

F^1 -transform of Functions of Two Variables

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Abstract

In this contribution, the F^1 -transform of functions of two variables is introduced. It combines properties of the F-transform of functions of two variables and the F^1 -transform of functions of one variable. The aim of this study is to prove approximation properties of the F^1 -transform components and of the inverse F^1 -transform in this particular case.

Keywords: F-transform, F^1 -transform, partial derivatives

1. Introduction

The technique of F-transform was developed as a tool for a fuzzy modeling [1]. Similar to conventional integral transforms (the Fourier and Laplace transforms, for example), the F-transform performs a transformation of an original universe of functions into a universe of their “skeleton models”. More specifically, the F-transform establishes a correspondence between a set of continuous functions on an interval of real numbers (space of reals) and the set of n -dimensional real vectors (matrices). Each component of the resulting vector (matrix) is a weighted local mean of a corresponding function over an area covered by a corresponding basic function (“kernel” of transformation). The vector (matrix) of the F-transform components is a simplified representation of an original function that can be used instead of the original function and for which further computations are easier. In this respect, the F-transform can be as useful in applications as traditional transforms. Moreover, sometimes the F-transform is more efficient than its counterparts.

Initially, the F-transform was introduced for functions of one or two variables. This method turned out to be very general and powerful in many applications. Especially, the F-transform of functions of two variables shows a great potential in applications to image processing, particularly, image compression [2], image fusion [3], edge detection [4].

Generalization of the ordinary F-transform to the F-transform of a higher degree in the case of functions of one variable was introduced in [5]. Many interesting properties of the F-transform of a higher degree have been proved there, and among others, a property of approximation of the first derivative of the original function.

In this contribution, we focus on the F-transform of the first degree (F^1 -transform) and its extension to functions of many variables. We discuss properties of the F^1 -transform and their applications. It turned out that the F^1 -transform has successful applications in image processing, especially in edge detection [6]. The edge detection problem consists in a specification of an area with significant changes of image intensity. In other words, the latter can be characterized by saying that the corresponding absolute value of a gradient reaches its maximum.

The goal of this paper is to characterize the F^1 -transform of functions of two variables and show how their partial first order derivatives can be approximated by the corresponding F^1 -transform components. Moreover, we aim at giving estimates of the qualities of the approximation of partial first order derivatives and of an original function by the inverse F^1 -transform.

The paper is organized as follows: Section 1 recalls the techniques of F-transform of functions of two variables and F^1 -transform of functions of one variable. In Section 2, the F^1 -transform of functions of two variables is introduced and partial derivatives are approximated. The inverse F^1 -transform of functions of two variables is established in Section 3. In Section 4, a short introduction to the edge detection problem based on the F^1 -transform is mentioned. Finally, conclusions and comments are in Section 5.

1.1. F-transform of functions of two variables

Let us recall the basic conceptions of the F-transform of function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ of two variables and refer to [1] for more details. At first, we introduce the notion of fuzzy partition. It is defined for the interval $[a, b]$ and then extended to $[a, b] \times [c, d]$.

Let $[a, b]$ be an interval on the real line \mathbb{R} and let $n \geq 2$ a number of fuzzy sets in a fuzzy partition of $[a, b]$. Let $x_0, x_1, \dots, x_n, x_{n+1} \in [a, b]$ be nodes such that $a = x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1} = b$. Fuzzy sets $A_1, \dots, A_n : [a, b] \rightarrow [0, 1]$ establish a *fuzzy partition* of $[a, b]$ if the following requirements are fulfilled:

1. for every $k = 1, \dots, n$, $A_k(x) = 0$ if $x \in [a, b] \setminus [x_{k-1}, x_{k+1}]$;

2. for every $k = 1, \dots, n$, A_k is continuous on $[x_{k-1}, x_{k+1}]$;
3. for every $k = 1, \dots, n$, $A_k(x) > 0$ if $x \in (x_{k-1}, x_{k+1})$;
4. for all $x \in [a, b]$, $\sum_{k=1}^n A_k(x) = 1$ (Ruspini condition).

