# Expectile smoothing of time series using F-transform

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## Abstract

In this paper, we will illustrate the F-transform based on generalized fuzzy partitions as a tool for expectile smoothing. This allows to represent a time series in terms of a fuzzy-valued function whose level-cuts are modeled by F-transform and estimated by expectile regression. The proposed methodology is illustrated on real economic and financial time series.

**Keywords**: Fuzzy Transform, Expectile Smoothing, Fuzzy Time Series

## 1. F-transform and its properties

The fuzzy transform (F-transform) has recently been introduced by I. Perfilieva in [3] (see also [4], [5], [6]) and its properties as a general smoothing tool have been illustrated in [8], [10], [1].

We briefly recall the basic definitions and properties of the F-transform (see [3], [10]).

Given a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ and given a finite family of fuzzy sets (in particular fuzzy numbers) $\mathbb{A} = \{A_1, A_2, ..., A_n\}$ forming a fuzzy partition of $[a, b]$, the F-transform produces a vector of real numbers $\mathbf{F} = (F_1, F_2, ..., F_n)$ (called the *direct* F-transform). Each $F_k$ is the minimizer of a weighted squared error between the values $f(x)$ and $F_k$ on the $k-$th subinterval of $[a, b]$. The direct F-transform $\mathbf{F}$ is then used to define the *inverse* F-transform function $\widehat{f} : [a, b] \longrightarrow \mathbb{R}$ and the main result is that $\widehat{f}$ is an approximating function of $f$ on $[a, b]$.

In the basic setting, each basic function $A_k$ of the fuzzy partition $(\mathbb{P}, \mathbb{A})$ has been considered to be zero outside the union of the two adjacent subintervals $[x_{k-1}, x_k] \cup [x_k, x_{k+1}]$; we can generalize (see details in [10]) the concept of a fuzzy partition by taking basic functions that cover more than two consecutive subintervals. Consider an integer $r \geq 1$ and $2r + 1$ consecutive points (and consequently $2r$ subintervals) of $\mathbb{P}$, $x_{k-r}, ..., x_k, ..., x_{k+r}$ for all $k = 1, 2, ..., n$; to complete the notation, we extend the points to $x_{1-r} < ... < x_0 < a$ and $b < x_{n+1} < ... < x_{n+r}$.

**Definition 1:** ([3], [10]) *Let $r \geq 1$ be a fixed integer number; a fuzzy $r$-partition of $[a, b]$ is* given by a pair $(\mathbb{P}, \mathbb{A}^{(r)})$ where $\mathbb{P} = \{a = x_1 < x_2 < ... < x_n = b\}$ is a decomposition of $[a, b]$, and $\mathbb{A}^{(r)}$ is a family of $n + 2r - 2$ continuous, normal, convex fuzzy numbers

$$\mathbb{A}^{(r)} = \{A_k^{(r)} : [a, b] \longrightarrow [0, 1]|$$
$$k = -r + 2, ..., n + r - 1\}$$

*such that*
*a. for $k = 1, 2, ..., n$, $A_k^{(r)}$ is a continuous fuzzy number with $A_k^{(r)}(x_k) = 1$ and $A_k^{(r)}(x) = 0$ for $x \notin [x_{k-r}, x_{k+r}]$;*
*b. for $k = 1, 2, ..., n$, $A_k^{(r)}$ is increasing on $[x_{k-r}, x_k]$ and decreasing on $[x_k, x_{k+r}]$;*
*c. for $k = -r + 2, ..., 0$, $A_k^{(r)}$ is decreasing on $[x_k, x_{k+r}]$;*
*d. for $k = n + 1, ...n + r - 1$, $A_k^{(r)}$ is increasing on $[x_{k-r}, x_k]$;*
*e. for all $x \in [a, b]$, the following partition-of-$r$ condition holds $\sum\limits_{k=-r+2}^{n+r-1} A_k^{(r)}(x) = r$.*

The integer $r \geq 1$ will be called the *bandwidth* of the partition $(\mathbb{P}, \mathbb{A}^{(r)})$.

