

# Solutions of the Distributivity Equation $\mathcal{I}(\mathcal{T}(x, y), z) = \mathcal{S}(\mathcal{I}(x, z), \mathcal{I}(y, z))$ for Some t-Representable T-Norms and T-Conorms

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## Abstract

Recently, in [1], [2], [3], and [4] we have discussed the distributivity equation of implications  $\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))$  over t-representable t-norms, generated from (classical) continuous Archimedean t-norms, in interval-valued fuzzy sets theory. In [5] we discussed similar methods, but for the following distributivity functional equation  $\mathcal{I}(x, \mathcal{S}_1(y, z)) = \mathcal{S}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))$ , when  $\mathcal{S}_1, \mathcal{S}_2$  are t-representable t-conorms. In this article we continue investigations presented at previous EUSFLAT-LFA 2011, i.e., we will show all solutions for the following distributivity equation  $\mathcal{I}(\mathcal{T}(x, y), z) = \mathcal{S}(\mathcal{I}(x, z), \mathcal{I}(y, z))$ , where  $\mathcal{I}$  is an unknown function,  $\mathcal{T}$  is a t-representable t-norm on  $\mathcal{L}^I$  generated from nilpotent t-norms  $T_1, T_2$  and  $\mathcal{S}$  is a t-representable t-conorm on  $\mathcal{L}^I$  generated from strict t-conorms  $S_1, S_2$ .

**Keywords:** triangular norm, t-norm, t-conorm, fuzzy implication, interval-valued fuzzy sets, distributivity equations, functional equations

## 1. Introduction

Distributivity of (classical) fuzzy implications over different fuzzy logic connectives has been studied in the recent past by many authors (see [6], [7], [8], [9], [10], [11], [12]). These equations have a very important role to play in efficient inferencing in approximate reasoning, especially in fuzzy control systems. Since all the rules of an inference engine are exercised during every inference cycle, the number of rules directly affects the computational duration of the overall application. To reduce the complexity of fuzzy “IF-THEN” rules, Combs and Andrews [13] required of the following classical tautology

$$(p \wedge q) \rightarrow r = (p \rightarrow r) \vee (q \rightarrow r).$$

Subsequently, there were many discussions (see [14], [15], [16], [17]), most of them pointed out the need for a theoretical investigation required for employing such equations. An overview of the most important methods that reduce the complexity of different inference systems can be found in [18, Chapter 8].

Recently, in [1], [2], [3] and [4] we have discussed the distributivity equation of implications

$$\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))$$

over t-representable t-norms generated from continuous Archimedean t-norms, in interval-valued fuzzy sets theory. In these articles, as a byproduct, we have obtained the solutions for each of the following functional equations, respectively:

$$f(u_1 + v_1, u_2 + v_2) = f(u_1, u_2) + f(v_1, v_2),$$

$$\begin{aligned} g(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) \\ = g(u_1, u_2) + g(v_1, v_2), \end{aligned}$$

$$\begin{aligned} h(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) \\ = \min(h(u_1, u_2) + h(v_1, v_2), b), \end{aligned}$$

$$\begin{aligned} k(u_1 + v_1, u_2 + v_2) \\ = \min(k(u_1, u_2) + k(v_1, v_2), b) \end{aligned}$$

where  $a, b > 0$  are fixed real numbers,  $f: L^\infty \rightarrow [0, \infty]$ ,  $g: L^a \rightarrow [0, \infty]$ ,  $h: L^a \rightarrow [0, b]$ , and  $k: L^\infty \rightarrow [0, b]$  are unknown functions. The above we use the following notation  $L^\infty = \{(u_1, u_2) \in [0, \infty]^2 \mid u_1 \geq u_2\}$ ,  $L^a = \{(u_1, u_2) \in [0, a]^2 \mid u_1 \geq u_2\}$ . In this paper, using solutions of the second equation above (presented on previous EUSFLAT-LFA 2011 conference), we continue these investigations, but for the following functional equation

$$\mathcal{I}(\mathcal{T}(x, y), z) = \mathcal{S}(\mathcal{I}(x, z), \mathcal{I}(y, z)), \quad (\text{D-TS})$$

satisfied for all  $x, y, z \in L^I$ , where  $\mathcal{I}$  is an unknown function,  $\mathcal{T}$  is a t-representable t-norm on  $\mathcal{L}^I$  generated from nilpotent t-norms  $T_1, T_2$  and  $\mathcal{S}$  is a t-representable t-conorm on  $\mathcal{L}^I$  generated from strict t-conorms  $S_1, S_2$ .