It follows from the conditions 1 – 4 that for each $k = 1, \dots, n$,

$$A_k(x_k) = 1.$$

If the nodes $x_0, x_1, \dots, x_n, x_{n+1}$ are h -equidistant, i.e. for all $k = 1, \dots, n+1$, $x_k = x_{k-1} + h$, where $h = (b - a)/(n + 1)$, we say that the fuzzy partition A_1, \dots, A_n is h -uniform.

If fuzzy sets A_1, \dots, A_n establish an h_x -uniform fuzzy partition of $[a, b]$ and B_1, \dots, B_m establish an h_y -uniform fuzzy partition of $[c, d]$ then the Cartesian product $\{A_1, \dots, A_n\} \times \{B_1, \dots, B_m\}$ of these fuzzy partitions is the set of all fuzzy sets $A_k \times B_l$, $k = 1, \dots, n$, $l = 1, \dots, m$. The membership function $A_k \times B_l : [a, b] \times [c, d] \rightarrow [0, 1]$ is equal to the product $A_k \cdot B_l$ of the corresponding membership functions. We say that fuzzy sets $A_k \times B_l$, $k = 1, \dots, n$, $l = 1, \dots, m$ establish an $h_x h_y$ -uniform fuzzy partition of the Cartesian product $[a, b] \times [c, d]$.

Once the fuzzy partition $A_k \times B_l$, $k = 1, \dots, n$, $l = 1, \dots, m$ is selected, we define the F -transform of a function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ with respect to the chosen partition of $[a, b] \times [c, d]$. The (direct) F -transform of f is an image of the mapping $F_{nm}[f] : \{A_1, \dots, A_n\} \times \{B_1, \dots, B_m\} \rightarrow \mathbb{R}$ represented by a matrix

$$F_{nm}[f] = \begin{pmatrix} F_{11} & \dots & F_{1m} \\ \vdots & \ddots & \vdots \\ F_{n1} & \dots & F_{nm} \end{pmatrix},$$

where

$$F_{kl} = \frac{\int_a^b \int_c^d f(x, y) A_k(x) B_l(y) dx dy}{\int_a^b \int_c^d A_k(x) B_l(y) dx dy}, \quad (1)$$

$k = 1, \dots, n$, $l = 1, \dots, m$. The value F_{kl} is called an F -transform component.

The inverse F -transform of f is a function on $[a, b] \times [c, d]$ which is defined by the following inversion formula:

$$\hat{f}_{nm}(x, y) = \sum_{k=1}^n \sum_{l=1}^m F_{kl} A_k(x) B_l(y),$$

$$x \in [a, b], y \in [c, d].$$

It was proved that the inverse F -transform $\hat{f}_{nm}(x, y)$ approximates the original function f with an arbitrary precision.

1.2. F^1 -transform of functions of one variable

The F^m -transform of a higher degree $m \geq 1$ was introduced in [5]. In this section we give a short description of the F^1 -transform of functions of one variable. The F^1 -transform is a generalization of the F -transform where the constant components are replaced by linear components (polynomials of the first degree).

Throughout this section, we assume that A_1, \dots, A_n , $n > 2$ is a fuzzy partition of $[a, b]$ with nodes x_k , $k = 0, \dots, n+1$. Let k be a fixed integer from $\{1, \dots, n\}$ and let $L_2(A_k)$ be a normed space of square-integrable functions $f : [x_{k-1}, x_{k+1}] \rightarrow \mathbb{R}$, $k = 1, \dots, n$.

By $L_2(A_1, \dots, A_n)$ we denote a set of functions $f : [a, b] \rightarrow \mathbb{R}$ such that for all $k = 1, \dots, n$, $f|_{[x_{k-1}, x_{k+1}]} \in L_2(A_k)$, where $f|_{[x_{k-1}, x_{k+1}]}$ is the restriction of f on $[x_{k-1}, x_{k+1}]$.

For any function f from $L_2(A_1, \dots, A_n)$ we define the F^1 -transform of f with respect to A_1, \dots, A_n as a vector

$$F^1[f] = [F_1^1, \dots, F_n^1]$$

where the components F_k^1 , $k = 1, \dots, n$ are linear functions

$$F_k^1 = c_{k,0}^1 + c_{k,1}^1(x - x_k),$$

$$k = 1, \dots, n,$$

with coefficients $c_{k,0}^1, c_{k,1}^1$ given by

$$c_{k,0}^1 = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x) A_k(x) dx}{\left(\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx\right)},$$

$$c_{k,1}^1 = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)(x - x_k) A_k(x) dx}{\left(\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x) dx\right)}.$$

Here we emphasize the following two properties of the F^1 -transform, more can be found in [5].

