \leftrightarrow a = 0 \leftrightarrow 1 = a \leftrightarrow 1 = 0$. Then, by Corollary 5 we have that $\langle H_3, \rightarrow \rangle$ is a functionally complete algebra, i.e., every n -ary function is compatible.

Example 2. Let H be the poset given by



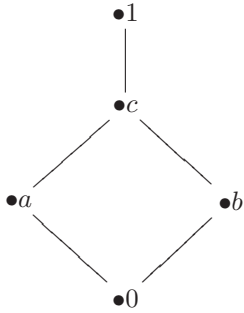
Consider the following binary operation:

\rightarrow	0	a	e	1
0	1	1	1	1
a	e	1	e	1
b	0	0	1	1
1	0	0	e	1

Then $\langle H, \rightarrow \rangle \in \text{SRL}$ and it is not a Heyting algebra.

Let $n \geq 1$. Hence for every $x \neq a$ we have that $\Box^n(x) = x$ and $\Box^n(a) = 0$. Suppose that there is s_n , so $s_n(a) = \min \{x \in H : \Box^n(x) \rightarrow a \leq x\} = e$. However $a \leq s_n(a) = e$, which is a contradiction. Hence, there is not function s_n . Moreover, the converse of Proposition 5 is not true in general. Now suppose that there is t_n , so $t_n(a) = a \vee t_n(0) = a \vee e = 1$. It follows from **(Fn)** that $1 = e \vee (b \rightarrow a) = e$, which is an absurd. Thus, there is not t_n . In a similar way we can prove that it is not possible define the functions s_0 and t_0 .

Example 3. Let H be the following poset:



We define the following binary operation:

\rightarrow	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	0	0	0	1

Then $\langle H, \rightarrow \rangle \in \text{RWH}$ and it is not a subresiduated lattice because for instance we have that $b = a \rightarrow 0 \not\leq 1 \rightarrow (a \rightarrow 0) = 1 \rightarrow b = 0$.

We obtain the following table:

x	$s_0(x)$	$t_0(x)$
0	c	c
a	c	c
b	c	c
c	1	1
1	1	1

Moreover, for every $n \geq 1$ it holds that $s_n = t_n = 1$.

6. Conclusions

In this paper we have generalized some well known results about compatible functions on Heyting algebras. It was done on the basis of the good description of the lattice of congruences of any RWH -algebra.

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