Please note that such theoretical developments connected with solutions of different functional equations can be also useful in other topics like fuzzy mathematical morphology (see [19]) or similarity measures (cf. [20]).

## 2. Intuitionistic and interval-valued fuzzy sets theories

Intuitionistic fuzzy sets theory introduced in 1983 by Atanassov [21] assign to each element of the universe not only a membership degree, but also a non-membership degree (for the discussion connected with the proposed terminology see [22]).

**Definition 2.1.** An intuitionistic fuzzy set  $A$  on  $X$  is a set

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where  $\mu_A, \nu_A: X \rightarrow [0, 1]$  are called, respectively, the membership function and the non-membership function. Moreover they satisfy the condition

$$\mu_A(x) + \nu_A(x) \leq 1, \quad x \in X.$$

Let us define

$$L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \geq y_2.$$

One can easily observe that  $\mathcal{L}^* = (L^*, \leq_{L^*})$  is a complete lattice with units  $0_{L^*} = (0, 1)$  and  $1_{L^*} = (1, 0)$ . Moreover, an intuitionistic fuzzy set  $A$  on  $X$  can be represented by the  $\mathcal{L}^*$ -fuzzy set given by  $A: X \rightarrow L^*$ .

Another extension of fuzzy sets theory is interval-valued fuzzy sets theory introduced, independently, by Sambuc [23] and Gorzalczyński [24], in which to each element of the universe a closed subinterval of the unit interval is assigned – it can be used as an approximation of the unknown membership degree. Let us define

$$L^I = \{(x_1, x_2) \in [0, 1]^2 : x_1 \leq x_2\},$$

$$(x_1, x_2) \leq_{L^I} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \leq y_2.$$

In the sequel, if  $x \in L^I$ , then we denote it by  $x = [x_1, x_2]$ . One can easily observe that  $\mathcal{L}^I = (L^I, \leq_{L^I})$  is also a complete lattice with units  $0_{\mathcal{L}^I} = [0, 0]$  and  $1_{\mathcal{L}^I} = [1, 1]$ .

**Definition 2.2.** An interval-valued fuzzy set on  $X$  is a mapping  $A: X \rightarrow L^I$ .

It is important to notice that in [25] it is shown that intuitionistic fuzzy sets theory is equivalent, from the mathematical point of view, to interval-valued fuzzy sets theory. In fact, we can see the point  $(x_1, x_2) \in L^*$  as the interval  $[x_1, 1 - x_2] \in L^I$  (and vice-verse). Since we are limited in number of pages, in this article we will discuss main results in the language of interval-valued fuzzy sets, but they can be easily transformed to the intuitionistic case.

### 3. Basic fuzzy connectives

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but we briefly mention some of the results employed in the rest of the work.

**Definition 3.1.** Let  $\mathcal{L} = (L, \leq_L)$  be a complete lattice. An associative, commutative operation  $\mathcal{T}: L^2 \rightarrow L$  is called a t-norm if it is increasing and  $1_{\mathcal{L}}$  is the neutral element of  $\mathcal{T}$ . An associative, commutative operation  $\mathcal{S}: L^2 \rightarrow L$  is called a t-conorm if it is increasing and  $0_{\mathcal{L}}$  is the neutral element of  $\mathcal{S}$ .

**Definition 3.2.** A t-norm  $T$  on  $([0, 1], \leq)$  is said to be

- (i) strict, if it is continuous and strictly monotone, i.e.,  $T(x, y) < T(x, z)$  whenever  $x > 0$  and  $y < z$ .
- (ii) nilpotent, if it is continuous and if each  $x \in (0, 1)$  is a nilpotent element of  $T$ , i.e., if there exists  $n \in \mathbb{N}$  such that  $x_T^{[n]} = 0$ , where

$$x_T^{[n]} := \begin{cases} x, & \text{if } n = 1, \\ T(x, x_T^{[n-1]}), & \text{if } n > 1. \end{cases}$$

**Definition 3.3.** A t-conorm  $S$  on  $([0, 1], \leq)$  is said to be

- (i) strict, if  $S$  is continuous and strictly monotone, i.e.,  $S(x, y) < S(x, z)$  whenever  $x < 1$  and  $y < z$ ,
- (ii) nilpotent, if  $S$  is continuous and if for each  $x \in (0, 1)$  there exists  $n \in \mathbb{N}$  such that  $x_S^{[n]} = 1$ ,

$$\text{where } x_S^{[n]} := \begin{cases} x, & \text{if } n = 1, \\ S(x, x_S^{[n-1]}), & \text{if } n > 1. \end{cases}$$

The following characterizations of nilpotent and strict t-norms and t-conorms are well-known in the literature.

