

$h - k$ -aggregation functions, measures and integrals

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Abstract

In many decision making problems evaluations of possible alternatives of choice, with respect to several points of view (criteria) are expressed by means of h -interval (or fuzzy numbers). For example a pessimistic and an optimistic evaluation generate an interval containing the exact evaluation. These situations reflect lack of information or uncertainty on the same evaluations. In this paper we discuss $h - k$ -aggregation functions that aggregate several h -interval evaluations into an overall evaluation, again expressed in terms of a k -interval.

1. Introduction

In Decision Analysis and, especially, in multiple-criteria Decision Analysis (MCDA) the aggregation of information is a fundamental process [6] and, consequently, different types of aggregation operators are found in the literature [7]. However, while in the theory it is often assumed that the available information are expressed by means of exact numbers, in many real situations found in MCDA the available information is vague or imprecise. In order to assess the uncertainty a good method is the use of fuzzy numbers. To express the evaluation of possible alternatives of choice by means of fuzzy numbers means that we are able to consider the best and worst possible scenario and also the possibility that the internal values of the fuzzy intervals will occur.

We consider h -intervals $[a_1, \dots, a_h]$, $a_1, \dots, a_h \in \mathbb{R}$ such that $a_1 \leq \dots \leq a_h$ that express evaluations with respect to a considered point of view by means of the h values a_1, \dots, a_h . For example, if $h = 2$, then evaluations are 2-intervals assigning to each criterion two evaluations corresponding to a pessimistic and an optimistic evaluation. If $h = 3$, then evaluations are 3-intervals $[a_1, a_2, a_3]$ assigning to each criterion three evaluations such that a_1 corresponds to a pessimistic evaluation, a_2 corresponds to an average evaluation and a_3 corresponds to an optimistic evaluation. If $h = 4$, then evaluations are 4-intervals $[a_1, a_2, a_3, a_4]$ assigning to each criterion four evaluations such that a_1 corresponds to a pessimistic evaluation, a_2 and a_3 to two evaluations defining an interval $[a_2, a_3]$ of average evaluation and a_4 corresponds to an optimistic evaluation. Observe that 2-interval evaluations can be seen as usual intervals of evaluations, 3-interval evaluations can

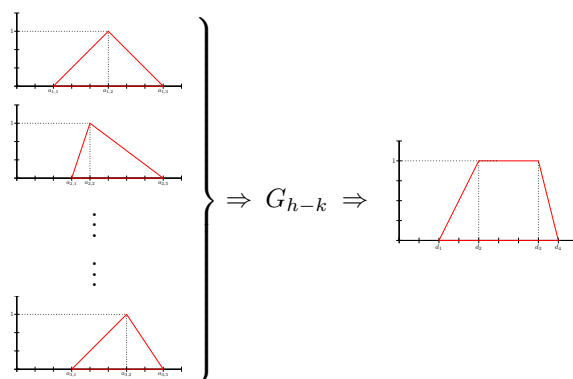
be seen as triangular fuzzy numbers and 4-intervals evaluations can be seen as trapezoidal fuzzy numbers. We have similar situations with $h \geq 5$. Let us denote by \mathcal{I}_h the set of all h -intervals, i.e.

$$\mathcal{I}_h = \{[a_1, \dots, a_h] | a_1, \dots, a_h \in \mathbb{R}, a_1 \leq \dots \leq a_h\}.$$

In [10], a general framework for the comparison of h -intervals has been presented. Here we introduce $h - k$ -aggregation functions that assigns to vectors

$$\mathbf{x} = ([x_{11}, \dots, x_{1h}], \dots, [x_{n1}, \dots, x_{nh}]) \in \mathcal{I}_h^n$$

of h -interval evaluations with respect to a set $N = \{1, \dots, n\}$ of considered criteria an overall evaluation in terms of a k -interval.



Formally an $h - k$ -aggregation function is a function $g : \mathcal{I}_h^n \rightarrow \mathcal{I}_k$ with $g(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_k(\mathbf{x}))$, satisfying the following properties:

- monotonicity: for all $\mathbf{x}, \mathbf{y} \in \mathcal{I}_h^n$, if $x_{i,j} \geq y_{i,j}$ for all $i = 1, \dots, n$ and for all $j = 1, \dots, h$, then $g_r(\mathbf{x}) \geq g_r(\mathbf{y})$ for all $r = 1, \dots, k$;
- left boundary condition: if $x_{i,h} \rightarrow -\infty$ for all $i = 1, \dots, n$, then $g_r(\mathbf{x}) \rightarrow -\infty$ for all $r = 1, \dots, k$;
- right boundary condition if $x_{i,1} \rightarrow +\infty$ for all $i = 1, \dots, n$, then $g_r(\mathbf{x}) \rightarrow +\infty$ for all $r = 1, \dots, k$.

2. The $h - k$ -weighted average

Let us consider k vectors of \mathcal{I}_h^n

$$\mathbf{a}^{(r)} = ([a_{11}^{(r)}, \dots, a_{1h}^{(r)}], \dots, [a_{n1}^{(r)}, \dots, a_{nh}^{(r)}]),$$

$r = 1, \dots, k$, such that

- $\sum_{j=h-t}^h a_{i,j}^{(r_1)} \geq \sum_{j=h-t}^h a_{i,j}^{(r_2)}$, for all $i = 1, \dots, n$, $t = 1, \dots, h-1$ and $r_1, r_2 = 1, \dots, k$, such that $r_1 \geq r_2$;
- $\sum_{i=1}^n \sum_{j=1}^h a_{i,j}^{(r)} = 1$, for all $r = 1, \dots, k$.