- If f is sufficiently smooth on $[a, b]$, then for each $k = 1, \dots, n$

$$c_{k,0}^1 = f(x_k) + O(h^2),$$

$$c_{k,1}^1 = f'(x_k) + O(h^2).$$

- Coefficient $c_{k,0}^1$ is equal to the k -th F -transform component of f , i.e.

$$c_{k,0}^1 = F_k.$$

In the following section we extend the F^1 -transform to functions of two variables and show some properties.

2. Introduction of the F^1 -transform of functions of two variables

2.1. Space $L_2(A_k \times B_l)$

Let us assume a rectangle $[a, b] \times [c, d]$ and fix a fuzzy partition $\{A_1, \dots, A_n\} \times \{B_1, \dots, B_m\}$, $n, m \geq 2$

of this rectangle. Let k, l be fixed integers from $\{1, \dots, n\}, \{1, \dots, m\}$ respectively, and let $L_2(A_k \times B_l)$ be a set of square-integrable functions $f : [x_{k-1}, x_{k+1}] \times [y_{l-1}, y_{l+1}] \rightarrow \mathbb{R}$, $k = 1, \dots, n$; $l = 1, \dots, m$, on their domain.

Let $\langle f, g \rangle_{kl}$ be an inner product of functions f and g in $L_2(A_k \times B_l)$ defined as follows

$$\langle f, g \rangle_{kl} = \int_a^b \int_c^d f(x, y)g(x, y)A_k(x)B_l(y)dx dy$$

and let

$$\|f\|_{kl} = \sqrt{\langle f, f \rangle_{kl}}$$

be a corresponding norm. Then $L_2(A_k \times B_l)$ is a Hilbert space.

In the sequel, we assume that our functions $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ are such that for all $k = 1, \dots, n$, $l = 1, \dots, m$, $f|_{[x_{k-1}, x_{k+1}] \times [y_{l-1}, y_{l+1}]} \in L_2(A_k \times B_l)$, where $f|_{[x_{k-1}, x_{k+1}] \times [y_{l-1}, y_{l+1}]}$ is the restriction of f on $[x_{k-1}, x_{k+1}] \times [y_{l-1}, y_{l+1}]$.

2.2. F^1 -transform of functions of two variables

Let $L_2^1(A_k)$, $k = 1, \dots, n$ be a linear subspace of $L_2(A_k)$ with the orthogonal basis given by two polynomials

$$P_k^0(x) = 1, \quad P_k^1(x) = x - x_k,$$

and $L_2^1(B_l)$, $l = 1, \dots, m$ be a linear subspace of $L_2(B_l)$ with the basis

$$Q_l^0(y) = 1, \quad Q_l^1(y) = y - y_l.$$

In order to introduce the F^1 -transform of a function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ we choose the linear subspace $L_2^1(A_k \times B_l)$ of $L_2(A_k \times B_l)$ with the orthogonal basis given by $S_{kl}^0(x, y) = 1$, $S_{kl}^1(x, y) = x - x_k$, $S_{kl}^2(x, y) = y - y_l$.

$$\begin{aligned} S_{kl}^0(x, y) &= 1, \\ S_{kl}^1(x, y) &= x - x_k, \\ S_{kl}^2(x, y) &= y - y_l. \end{aligned} \quad (2)$$

Lemma 1 *Let for all $k = 1, \dots, n$, $l = 1, \dots, m$ $f \in L_2(A_k \times B_l)$ and let the polynomial F_{kl}^1 be an orthogonal projection of f on the linear subspace $L_2^1(A_k \times B_l)$ with the orthogonal basis $\{S_{kl}^i\}_{i=0,1,2}$. Then*

$$F_{kl}^1 = c_{kl,0}^1 S_{kl}^0 + c_{kl,1}^1 S_{kl}^1 + c_{kl,2}^1 S_{kl}^2, \quad (3)$$

where

$$c_{kl,i}^1 = \frac{\langle f, S_{kl}^i \rangle_{kl}}{\langle S_{kl}^i, S_{kl}^i \rangle_{kl}} = \frac{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} f(x, y) S_{kl}^i(x, y) A_k(x) B_l(y) dx dy}{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} (S_{kl}^i(x, y))^2 A_k(x) B_l(y) dx dy},$$

$i = 0, 1, 2$.