A parametric form of a *fuzzy $r$-partition* of $[a, b]$ is obtained by considering $n + r - 2$ shape functions of type $L_k^{(r)}(x)$ $k = 2, ..., n + r - 1$; the basic functions are

$$A_k^{(r)}(x) = \begin{cases} L_k^{(r)}(x) & \text{if } x \in [x_{k-r}, x_k] \\ 1 - L_{k+r}^{(r)}(x) & \text{if } x \in [x_k, x_{k+r}] \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$\text{for } k = 2, ..., n - 1$$

$$A_k^{(r)}(x) = \begin{cases} 1 - L_{k+r}^{(r)}(x) & \text{if } x \in [x_k, x_{k+r}] \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$\text{for } k = -r + 2, ..., 1$$

$$A_k^{(r)}(x) = \begin{cases} L_{k-r}^{(r)}(x) & \text{if } x \in [x_{k-r}, x_k] \\ 0 & \text{otherwise} \end{cases}, (3)$$

$$\text{for } k = n, ..., n + r - 1.$$

Figure 1 illustrates a fuzzy 2-partition $(\mathbb{P}, \mathbb{A}^{(2)})$; each $A_k^{(2)}(x)$ covers four intervals.
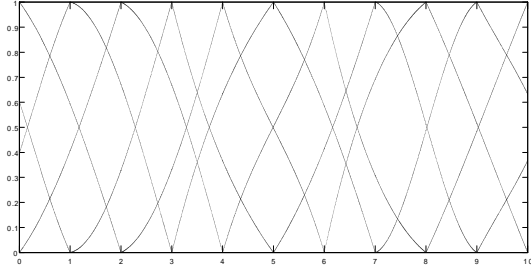
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Figure 1. Generalized fuzzy partition, r=2.

**Definition 2:** ([3], [10]) *The direct $F^{(r)}$-transform (of integer bandwidth $r \geq 1$) based on the given generalized fuzzy r-partition $(\mathbb{P}, \mathbb{A}^{(r)})$ is defined by the vector* $\mathbf{F}^{(r)} = (F_1^{(r)}, F_2^{(r)}, ..., F_n^{(r)})^T$, *where*

$$F_k^{(r)} = \frac{1}{I_k^{(r)}} \int_a^b f(x) A_k^{(r)}(x) dx \text{ for } k = 1, 2, ..., n \quad (4)$$

$$I_k^{(r)} = \int_a^b A_k^{(r)}(x) dx. \quad (5)$$

*Correspondingly, the $iF^{(r)}$-transform function (of bandwidth r) is*

$$\widehat{f}^{(r)}(x) = \frac{1}{r} \sum_{k=1}^n F_k^{(r)} A_k^{(r)}(x). \quad (6)$$

We see that $\widehat{f}^{(r)}(x)$ has the structure of a moving average of the values $\{F_j^{(r)}, j = 1, ..., n\}$; in fact, assuming $F_k^{(r)} = 0$ if $k < 1$ or $k > n$, we have

$$\widehat{f}^{(r)}(x) = \frac{1}{r} \sum_{j=k-r}^{k+r} F_j^{(r)} A_j^{(r)}(x), \quad (7)$$

i.e., a weighted average of $F_{k-r}^{(r)}, ..., F_k^{(r)}, ..., F_{k+r}^{(r)}$ with weights $\frac{A_{k-r}^{(r)}(x)}{r}, ..., \frac{A_k^{(r)}(x)}{r}, ..., \frac{A_{k+r}^{(r)}(x)}{r}$.

The main properties of the $F^{(r)}$-transform are analogues to the properties of the standard F-transform proved in [3].