**Theorem 3.4** ([26]). *A function  $T: [0, 1]^2 \rightarrow [0, 1]$  is a strict t-norm if and only if there exists a continuous, strictly decreasing function  $t: [0, 1] \rightarrow [0, \infty]$  with  $t(1) = 0$  and  $t(0) = \infty$ , which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x, y) = t^{-1}(t(x) + t(y)), \quad x, y \in [0, 1].$$

**Theorem 3.5** ([26]). *A function  $T: [0, 1]^2 \rightarrow [0, 1]$  is a nilpotent t-norm if and only if there exists a continuous, strictly decreasing function  $t: [0, 1] \rightarrow [0, \infty)$  with  $t(1) = 0$ , which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x, y) = t^{-1}(\min(t(x) + t(y), t(0))), \quad x, y \in [0, 1].$$

**Theorem 3.6** ([26]). *A function  $S: [0, 1]^2 \rightarrow [0, 1]$  is a strict t-conorm if and only if there exists a continuous, strictly increasing function  $s: [0, 1] \rightarrow [0, \infty]$  with  $s(0) = 0$  and  $s(1) = \infty$ , which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x, y) = s^{-1}(s(x) + s(y)), \quad x, y \in [0, 1].$$

**Theorem 3.7** ([26]). *A function  $S: [0, 1]^2 \rightarrow [0, 1]$  is a nilpotent t-conorm if and only if there exists a continuous, strictly increasing function  $s: [0, 1] \rightarrow [0, \infty)$  with  $s(0) = 0$ , which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x, y) = s^{-1}(\min(s(x) + s(y), s(1))), \quad x, y \in [0, 1].$$

In our article we shall consider the following special classes of t-norms and t-conorms.

**Definition 3.8** (see [27]). (i) A t-norm  $\mathcal{T}$  on  $\mathcal{L}^I$  is called t-representable if there exist t-norms  $T_1$  and  $T_2$  on  $([0, 1], \leq)$  such that  $T_1 \leq T_2$  and

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)],$$

for all  $[x_1, x_2], [y_1, y_2] \in L^I$ .

(ii) A t-conorm  $\mathcal{S}$  on  $\mathcal{L}^I$  is called t-representable if there exist t-conorms  $S_1$  and  $S_2$  on  $([0, 1], \leq)$  such that  $S_1 \leq S_2$  and

$$\mathcal{S}([x_1, x_2], [y_1, y_2]) = [S_1(x_1, y_1), S_2(x_2, y_2)],$$

for all  $[x_1, x_2], [y_1, y_2] \in L^I$ .

It should be noted that not all t-norms and t-conorms on  $\mathcal{L}^I$  are t-representable (see [27]).

One possible definition of an implication on  $\mathcal{L}^I$  is based on the well-accepted notation introduced by Fodor and Roubens [28] (see also [18], [29] and [30]).

**Definition 3.9.** Let  $\mathcal{L} = (L, \leq_L)$  be a complete lattice. A function  $\mathcal{I}: L^2 \rightarrow L$  is called a fuzzy implication on  $\mathcal{L}$  if it is decreasing with respect to the first variable, increasing with respect to the second variable and fulfills the following conditions:  $\mathcal{I}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = \mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = \mathcal{I}(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$  and  $\mathcal{I}(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$ .

#### 4. Some results pertaining to functional equations

In this section we show two results related to functional equations, which will be crucial in obtaining main results.