PROOF: Let the assumptions of Lemma 1 be hold. From the properties of the Hilbert space $L_2(A_k \times B_l)$, $f - F_{kl}^1$ is orthogonal to the subspace $L_2^1(A_k \times B_l)$, i.e. the following holds

$$\langle f - F_{kl}^1, S_{kl}^i \rangle_{kl} = 0, \quad i = 0, 1, 2.$$

After substitution the expression for F_{kl}^1 , we obtain

$$\begin{aligned} \langle f - c_{kl,0}^1 S_{kl}^0 - c_{kl,1}^1 S_{kl}^1 - c_{kl,2}^1 S_{kl}^2, S_{kl}^i \rangle_{kl} &= \\ \langle f, S_{kl}^i \rangle_{kl} - c_{kl,i}^1 \langle S_{kl}^i, S_{kl}^i \rangle_{kl} &= 0, \end{aligned}$$

and therefore

$$c_{kl,i}^1 = \frac{\langle f, S_{kl}^i \rangle_{kl}}{\langle S_{kl}^i, S_{kl}^i \rangle_{kl}}, \quad i = 0, 1, 2.$$

□

Similarly to the F-transform of a function f of two variables, we define the F^1 -transform of f with components in the form of linear polynomials.

Definition 2 *Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ and let linear polynomials F_{kl}^1 (3) be the kl -th orthogonal projections of $f|_{[x_{k-1}, x_{k+1}] \times [y_{l-1}, y_{l+1}]}$ on $L_2^1(A_k \times B_l)$, $k = 1, \dots, n$, $l = 1, \dots, m$. We define the (direct) F^1 -transform of f as a matrix $F_{nm}^1[f] = (F_{kl}^1)$ of linear components F_{kl}^1 , $k = 1, \dots, n$, $l = 1, \dots, m$.*

By Lemma 1, the F^1 -transform components have the following representation

$$F_{kl}^1 = c_{kl,0}^1 + c_{kl,1}^1(x - x_k) + c_{kl,2}^1(y - y_l),$$

$$k = 1, \dots, n, \quad l = 1, \dots, m,$$

where coefficients are given by

$$\begin{aligned} c_{kl,0}^1 &= \frac{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} f(x, y) A_k(x) B_l(y) dx dy}{\left(\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx\right) \left(\int_{y_{l-1}}^{y_{l+1}} B_l(y) dy\right)}, \\ c_{kl,1}^1 &= \frac{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} f(x, y)(x - x_k) A_k(x) B_l(y) dx dy}{\left(\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x) dx\right) \left(\int_{y_{l-1}}^{y_{l+1}} B_l(y) dy\right)}, \\ c_{kl,2}^1 &= \frac{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} f(x, y)(y - y_l) A_k(x) B_l(y) dx dy}{\left(\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx\right) \left(\int_{y_{l-1}}^{y_{l+1}} (y - y_l)^2 B_l(y) dy\right)}. \end{aligned}$$

Here we made use of the fact that polynomials $\{S_{kl}^i\}_{i=0,1,2}$ are represented by (2).

Remark 3 *The coefficients $c_{kl,0}^1$ are equal to the components F_{kl} (1) of the ordinary F-transform of the given function f . Therefore, we can write for each $k = 1, \dots, n$, $l = 1, \dots, m$:*

$$F_{kl}^1 = F_{kl} + c_{kl,1}^1(x - x_k) + c_{kl,2}^1(y - y_l),$$

where the coefficients $c_{kl,1}^1, c_{kl,2}^1$ are given as above.

Lemma 4 Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ and $A_k \times B_l$, $k = 1, \dots, n$, $l = 1, \dots, m$ be an $h_x h_y$ -uniform fuzzy partition of $[a, b] \times [c, d]$. Moreover, let functions f , A_k , B_l be twice continuously differentiable on $[a, b] \times [c, d]$. Then for every k, l the following holds:

$$c_{kl,0}^1 = f(x_k, y_l) + O(h)$$

where we assume that $h = h_x = h_y$.