**Proposition 1:** ([3], [10]) *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, and let $\mathbf{F}^{(r)} = (F_1^{(r)}, F_2^{(r)}, ..., F_n^{(r)})^T$ be its $F^{(r)}$-transform with respect to a given r-partition $(\mathbb{P}, \mathbb{A}^{(r)})$. Then, for any $k = 1, 2, ..., n$, the function $\phi_k(y) = \int_a^b (f(x) - y)^2 A_k^{(r)}(x) dx$ is minimized by $y = F_k^{(r)}$.*

The discrete version of the $F^{(r)}$-transform is analogous to the standard case as in [3]. Consider a function $f : [a, b] \longrightarrow \mathbb{R}$ and $m$ points $t_1 < t_2 < ... <$

$t_m$ where $f(t_i) = f_i$ is known, $i = 1, 2, ..., m$. Let $(\mathbb{P}, \mathbb{A}^{(r)})$ be a fuzzy r-partition of $[a, b]$, and assume that the set of points $\{t_i | i = 1, ..., m\}$ is *sufficiently dense* with respect to $(\mathbb{P}, \mathbb{A}^{(r)})$, i.e. that each set $T_k = \{t_j | A_k^{(r)}(t_j) > 0\}$, $k = 1, ..., n$, is nonempty.

**Definition 3:** *The discrete (direct) $F^{(r)}$-transform of the data set $\{(t_i, f_i) | i = 1, ..., m\}$ with the fuzzy r-partition $(\mathbb{P}, \mathbb{A}^{(r)})$ (assuming $\{t_i | i = 1, ..., m\}$ sufficiently dense) is defined by the vector*

$$\mathbf{F}^{(r)} = \left( F_1^{(r)}, F_2^{(r)}, ..., F_n^{(r)} \right)^T , \text{ where} \quad (8)$$

$$F_k^{(r)} = \frac{g_k^{(r)}}{s_k^{(r)}} \text{ for } k = 1, 2, ..., n \text{ , and}$$

$$g_k^{(r)} = \sum_{i=1}^m f_i A_k^{(r)}(t_i) , \quad s_k^{(r)} = \sum_{i=1}^m A_k^{(r)}(t_i) > 0.$$

*The discrete (inverse) $iF^{(r)}$-transform is the function $\widehat{f}^{(r)} : [a, b] \longrightarrow \mathbb{R}$ given by*

$$\widehat{f}^{(r)}(t) = \frac{1}{r} \sum_{k=1}^n F_k^{(r)} A_k^{(r)}(t), \, t \in [a, b]. \quad (9)$$

In matrix notation, define the following vectors:

$$\mathbf{f} = (f_1, f_2, ..., f_m)^T \quad (10)$$

$$\mathbf{g}^{(r)} = \left( g_1^{(r)}, g_2^{(r)}, ..., g_n^{(r)} \right)^T$$

$$\widehat{\mathbf{f}}^{(r)} = \left( \widehat{f}^{(r)}(t_1), \widehat{f}^{(r)}(t_2), ..., \widehat{f}^{(r)}(t_m) \right)^T$$

and the following $m \times n$ matrix:

$$\mathbf{M}^{(r)} = [m_{i,k}] \text{ with } m_{i,k} = A_k^{(r)}(x_i). \quad (11)$$

It follows that

$$\widehat{\mathbf{f}}^{(r)} = \frac{1}{r} \mathbf{M}^{(r)} \mathbf{F}^{(r)} \quad (12)$$

and $\frac{1}{r} \mathbf{M}^{(r)}$ represents the *moving average operator* acting on $\mathbf{F}^{(r)}$ to produce $\widehat{\mathbf{f}}^{(r)}$. If the elements $s_k^{(r)}$ are organized into a diagonal $n$−matrix

$$\mathbf{S}_r = diag(s_1^{(r)}, s_2^{(r)}, ..., s_n^{(r)}), \quad (13)$$

then we have $(\mathbf{M}^{(r)})^T \mathbf{f} = \mathbf{g}^{(r)}$ so that $\mathbf{F}^{(r)} = \mathbf{S}_r^{-1}(\mathbf{M}^{(r)})^T \mathbf{f}$ and

$$\widehat{\mathbf{f}}^{(r)} = \frac{1}{r} \mathbf{M}^{(r)} \mathbf{S}_r^{-1} (\mathbf{M}^{(r)})^T \mathbf{f}. \quad (14)$$