**Proposition 4.1** ([12, Proposition 3.6]). *Fix real  $a > 0$ . For a function  $f: [0, a] \rightarrow [0, \infty]$  the following statements are equivalent:*

(i)  *$f$  satisfies the functional equation*

$$f(\min(x + y, a)) = f(x) + f(y),$$

*for all  $x, y \in [0, a]$ .*

(ii) *Either  $f = 0$ , or  $f = \infty$ , or*

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{if } x > 0, \end{cases} \quad \text{for all } x \in [0, a].$$

**Proposition 4.2** ([2, Proposition 4.2]). *Fix real  $a > 0$ . Let  $L^a = \{(u_1, u_2) \in [0, a]^2 : u_1 \geq u_2\}$ . For a function  $f: L^a \rightarrow [0, \infty]$  the following statements are equivalent:*

(i)  *$f$  satisfies the functional equation*

$$\begin{aligned} f(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) \\ = f(u_1, u_2) + f(v_1, v_2), \end{aligned} \quad (\text{A})$$

*for all  $(u_1, u_2), (v_1, v_2) \in L^a$ .*

(ii) *Either*

$$f = 0, \quad (\text{S1})$$

*or*

$$f = \infty, \quad (\text{S2})$$

*or*

$$f(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0, \end{cases} \quad (\text{S3})$$

*or*

$$f(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0, \end{cases} \quad (\text{S4})$$

*for all  $(u_1, u_2) \in L^a$ .*

#### 5. Distributive equations for t-representable t-norms and t-conorms

In this section we will show how we can use solutions presented in Proposition 4.2 to obtain all solutions of our main distributivity equations

$$\mathcal{I}(\mathcal{T}(x, y), z) = \mathcal{S}(\mathcal{I}(x, z), \mathcal{I}(y, z))$$

satisfied for all  $x, y, z \in L^I$ , where  $\mathcal{I}$  is an unknown function,  $\mathcal{T}$  is a t-representable t-norm on  $\mathcal{L}^I$  generated from nilpotent t-norms  $T_1, T_2$  and  $\mathcal{S}$  is a t-representable t-conorm on  $\mathcal{L}^I$  generated from strict t-conorms  $S_1, S_2$ .

Assume that projection mappings on  $\mathcal{L}^I$  are defined as the following:

$$pr_1([x_1, x_2]) = x_1, \quad pr_2([x_1, x_2]) = x_2,$$

for  $[x_1, x_2] \in L^I$ .

At this situation our distributivity equation has the following form

$$\begin{aligned} \mathcal{I}([T_1(x_1, y_1), T_2(x_2, y_2)], [z_1, z_2]) \\ = [S_1(pr_1(\mathcal{I}([x_1, x_2], [z_1, z_2])), pr_1(\mathcal{I}([y_1, y_2], [z_1, z_2]))), \\ S_2(pr_2(\mathcal{I}([x_1, x_2], [z_1, z_2])), pr_2(\mathcal{I}([y_1, y_2], [z_1, z_2])))] \end{aligned}$$

for all  $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in L^I$ . As a consequence we obtain the following two equations

$$\begin{aligned} pr_1(\mathcal{I}([T_1(x_1, y_1), T_2(x_2, y_2)], [z_1, z_2])) \\ = S_1(pr_1(\mathcal{I}([x_1, x_2], [z_1, z_2])), pr_1(\mathcal{I}([y_1, y_2], [z_1, z_2]))), \\ pr_2(\mathcal{I}([T_1(x_1, y_1), T_2(x_2, y_2)], [z_1, z_2])) \\ = S_2(pr_2(\mathcal{I}([x_1, x_2], [z_1, z_2])), pr_2(\mathcal{I}([y_1, y_2], [z_1, z_2]))), \end{aligned}$$

which are satisfied for all  $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in L^I$ . Now, let us fix arbitrarily  $[z_1, z_2] \in L^I$  and define two functions  $g_1^{[z_1, z_2]}, g_2^{[z_1, z_2]}: L^I \rightarrow L^I$  by

$$\begin{aligned} g_1^{[z_1, z_2]}(\cdot) &:= pr_1 \circ \mathcal{I}(\cdot, [z_1, z_2]), \\ g_2^{[z_1, z_2]}(\cdot) &:= pr_2 \circ \mathcal{I}(\cdot, [z_1, z_2]). \end{aligned}$$

Thus we have shown that if  $\mathcal{T}$  and  $\mathcal{S}$  on  $\mathcal{L}^I$  are t-representable, then

$$\begin{aligned} g_1^{[z_1, z_2]}([T_1(x_1, y_1), T_2(x_2, y_2)]) \\ = S_1(g_1^{[z_1, z_2]}([x_1, x_2]), g_1^{[z_1, z_2]}([y_1, y_2])), \\ g_2^{[z_1, z_2]}([T_1(x_1, y_1), T_2(x_2, y_2)]) \\ = S_2(g_2^{[z_1, z_2]}([x_1, x_2]), g_2^{[z_1, z_2]}([y_1, y_2])), \end{aligned}$$