PROOF: We will give the proof for fixed values of k from $1, \dots, n$, and l from $1, \dots, m$.

$$c_{kl,0}^1 = \frac{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} f(x, y) A_k(x) B_l(y) dx dy}{\left(\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx \right) \left(\int_{y_{l-1}}^{y_{l+1}} B_l(y) dy \right)}.$$

We denote the below given integrals as follows:

$$\begin{aligned} I_1 &= \int_{x_{k-1}}^{x_{k+1}} f(x, y) A_k(x) dx, \\ I_2 &= \int_{y_{l-1}}^{y_{l+1}} I_1 B_l(y) dy, \\ I_3 &= \int_{x_{k-1}}^{x_{k+1}} A_k(x) dx, \\ I_4 &= \int_{y_{l-1}}^{y_{l+1}} B_l(y) dy. \end{aligned}$$

Then we apply the trapezoidal rule with nodes x_{k-1}, x_k, x_{k+1} to the integral I_1 and the same rule with nodes y_{l-1}, y_l, y_{l+1} to the integral I_2 . We use the properties:

$$\begin{aligned} A_k(x_{k-1}) &= A_k(x_{k+1}) = 0, \quad A_k(x_k) = 1, \\ B_l(y_{l-1}) &= B_l(y_{l+1}) = 0, \quad B_l(y_l) = 1. \end{aligned}$$

$$\begin{aligned} I_1 &= \int_{x_{k-1}}^{x_{k+1}} f(x, y) A_k(x) dx = \\ &h[1/2 f(x_{k-1}, y) A_k(x_{k-1}) + f(x_k, y) A_k(x_k) + \\ &1/2 f(x_{k+1}, y) A_k(x_{k+1})] + R_1 = \\ &h[f(x_k, y)] + R_1, \\ I_2 &= \int_{y_{l-1}}^{y_{l+1}} I_1 B_l(y) dy = \\ &\int_{y_{l-1}}^{y_{l+1}} [h(f(x_k, y)) + R_1] B_l(y) dy = \\ &h[h(f(x_k, y_l)) + R_1] + R_2 = \\ &h^2(f(x_k, y_l)) + hR_1 + R_2. \end{aligned}$$

Integrals I_3, I_4 are:

$$\begin{aligned} I_3 &= \int_{x_{k-1}}^{x_{k+1}} A_k(x) dx = h, \\ I_4 &= \int_{y_{l-1}}^{y_{l+1}} B_l(y) dy = h. \end{aligned}$$

By the trapezoidal rule, the estimations of errors R_1, R_2 are as follows:

$$R_i = -\frac{h^3 M_i}{6}, \quad i = 1, 2,$$

where for $\xi \in (x_{k-1}, x_{k+1})$ and $\eta \in (y_{l-1}, y_{l+1})$

$$\begin{aligned} M_1 &= (f(\xi, y) A_k(\xi))^{(2)}, \\ M_2 &= (f(x_k, \eta) B_l(\eta))^{(2)}. \end{aligned}$$

Then the final estimation of the coefficient $c_{kl,0}^1$ is following:

$$c_{kl,0}^1 = \frac{h^2(f(x_k, y_l)) + hR_1 + R_2}{h^2} = f(x_k, y_l) + O(h).$$

□

The goal of this contribution is to show that the coefficients $c_{kl,1}^1$ and $c_{kl,2}^1$ approximate the corresponding partial derivatives $\frac{\partial f}{\partial x}|_{(x_k, y_l)}$, $\frac{\partial f}{\partial y}|_{(x_k, y_l)}$ and to estimate the qualities of the approximation of the partial derivatives of the original function. The below given theorem proves the goal.