Matrix $\mathbf{S}_r$ is invertible, and matrix $\mathbf{H}^{(r)} = \frac{1}{r} \mathbf{M}^{(r)} \mathbf{S}_r^{-1} (\mathbf{M}^{(r)})^T$ is positive semidefinite, called the *hat-matrix* in the resulting equation

$$\widehat{\mathbf{f}}^{(r)} = \mathbf{H}^{(r)} \mathbf{f}. \quad (15)$$

## 2. F-transform in expectile smoothing

In order to investigate the role of F-transform in expectile smoothing, let's first introduce some basic facts on quantile and expectile regression and autoregression, a recent interesting field in nonparametric regression (see e.g. [7], [2] and the references therein).

Consider a real-valued random variable $\xi$; a given $r$-quantile $\xi(r)$ is defined by the property that the probability that an observation is less than $\xi(r)$ is $r$, with $r \in \,]0,1[$

$$Prob(\xi \le \xi(r)) = r.$$

Given a set of $T$ observations $x_t$, $t = 1, ..., T$, the sample quantile $\overline{\xi}(r)$ can be obtained as the solution to minimize the function (with respect to $m$)

$$S_r(m) = \sum_{t=1}^{T} \rho_r(x_t - m)$$

where $\rho_r(y)$ is the *check function* (see e.g. [7]) defined for quantiles as

$$\rho_r(y) = (r - I(y < 0)) y$$

and the function $I(.)$ is defined by

$$I(y < 0) = \begin{cases} 1 \text{ if } y < 0 \\ 0 \text{ otherwise} \end{cases}.$$

We can write

$$S_r(m) = (r-1) \sum_{x_t < m} (x_t - m) + r \sum_{x_t > m} (x_t - m)$$

and it is immediate to see that $S_r(m) \ge 0$ for all real $m$.

If $r = \frac{1}{2}$, then the $r$-quantile gives the median $m_e$ of the (empirical) distribution of $\xi$, i.e. the minimizer of the functional

$$m_e = \arg\min_m S_{1/2}(m) = \sum_{t=1}^{T} |x_t - m|.$$

The expectiles are defined in a similar way as for quantiles, in particular by using the mean instead of the median.

The population expectiles $\widetilde{\mu}(\omega)$, for $\omega \in \,]0,1[$, are defined by tail expectations rather than tail probabilities. For a given value of $\omega \in \,]0,1[$, the sample expectile $\widetilde{\mu}(\omega)$ is obtained by minimizing

$$S_\omega(\mu) = \sum_{t=1}^{T} \rho_\omega(x_t - \mu) \ , \ \omega \in \,]0,1[$$

where the check function is now given by

$$\rho_\omega(y) = y^2 |\omega - I(y < 0)|.$$

This is equivalent to

$$S_\omega(\mu) = \sum_{\substack{t=1 \\ x_t < \mu}}^{T} (1-\omega)(x_t - \mu)^2 + \sum_{\substack{t=1 \\ x_t > \mu}}^{T} \omega(x_t - \mu)^2.$$

If $\omega = \frac{1}{2}$ we obtain the mean value $\mu_e$ of the observations

$$\mu_e = \arg\min_\mu S_{\frac{1}{2}}(\mu) = \frac{1}{2} \sum_{t=1}^{T} (x_t - \mu)^2$$

$$\mu_e = \frac{1}{T} \sum_{t=1}^{T} x_t$$

The ordinary least squares (OLS) estimation of $\mu$ is obtained by minimizing

$$S_{OLS} = \sum_{t=1}^{T} (x_t - \mu)^2 \,;$$

when an asymmetry parameter $\omega \in \,]0,1[$ is chosen, then the least asymmetrical weighted squares (LAWS) function

$$S_{LAWS}^{(\omega)}(\mu) = \sum_{t=1}^{T} w_\omega(t)(x_t - \mu)^2 \qquad (16)$$

is minimized, where the weights $w_\omega(t)$ are

$$w_\omega(t) = \begin{cases} \omega & \text{if } x_t > \mu \\ 1 - \omega & \text{if } x_t \le \mu \end{cases}$$

The value $\mu = \mu(\omega)$ (depending on $\omega$) is the population expectile for different values of the asymmetry parameter $\omega \in \,]0,1[$. The model in (16) can be fitted by applying iteratively a weighted least squares minimization (see e.g. [11]).