Let us assume that  $T_1 = T_2$  is a nilpotent t-norm generated from additive generator  $t$  and  $S_1 = S_2$  is a strict t-conorm generated from additive generator  $s$ . Using the representations of nilpotent t-norms (Theorem 3.5) and strict t-conorms (Theorem 3.6) we can transform our problem to the following equations:

$$\begin{aligned} & g_1^{[z_1, z_2]}([t^{-1}(\min(t(x_1) + t(y_1), t(0))), \\ & \quad t^{-1}(\min(t(x_2) + t(y_2), t(0)))] \\ & = s^{-1}(s(g_1^{[z_1, z_2]}([x_1, x_2])) \\ & \quad + s(g_1^{[z_1, z_2]}([y_1, y_2]))). \end{aligned}$$

Hence

$$\begin{aligned} & s \circ g_1^{[z_1, z_2]}([t^{-1}(\min(t(x_1) + t(y_1), t(0))), \\ & \quad t^{-1}(\min(t(x_2) + t(y_2), t(0)))] \\ & = s \circ g_1^{[x_1, x_2]}([x_1, x_2]) \\ & \quad + s \circ g_1^{[y_1, y_2]}([y_1, y_2]). \end{aligned}$$

Let us put  $t(x_1) = u_1$ ,  $t(x_2) = u_2$ ,  $t(y_1) = v_1$  and  $t(y_2) = v_2$ . Of course  $u_1, u_2, v_1, v_2 \in [0, t(0)]$ . Moreover  $[x_1, x_2], [y_1, y_2] \in L^I$ , thus  $x_1 \leq x_2$  and  $y_1 \leq y_2$ . The generator  $t$  is strictly decreasing, so  $u_1 \geq u_2$  and  $v_1 \geq v_2$ . If we put

$$f_{[z_1, z_2]}(u, v) := s \circ pr_1 \circ \mathcal{I}([t^{-1}(u), t^{-1}(v)], [z_1, z_2]),$$

where  $u, v \in [0, t(0)]$ ,  $u \geq v$ , then we get the following functional equation

$$\begin{aligned} & f_{[z_1, z_2]}(\min(u_1 + v_1, t(0)), \min(u_2 + v_2, t(0))) \\ & = f_{[z_1, z_2]}(u_1, u_2) + f_{[z_1, z_2]}(v_1, v_2), \quad (1) \end{aligned}$$

satisfied for all  $(u_1, u_2), (v_1, v_2) \in L^{t(0)}$ . Of course function  $f_{[z_1, z_2]}: L^{t(0)} \rightarrow [0, \infty]$  is unknown above. In a same way we can repeat all the above calculations, but for the function  $g_2$ , to obtain the following functional equation

$$\begin{aligned} & f^{[z_1, z_2]}(\min(u_1 + v_1, t(0)), \min(u_2 + v_2, t(0))) \\ & = f^{[z_1, z_2]}(u_1, u_2) + f^{[z_1, z_2]}(v_1, v_2), \quad (2) \end{aligned}$$

satisfied for all  $(u_1, u_2), (v_1, v_2) \in L^{t(0)}$ , where

$$f^{[z_1, z_2]}(u, v) := s \circ pr_2 \circ \mathcal{I}([t^{-1}(u), t^{-1}(v)], [z_1, z_2])$$

is an unknown function. Observe that (1) and (2) are exactly our functional equation (A). Therefore, using solutions of Proposition 4.2, we are able to obtain the description of the horizontal section  $\mathcal{I}(\cdot, [z_1, z_2])$  for a fixed  $[z_1, z_2] \in L^I$ . Since in this proposition we have 4 possible solutions, we should have 16 different solutions of (D-TS). Observe now that some of these solutions maybe incorrect, since the range of  $\mathcal{I}$  is  $L^I$ . Now, we will check all possibilities. Let us fix arbitrarily  $[z_1, z_2] \in L^I$  and consider 16 different cases:

1.  $f_{[z_1, z_2]} = 0$  and  $f^{[z_1, z_2]} = 0$ .  
This implies that

$$s \circ pr_1 \circ \mathcal{I}([t^{-1}(u_1), t^{-1}(u_2)], [z_1, z_2]) = 0,$$

for all  $u_1, u_2 \in [0, t(0)]$ ,  $u_1 \geq u_2$ , thus

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 0, \quad [x_1, x_2] \in L^I.$$