Theorem 5 Let $A_k \times B_l$, $k = 1, \dots, n$, $l = 1, \dots, m$ be an $h_x h_y$ -uniform fuzzy partition of $[a, b] \times [c, d]$ and let $F_{nm}^1[f] = (F_{11}^1, \dots, F_{nm}^1)$ where $F_{kl}^1 = c_{kl,0}^1 + c_{kl,1}^1(x - x_k) + c_{kl,2}^1(y - y_k)$, be the F^1 -transform of f with respect to the given partition. Let functions f , A_k , B_l be four times continuously differentiable on $[a, b] \times [c, d]$. Then for every k, l the following holds:

$$\frac{\partial F_{kl}^1}{\partial x}|_{(x_k, y_l)} = c_{kl,1}^1 = \frac{\partial f}{\partial x}|_{(x_k, y_l)} + O(h), \quad (4)$$

$$\frac{\partial F_{kl}^1}{\partial y}|_{(x_k, y_l)} = c_{kl,2}^1 = \frac{\partial f}{\partial y}|_{(x_k, y_l)} + O(h) \quad (5)$$

where $h = h_x = h_y$.

PROOF: We will give the proof for (4) only, because the proof of (5) can be obtained analogously. Let $k = 1, \dots, n$, $l = 1, \dots, m$ and

$$c_{kl,1}^1 = \frac{\int_{y_{l-1}}^{y_{l+1}} \int_{x_{k-1}}^{x_{k+1}} f(x, y)(x - x_k) A_k(x) B_l(y) dx dy}{\left(\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x) dx \right) \left(\int_{y_{l-1}}^{y_{l+1}} B_l(y) dy \right)}.$$

Let us denote $h_1 = h/2$. We assume the following integrals:

$$\begin{aligned} I_1 &= \int_{y_{l-1}}^{y_{l+1}} f(x, y) B_l(y) dy, \\ I_2 &= \int_{x_{k-1}}^{x_{k+1}} I_1 (x - x_k) A_k(x) dx, \\ I_3 &= \int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x) dx, \\ I_4 &= \int_{y_{l-1}}^{y_{l+1}} B_l(y) dy. \end{aligned}$$

Then we apply the trapezoidal rule with three nodes y_{l-1}, y_l, y_{l+1} to the integral I_1 and we use the following:

$$B_l(y_{l-1}) = B_l(y_{l+1}) = 0, \quad B_l(y_l) = 1.$$

$$\begin{aligned}
I_1 &= \int_{y_{l-1}}^{y_{l+1}} f(x, y) B_l(y) dy = \\
&h[1/2f(x, y_{l-1})B_l(y_{l-1}) + f(x, y_l)B_l(y_l) + \\
&1/2f(x, y_{l+1})B_l(y_{l+1})] + R_1 = \\
&h[f(x, y_l)] + R_1,
\end{aligned}$$

We apply the Simpson's rule with five nodes $x_{k-1}, x_k - h_1, x_k, x_k + h_1, x_{k+1}$ to the integrals I_2, I_3 and we use the properties:

$$A_k(x_{k-1}) = A_k(x_{k+1}) = 0,$$

$$A_k(x_k - h_1) = A_k(x_k + h_1).$$

$$\begin{aligned}
I_2 &= \int_{x_{k-1}}^{x_{k+1}} I_1(x - x_k) A_k(x) dx = \\
&\int_{x_{k-1}}^{x_{k+1}} [h(f(x, y_l)B_l(y_l)) + R_1](x - x_k) A_k(x) dx = \\
&h_1/3[4hf(x_k - h_1, y_l)(x_k - h_1 - x_k)A_k(x_k - h_1) + \\
&4R_1(x_k - h_1 - x_k)A_k(x_k - h_1) + \\
&4h[f(x_k + h_1, y_l)](x_k + h_1 - x_k)A_k(x_k + h_1) + \\
&4R_1(x_k + h_1 - x_k)A_k(x_k + h_1)] + R_2 = \\
&\frac{4h_1^2 A_k(x_k + h_1)}{3}[f(x_k + h_1, y_l) - f(x_k - h_1, y_l)] + R_2 = \\
&\frac{8h_1^3 A_k(x_k + h_1)}{3}[f(x_k + h_1, y_l) - f(x_k - h_1, y_l)] + R_2,
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x) dx = \\
&h_1/3[4(x_k - h_1 - x_k)^2 A_k(x_k - h_1) + \\
&4(x_k + h_1 - x_k)^2 A_k(x_k + h_1)] + R_3 = \\
&\frac{8h_1^3 A_k(x_k + h_1)}{3} + R_3.
\end{aligned}$$