**Iterated LAWS Algorithm:**

1. Chose a value $\omega \in \,]0,1[$ and start with estimated (initial) $w_i^l$, $l = 0$ (e.g. $w_t^0 = \frac{1}{2}$ for all $t = 1, ..., T$);

2. Compute the minimizer $\mu^l$ of $S_{LAWS}^{(\omega)}(\mu) = \sum_{t=1}^{T} w_t^l(x_t - \mu)^2$ and compute new weights

$$w_t^{l+1} = \begin{cases} \omega & \text{if } x_t > \mu^l \\ 1 - \omega & \text{if } x_t \le \mu^l \end{cases}$$

increase $l = l + 1$;

3. Continue iteratively with step 2 until the weights $w_t^{l+1}$ become "stable" with respect to $w_t^l$ (i.e. $w_t^{l+1} = w_t^l$ for all $t$).

Note that ordinary least squares is a special case of LAWS when $\omega = \frac{1}{2}$, and the solution of (16) in step 2 is given by

$$\mu^l = \sum_{t=1}^{T} w_t^l x_t \bigg/ \sum_{t=1}^{T} w_t^l \,.$$

The expectile F-transform, for a fixed generalized fuzzy $r$-partition $(\mathbb{P}, \mathbb{A}^{(r)})$ and for a given value of

$\omega \in ]0,1]$, can be defined, according to the expectiles setting described above, to be the minimizer of the following operators, for $k = 1, ..., n$,

$$\Phi_{k,\omega}(F) = \int_a^b w_\omega(x)(f(x) - F)^2 A_k^{(r)}(x) dx$$
where
$$w_\omega(x) = \begin{cases} \omega & \text{if} \quad f(x) \leq F \\ 1 - \omega & \text{if} \quad f(x) > F \end{cases}.$$

Remark that if $\omega = 0.5$ the minimization of $\Phi_{k,0.5}(F)$ with respect to $F$ gives the usual F-transform component $F_{k,0.5}$ in Definition 3.

According to the fact that, corresponding to $\omega > \frac{1}{2}$ we obtain a value of $F$ greater than $F_{k,0.5}$, and that, corresponding to $\omega < \frac{1}{2}$ we obtain a value of $F$ less than $F_{k,0.5}$, we suggest the following procedure to obtain a fuzzy-valued version of the direct F-transform:

Choose $\alpha \in ]0,1]$ and consider the two operators $\Phi_{k,\alpha}^-(F)$ and $\Phi_{k,\alpha}^+(F)$, defined by

$$\Phi_{k,\alpha}^-(F) = \int_a^b w_\alpha^-(x)(f(x) - F)^2 A_k^{(r)}(x) dx$$
where
$$w_\alpha^-(x) = \begin{cases} \frac{\alpha}{2} & \text{if} \quad f(x) \leq F \\ 1 - \frac{\alpha}{2} & \text{if} \quad f(x) > F \end{cases}$$

and

$$\Phi_{k,\alpha}^+(F) = \int_a^b w_\alpha^+(x)(f(x) - F)^2 A_k^{(r)}(x) dx$$
where
$$w_\alpha^+(x) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if} \quad f(x) \leq F \\ \frac{\alpha}{2} & \text{if} \quad f(x) > F \end{cases}.$$

If $\alpha = 1$ we obtain $\Phi_{k,0.5}(F)$.

The minimization of $\Phi_{k,\alpha}^-(F)$ and $\Phi_{k,\alpha}^+(F)$ produces, respectively $F_{k,\alpha}^-$ and $F_{k,\alpha}^+$ so that $\left[ F_{k,\alpha}^-, F_{k,\alpha}^+ \right]$ is the $\alpha$−cut of $F_k$.