Similarly we get

$$pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 0, \quad [x_1, x_2] \in L^I.$$

In summary, we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = [0, 0] = 0_{L^I}.$$

2.  $f_{[z_1, z_2]} = 0$  and  $f^{[z_1, z_2]} = \infty$ .  
On one side we get

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 0, \quad [x_1, x_2] \in L^I.$$

On the other side we have

$$s \circ pr_2 \circ \mathcal{I}([t^{-1}(u_1), t^{-1}(u_2)], [z_1, z_2]) = \infty,$$

thus

$$pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 1, \quad [x_1, x_2] \in L^I.$$

In summary we get the following correct solution

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = [0, 1].$$

3.  $f_{[z_1, z_2]} = 0$  and  $f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$   
On one side we get

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 0, \quad [x_1, x_2] \in L^I.$$

On the other side we have

$$\begin{aligned} & s \circ pr_2 \circ \mathcal{I}([t^{-1}(u_1), t^{-1}(u_2)], [z_1, z_2]) \\ & = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0, \end{cases} \end{aligned}$$

thus

$$\begin{aligned} & pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) \\ & = \begin{cases} 0, & \text{if } x_2 = 1, \\ 1, & \text{if } x_2 < 1, \end{cases} \quad [x_1, x_2] \in L^I. \end{aligned}$$

In summary we get the following correct solution

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_2 = 1, \\ [0, 1], & \text{if } x_2 < 1. \end{cases}$$

4.  $f_{[z_1, z_2]} = 0$  and  $f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$   
On one side we get

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 0, \quad [x_1, x_2] \in L^I.$$

On the other side we have

$$\begin{aligned} s \circ pr_2 \circ \mathcal{I}([t^{-1}(u_1), t^{-1}(u_2)], [z_1, z_2]) \\ = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0, \end{cases} \end{aligned}$$

thus

$$\begin{aligned} pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) \\ = \begin{cases} 0, & \text{if } x_1 = 1, \\ 1, & \text{if } x_1 < 1, \end{cases} \quad [x_1, x_2] \in L^I. \end{aligned}$$

In summary we get the following correct solution

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_1 = 1, \\ [0, 1], & \text{if } x_1 < 1. \end{cases}$$

5.  $f_{[z_1, z_2]} = \infty$  and  $f^{[z_1, z_2]} = 0$ .  
This implies that

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 1, \quad [x_1, x_2] \in L^I,$$

while

$$pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 0, \quad [x_1, x_2] \in L^I.$$

In summary we obtain the following function:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = [1, 0]$$

which is incorrect, since  $[1, 0] \notin L^I$ .

6.  $f_{[z_1, z_2]} = \infty$  and  $f^{[z_1, z_2]} = \infty$ .  
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = [1, 1] = 1_{L^I}.$$

7.  $f_{[z_1, z_2]} = \infty$  and  $f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$   
On one side we get

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 1, \quad [x_1, x_2] \in L^I.$$

On the other side we have

$$\begin{aligned} pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) \\ = \begin{cases} 0, & \text{if } x_2 = 1, \\ 1, & \text{if } x_2 < 1, \end{cases} \quad [x_1, x_2] \in L^I. \end{aligned}$$

In summary we get the following function

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [1, 0], & \text{if } x_2 = 1, \\ [1, 1], & \text{if } x_2 < 1, \end{cases}$$

which is incorrect.

8.  $f_{[z_1, z_2]} = \infty$  and  $f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$   
On one side we get

$$pr_1 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) = 1, \quad [x_1, x_2] \in L^I.$$

On the other side we have

$$\begin{aligned} pr_2 \circ \mathcal{I}([x_1, x_2], [z_1, z_2]) \\ = \begin{cases} 0, & \text{if } x_1 = 1, \\ 1, & \text{if } x_1 < 1, \end{cases} \quad [x_1, x_2] \in L^I. \end{aligned}$$

In summary we get the following function

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [1, 0], & \text{if } x_1 = 1, \\ [1, 1], & \text{if } x_1 < 1, \end{cases}$$

which is incorrect.

9.  $f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$  and  $f^{[z_1, z_2]} = 0$ .