Analogously to the previous proof, integral I_4 is given as follows:

$$I_4 = \int_{y_{l-1}}^{y_{l+1}} B_l(y) dy = h = 2h_1.$$

The estimations of the errors R_2, R_3 with respect to Simpson's rule and R_1, R_4 with respect to the trapezoidal rule are following:

$$\begin{aligned}
R_1 &= -\frac{(y_{l+1} - y_{l-1})h^2 M_1}{12} = -\frac{h^3 M_1}{6} = -\frac{4h_1^3 M_1}{3}, \\
R_2 &= -\frac{(x_{k+1} - x_{k-1})h_1^4 M_2}{180} = -\frac{h_1^5 M_2}{45}, \\
R_3 &= -\frac{h_1^5 M_3}{45},
\end{aligned}$$

where M_1, M_2, M_3 for $\xi \in (x_{k-1}, x_{k+1})$ and $\eta, \zeta \in (y_{l-1}, y_{l+1})$ are given by

$$\begin{aligned}
M_1 &= (f(x, \xi)B_l(\xi))^{(2)}, \\
M_2 &= (f(\eta, y_l)A_k(\eta))^{(4)}, \\
M_3 &= ((\zeta - x_k)^2 A_k(\zeta))^{(4)}.
\end{aligned}$$

According to the integrals I_1, I_2, I_3, I_4 , errors R_1, R_2, R_3 and the fact that $A_k(x_k + h_1) > 0$, we come to the final estimation:

$$\begin{aligned}
c_{kl,1}^1 &= \\
&\frac{8h_1^3 A_k(x_k + h_1)[f(x_k + h_1, y_l) - f(x_k - h_1, y_l)] + 3R_2}{16h_1^4 A_k(x_k + h_1) + 6h_1 R_3} = \\
&\frac{[f(x_k + h_1, y_l) - f(x_k - h_1, y_l)]/2h_1 + O(h_1)}{1 + O(h_1^2)} = \\
&\frac{f(x_k + h_1, y_l) - f(x_k - h_1, y_l)}{2h_1} + O(h_1).
\end{aligned}$$

And therefore

$$c_{kl,1}^1 = \frac{\partial f}{\partial x}|_{(x_k, y_l)} + O(h_1).$$

According to the assumptions, $O(h_1)$ and $O(h)$ are counted to be same. \square

Corollary 6 Under the assumptions of Theorem 5, for every $k = 1, \dots, n$, $l = 1, \dots, m$, the following holds:

$$f(x, y) = F_{kl}^1 + O(h),$$

$$x \in [x_{k-1}, x_{k+1}], y \in [y_{l-1}, y_{l+1}].$$

PROOF: Let $x \in [x_{k-1}, x_{k+1}]$, $y \in [y_{l-1}, y_{l+1}]$, $k = 1, \dots, n$, $l = 1, \dots, m$. By the Taylor polynomial of the first degree we have

$$\begin{aligned}
f(x, y) &= f(x_k, y_l) + \frac{\partial f}{\partial x}|_{(x_k, y_l)}(x - x_k) + \\
&\frac{\partial f}{\partial y}|_{(x_k, y_l)}(y - y_l) + O(h^2).
\end{aligned} \tag{6}$$

According to Lemma 4 and Theorem 5, we can rewrite the formula (6) to the following form:

$$\begin{aligned}
f(x, y) &= c_{kl,0}^1 + O(h) + (c_{kl,1}^1 + O(h))(x - x_k) + \\
&(c_{kl,2}^1 + O(h))(y - y_l) + O(h^2) = \\
&c_{kl,0}^1 + c_{kl,1}^1(x - x_k) + c_{kl,2}^1(y - y_l) + \\
&O(h) + O(h)(x - x_k) + O(h)(y - y_l) + O(h^2) = \\
&F_{kl}^1 + O(h).
\end{aligned}$$

\square

3. Inverse F^1 -transform

We define the *inverse F^1 -transform* of function f similar to the ordinary case as a linear combination of basic functions A_k, B_l and components F_{kl}^1 .