As a consequence, the iF-transform of $f$ is fuzzified by:
$$\widehat{f}(x) = \frac{1}{r} \sum_{k=1}^n F_k A_k^{(r)}(x)$$

with the corresponding $\alpha-cuts$ expressed as a linear combination of intervals $\left[ F_{k,\alpha}^-, F_{k,\alpha}^+ \right]$:

$$\left[ \widehat{f}(x) \right]_\alpha = \left[ \widehat{f_\alpha^-}(x), \widehat{f_\alpha^+}(x) \right]_\alpha \quad (17)$$
$$= \frac{1}{r} \sum_{k=1}^n \left[ F_{k,\alpha}^-, F_{k,\alpha}^+ \right] A_k^{(r)}(x)$$

When $\alpha = 1$ we obtain the standard $F$−transform and the corresponding $iF$−transform.

The discrete case can be handled in a similar way as for the standard discrete F-transform. The expectiles, in the discrete case, are obtained by minimizing the following functions:

$$\Phi_{k,\alpha}^-(F) = \sum_{k=1}^m w_\alpha^-(t_i)(f(t_i) - F)^2 A_k^{(r)}(t_i) \quad (18)$$

where
$$w_\alpha^-(t_i) = \begin{cases} \frac{\alpha}{2} & \text{if} \quad f(t_i) \leq F \\ 1 - \frac{\alpha}{2} & \text{if} \quad f(t_i) > F \end{cases}$$

and

$$\Phi_{k,\alpha}^+(F) = \sum_{k=1}^m w_\alpha^+(t_i)(f(t_i) - F)^2 A_k^{(r)}(t_i) \quad (19)$$

where
$$w_\alpha^+(t_i) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if} \quad f(t_i) \leq F \\ \frac{\alpha}{2} & \text{if} \quad f(t_i) > F \end{cases}$$

Consider that, for fixed values of $w_\alpha^\pm(t_i)$, the minimizers $F_{k,\alpha}^-$ and $F_{k,\alpha}^+$ of (18) and (19) are obtained, respectively, for $k = 1, ..., n$, by

$$F_{k,\alpha}^- = \frac{\sum_{k=1}^m w_\alpha^-(t_i) f(t_i) A_k^{(r)}(t_i)}{\sum_{k=1}^m w_\alpha^-(t_i) A_k^{(r)}(t_i)}$$

and

$$F_{k,\alpha}^+ = \frac{\sum_{k=1}^m w_\alpha^+(t_i) f(t_i) A_k^{(r)}(t_i)}{\sum_{k=1}^m w_\alpha^+(t_i) A_k^{(r)}(t_i)}.$$

An iterative procedure can be easily designed, similar to the Iterated LAWS Algorithm as described above.

Remark that if $\alpha = \frac{1}{2}$, then we have $w_\alpha^-(t_i) = w_\alpha^+(t_i) = \frac{1}{2}$ for all $i = 1, ..., m$ and we obtain exactly the standard discrete F-transform of $f$ based on the observations $f(t_i)$ at the points $t_1, t_2, ..., t_m$, assumed to be sufficiently dense with respect to the given fuzzy $r$-partition $(\mathbb{P}, \mathbb{A}^{(r)})$.
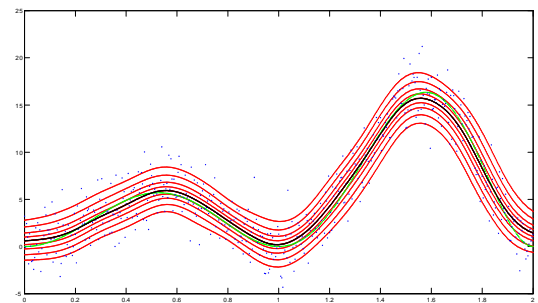


Figure 2. $\alpha$-cuts of a fuzzy-valued function by F-transform ($m = 501$, $n = 101$, $r = 6$) and $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

In figure 2 we illustrate the expectile smoothing to the same simulated example as in [10] (Section 4.3, example 1): $f(t_i) = 5e^{-0.5t_i^2}\sin^2(\pi t_i) + 2z_i$, $t_i \in [0,2]$, $i = 1, ..., m$, where $z_i \in N(0,1)$.