In this case we obtain the following function:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_2 = 1, \\ [1, 0], & \text{if } x_2 < 1. \end{cases}$$

which is incorrect.

10.  $f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$  and  $f^{[z_1, z_2]} = \infty$ .

In this case we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [1, 1], & \text{if } x_2 = 1, \\ [0, 1], & \text{if } x_2 < 1. \end{cases}$$

11.  $f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$  and  $f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$

In this case we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_2 = 1, \\ [1, 1], & \text{if } x_2 < 1. \end{cases}$$

12.  $f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$  and  $f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$

In this case we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_1 = 1, \\ [0, 1], & \text{if } x_1 < 1 \text{ \& } x_2 = 1, \\ [1, 1], & \text{if } x_2 < 1. \end{cases}$$

Please note that  $u_1 \geq u_2$ , so it is not possible that  $x_1 > x_2$ .

$$13. f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases} \quad \text{and} \\ f^{[z_1, z_2]} = 0.$$

In this case we obtain the following function:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_1 = 1, \\ [1, 0], & \text{if } x_1 < 1, \end{cases}$$

which is incorrect.

$$14. f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0, \end{cases} \quad \text{and} \\ f^{[z_1, z_2]} = \infty.$$

In this case we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_1 = 1, \\ [1, 1], & \text{if } x_1 < 1. \end{cases}$$

$$15. f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases} \quad \text{and} \\ f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$$

In this case we obtain the following function:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_1 = 1, \\ [1, 0], & \text{if } x_1 < 1 \text{ \& } x_2 = 1, \\ [1, 1], & \text{if } x_2 < 1, \end{cases}$$

which is incorrect.

$$16. f_{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases} \quad \text{and} \\ f^{[z_1, z_2]}(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$$

In this case we obtain the following correct solution:

$$\mathcal{I}([x_1, x_2], [z_1, z_2]) = \begin{cases} [0, 0], & \text{if } x_1 = 1, \\ [1, 1], & \text{if } x_1 < 1. \end{cases}$$

Therefore, we have obtained 10 correct horizontal sections in  $\mathcal{L}^I$ . Unfortunately, we need to notice that it is not possible to find at least one solution  $\mathcal{I}$  which is a fuzzy implication on  $\mathcal{L}^I$  in the sense of Definition 3.9. For solutions 1), 2), 3), 4), 11), 12), 14) and 16) we have

$$\mathcal{I}(1_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = \mathcal{I}([1, 1], [1, 1]) \neq [1, 1] = 1_{\mathcal{L}^I},$$

so it is not possible to find a horizontal solution, which is correct for  $[x_1, x_2] = [1, 1]$ . The horizontal sections 6) and 10) are incorrect in this situation since we have

$$\mathcal{I}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = \mathcal{I}([0, 0], [1, 1]) \neq [0, 0] = 0_{\mathcal{L}^I},$$

so it is not possible to find a horizontal solution, which is correct for  $[x_1, x_2] = [0, 0]$ .

## 6. Conclusion

In this article we have discussed the following distributivity equation

$$\mathcal{I}(\mathcal{T}(x, y), z) = \mathcal{S}(\mathcal{I}(x, z), \mathcal{I}(y, z)),$$

when both operations are t-representable and such that  $\mathcal{T}$  is generated from nilpotent t-norms, while  $\mathcal{S}$  is generated from strict t-conorms.

Using similar methods as in [31] we can easily obtain all solutions of dual functional equation

$$\mathcal{I}(\mathcal{S}(x, y), z) = \mathcal{T}(\mathcal{I}(x, z), \mathcal{I}(y, z)),$$

where  $\mathcal{I}$  is an unknown function,  $\mathcal{S}$  is a t-representable t-conorm on  $\mathcal{L}^I$  generated from nilpotent t-conorms  $S_1, S_2$  and  $\mathcal{T}$  is a t-representable t-norm on  $\mathcal{L}^I$  generated from strict t-norms  $T_1, T_2$ . In fact it is enough to consider for each t-norm  $\mathcal{T}$  on  $\mathcal{L}^I$  the function

$$\mathcal{S}(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$$

where  $\mathcal{N}$  is a strong negation on  $\mathcal{L}^I$  (see [27] and [32]). This function is a t-conorm on  $\mathcal{L}^I$  ( $\mathcal{N}$ -dual to  $\mathcal{T}$ ). In our future work we will concentrate on a situation, when both operations are t-representable uninorms.

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