Definition 7 Let $F_{nm}^1[f] = (F_{kl}^1)$, $k = 1, \dots, n$, $l = 1, \dots, m$ be the F^1 -transform of given function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ with respect to $A_k \times B_l$. We say that the function $\hat{f}_{nm}^1 : [a, b] \times [c, d] \rightarrow \mathbb{R}$ represented by

$$\hat{f}_{nm}^1(x, y) = \sum_{k=1}^n \sum_{l=1}^m F_{kl}^1 A_k(x) B_l(y),$$

$$x \in [a, b], y \in [c, d],$$

is the *inverse F^1 -transform* of function f .

The following theorem estimates the difference between the original function and its inverse F^1 -transform.

Theorem 8 Let $A_k \times B_l$, $k = 1, \dots, n$, $l = 1, \dots, m$ be an $h_x h_y$ -uniform fuzzy partition of $[a, b] \times [c, d]$ that fulfills the Ruspini condition on $[a, b] \times [c, d]$ and let \hat{f}_{nm}^1 be the inverse F^1 -transform of f with respect to the given partition. Moreover, let functions f, A_k, B_l be four times continuously differentiable on $[a, b] \times [c, d]$. Then, for all $(x, y) \in [a, b] \times [c, d]$ the following estimation holds:

$$f(x, y) - \hat{f}_{nm}^1(x, y) = O(h),$$

where we assume $h = h_x = h_y$.

PROOF: Let $(x, y) \in [a, b] \times [c, d]$ so that $(x, y) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]$ for some $k = 1, \dots, n$, $l = 1, \dots, m$. Using the Corollary 6 and the Ruspini condition we get

$$\begin{aligned} f(x, y) - \hat{f}_{nm}^1(x, y) &= \\ f(x, y) - \sum_{i=1}^n \sum_{j=1}^m F_{ij}^1 A_i(x) B_j(y) &= \\ f(x, y) \sum_{i=1}^n \sum_{j=1}^m A_i(x) B_j(y) - \sum_{i=1}^n \sum_{j=1}^m F_{ij}^1 A_i(x) B_j(y) &= \\ \sum_{i=1}^n \sum_{j=1}^m A_i(x) B_j(y) (f(x, y) - F_{ij}^1) &= \\ \sum_{i=k}^{k+1} \sum_{j=l}^{l+1} A_i(x) B_j(y) (F_{ij}^1 + O(h) - F_{ij}^1) &= O(h). \end{aligned}$$

□

4. Application to the edge detection problem

In this section, we will show that the theory introduced in this article can be applied in image processing. Particularly, we will briefly describe the main idea of using the F^1 -transform technique in the edge detection problem.

Edge detection is one of important problems in image processing. This is due to a wide spectrum of methods which use the edge detection as a preprocessing technique. There are many methods which solve the problem of edge detection. They differ one from another one by a specification of the notion of “edge”. Let us remark that there is no explicit definition of the term “edge”.

In our approach, we chose a characterization of an edge as an area with significant changes of image intensity. In this respect, the edge detection problem is connected with a searching of local maxima of the gradient magnitude. This means to find the first derivative of the image function and the local maxima of it.

If we keep the formalization of the theory described above, we assume a given function

$$f : [1, N] \times [1, M] \rightarrow \mathbb{R}$$

of two variables as an image function. The function f is then represented by $N \times M$ discrete points.

We apply our theory (adapted to the discrete case) to the image function f . By using the F^1 -transform we approximate partial derivatives of the function f and determine the gradient magnitude:

$$|\text{grad } f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

The advantage of the proposed method is that we can solve two problems in one step by using the technique of F^1 -transform:

- blur the image,

- compute the gradient magnitude.

Blurring (removing of high frequencies) is a general property of the F -transform technique. In image processing, blurring is used to smooth the image and filter out any noise.

The proposed method was successfully combined with the Canny’s method of detection one-pixel edges. In [6], it was proved that our technique detects more relevant edges, keeps all necessary details and gives a smoother output image.

5. Conclusion

In this paper, we extended the technique of F^1 -transform to functions of two variables and focused on the F^1 -transform components. We characterized coefficients in the F^1 -transform component. We showed how partial derivatives of an original function can be approximated by coefficients of the F^1 -transform components and moreover, we estimated the quality of this approximation. We also described the inverse F^1 -transform of functions of two variables and estimated the quality of approximation of the original function. Finally, we briefly described the main idea of applying the technique of F^1 -transform in the edge detection problem. In the future, we aim to approximate mixed partial derivatives and elaborate applications of these results.

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