The data are represented by points and 9 curves are generated, corresponding to the values of $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$; it is to be remarked that for any value of $\alpha \in ]0,1]$ we can obtain the $\alpha$−cut $\left[F_{k,\alpha}^{-}, F_{k,\alpha}^{+}\right]$ of $F_k$, $k = 1, 2, ..., n$. The curves are then constructed by inverse F-transform, equation (17).

## 3. Application to financial time series

In order to show how the F-transform can be used for expectile smoothing, we apply the proposed estimation on some real financial time series. In all the cases, the number $n$ of subintervals in the fuzzy partition $(\mathbb{P}, \mathbb{A}^{(r)})$ is approximately $\frac{m}{5}$ and the bandwidth $r$ is estimated by generalized cross validation as in [10]. In all cases, for simplicity, the basic functions $A_k(x)$, defined on the intervals $[x_{k-r}, x_{k+r}]$, are obtained by translating and rescaling the same symmetric triangular fuzzy number $T_0$, defined on $[-1,1]$ and centered at the origin, with membership

$$T_0(t) = \begin{cases} 1+t & \text{if} \quad t \in [-1,0] \\ 1-t & \text{if} \quad t \in [0,1] \\ 0 & \text{otherwise} \end{cases} .$$

The first time series is the daily London Gold Fixing, the usual benchmark for the gold price; it also provides a published benchmark price that is widely used as a pricing medium by producers, consumers, investors and central banks. The $m = 1317$ observations cover the period from 1 june 2007 to 31 august 2012.

Figures 3 and 3a represent the observations and the $\alpha$-cuts obtained by the repeated application of F-transform (18), (19) for the indicated values of $\alpha \in ]0,1]$.
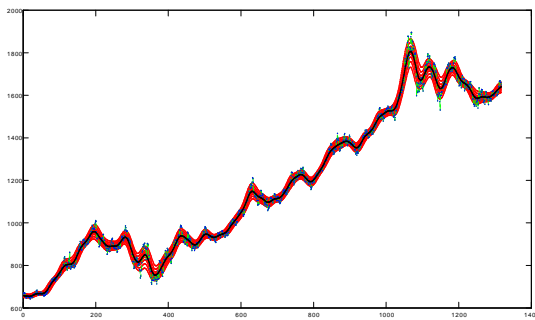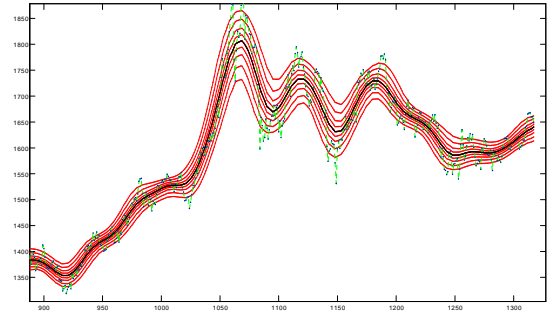


*Figure 3a.* Zoom of figure 3.

As a second collection of data we use 5 years of the FTSE 100, that is a share daily index of the 100 companies listed on the London Stock Exchange with the highest market capitalization. It is one of the most widely used stock indices and is seen as an indicator of business prosperity. The $m = 1264$ cover the period from 1 june 2007 to 31 may 2012.
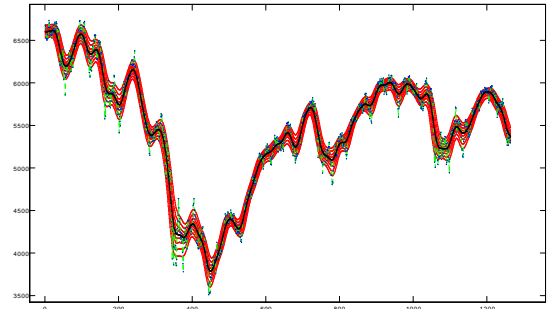


*Figure 4.* $\alpha$-cuts of a fuzzy-valued function by F-transform $(m = 1264, n = 250, r = 3)$ and $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

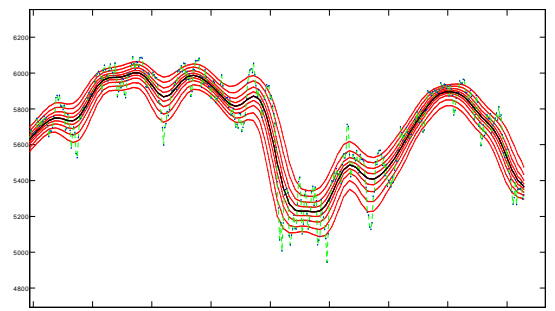The fuzzy values (17) can be better distinguished in figure 4a.



*Figure 3.* $\alpha$-cuts of a fuzzy-valued function by F-transform $(m = 1317, n = 250, r = 3)$ and $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$



*Figure 4a.* Zoom of figure 4.

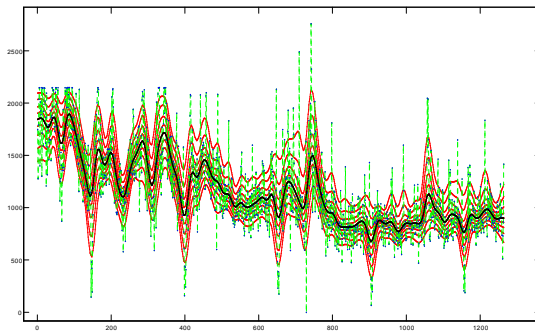The third series of data considers the traded volumes of FTSE 100 above.



*Figure 5.* $\alpha$-cuts of a fuzzy-valued function
by F-transform ($m = 1264$, $n = 250$, $r = 3$) and
$\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

The last time series is the daily ITRAXX Europe index, with $m = 1380$ data from 16/6/2004 to 01/10/2009.
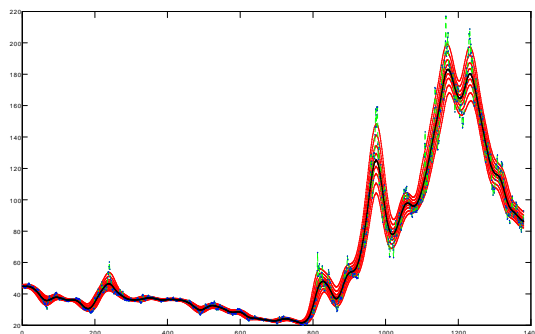


*Figure 6.* $\alpha$-cuts of a fuzzy-valued function
by F-transform ($m = 1380$, $n = 276$, $r = 4$) and
$\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

## 4. Conclusions

The F-transform (FT) setting is suggested as a tool for expectile smoothing of a time series. The discrete F-transform $F_k$ corresponding to different expectile smoothing estimations are used to obtain the $\alpha-$cuts $\left[ F_{k,\alpha}^-, F_{k,\alpha}^+ \right]$ of $F_k$, $k = 1, 2, ..., n$ and consequently, by inverse F-transform, a fuzzy-valued function is constructed representing the given time series. What seems to be of interest, is the fact that both direct and inverse F-transforms are able to reproduce (the direct FT) and to reconstruct (using inverse FT) the time series at different levels of precision.

Using an appropriate (generalized) fuzzy partition, the $\alpha-$cuts $\left[ F_{k,\alpha}^-, F_{k,\alpha}^+ \right]$ of $F_k$ have the same smoothing property inherited from F-transform, with a "degree of smoothness" depending on the bandwidth of the partition.

The preliminary results in section 3, obtained for the real financial time series, encourage to further work in the study and applications of F-transform as a tool to obtain a fuzzy-valued interpretation of a time series. In particular, we are interested to better understand the connections between the fuzzy partition $(\mathbb{P}, \mathbb{A}^{(r)})$ (the number and form of the basic functions, the length of the bandwidth,...) and the properties of the estimated fuzzy-valued (inverse) F-transform.

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