The $h-k$ -weighted average with respect to the weights $\mathbf{a}^{(r)}$, $r = 1, \dots, k$ is the $h-k$ -aggregation function

$$WA_{\mathbf{a}} : \mathcal{I}_h^n \rightarrow \mathcal{I}_k,$$

with $WA_{\mathbf{a}}(\mathbf{x}) = (WA_{\mathbf{a},1}(\mathbf{x}), \dots, WA_{\mathbf{a},k}(\mathbf{x}))$, defined as follows: for all $\mathbf{x} \in \mathcal{I}_h^n$ and $r = 1, \dots, k$,

$$WA_{\mathbf{a},r}(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^h a_{i,j}^{(r)} x_{i,j}. \quad (1)$$

The $h-k$ -weighted average can be formulated also as follows. Let us consider k vectors of \mathcal{I}_h^n

$$\mathbf{b}^{(r)} = ([b_{11}^{(r)}, \dots, b_{1h}^{(r)}], \dots, [b_{n1}^{(r)}, \dots, b_{nh}^{(r)}]),$$

$r = 1, \dots, k$, such that

- $b_{i,1}^{(r)} \geq b_{i,2}^{(r)} \geq \dots \geq b_{i,h}^{(r)} \geq 0$, for all $i = 1, \dots, n$ and $r = 1, \dots, k$;
- $b_{i,j}^{(1)} \geq b_{i,j}^{(2)} \geq \dots \geq b_{i,j}^{(k)} \geq 0$, for all $i = 1, \dots, n$ and $j = 1, \dots, h$;
- $\sum_{i=1}^n b_{i,1}^{(r)} = 1$, for all $i = 1, \dots, n$ and $r = 1, \dots, k$.

The $h-k$ -weighted average with respect to weights $\mathbf{b}^{(r)}$ is the $h-k$ -aggregation function $WA_{\mathbf{a}} : \mathcal{I}_h^n \rightarrow \mathcal{I}_k$ defined as follows: for all $\mathbf{x} \in \mathcal{I}_h^n$ and $r = 1, \dots, k$,

$$WA_{\mathbf{a},r}(\mathbf{x}) = \sum_{i=1}^n b_{i,1}^{(r)} x_{i,1} + \sum_{i=1}^n \sum_{j=2}^h b_{i,j}^{(r)} (x_{i,j} - x_{i,j-1}). \quad (2)$$

There is the following relation between weights $b_{i,j}^{(r)}$ and $a_{i,j}^{(r)}$: for all $i = 1, \dots, n$; $j = 1, \dots, h-1$, and $r = 1, \dots, k$,

$$\begin{cases} a_{i,j}^{(r)} = b_{i,j}^{(r)} - b_{i,j+1} \\ a_{i,h}^{(r)} = b_{i,h}^{(r)}. \end{cases} \quad (3)$$

Two very natural conditions for $h-k$ -aggregation functions are the following

- additivity: for all $\mathbf{x}, \mathbf{y} \in \mathcal{I}_h^n$, $g(\mathbf{x} + \mathbf{y}) = g(\mathbf{x}) + g(\mathbf{y})$, where $\mathbf{x} + \mathbf{y} = \mathbf{z}$ with $z_{i,j} = x_{i,j} + y_{i,j}$ for all $i = 1, \dots, n$ and for all $j = 1, \dots, h$;
- idempotence: for all $a \in \mathbb{R}$, $g(\mathbf{a}) = a$, where $\mathbf{a} \in \mathcal{I}_h^n$ is $\mathbf{a} = [a, \dots, a]$.

Theorem 1 An $h-k$ -aggregation function is additive and idempotent if and only if it is the $h-k$ weighted average.

3. The $h-k$ -Choquet integral

Given the set of criteria $N = \{1, \dots, n\}$ let us consider the set

$$\mathcal{Q} = \{(A_1, \dots, A_h) \mid A_1 \subseteq A_2 \subseteq \dots \subseteq A_h \subseteq N\}.$$

Elements of \mathcal{Q} are h -uple of coalitions of criteria A_j such that $A_j \subseteq A_{j+1}$, $j = 1, \dots, h-1$. We indicate a generic element $(A_1, \dots, A_h) \in \mathcal{Q}$ with the abbreviated form $(A_j)_1^h$, which means $(A_j)_{j=1}^h$. Regarding its algebraic structure, the set \mathcal{Q} is a lattice where $\sup\{(A_j)_1^h, (B_j)_1^h\} = (A_j \cup B_j)_1^h$ and $\inf\{(A_j)_1^h, (B_j)_1^h\} = (A_j \cap B_j)_1^h$, for all $(A_j)_1^h, (B_j)_1^h \in \mathcal{Q}$.

Regarding the significance of \mathcal{Q} in this work, let us consider a possible alternative of choice \mathbf{x} and suppose that on each criterion $i \in N$, \mathbf{x} is evaluated by means of an h -interval. Thus, such an alternative \mathbf{x} can be identified with a score vector

$$\mathbf{x} = ([x_{11}, \dots, x_{1h}], \dots, [x_{n1}, \dots, x_{nh}]) \in \mathcal{I}_h^n.$$

Now consider a fixed evaluation level $t \in \mathbb{R}$ (e.g. t could represent some satisfaction level). The set $\{i \in N \mid x_{i,j} \geq t\}$ (briefly indicated with $\{x_{i,j} \geq t\}$) for all $j = 1, \dots, h$ aggregates the criteria whose j th evaluation of \mathbf{x} is at least t and, obviously, the vector $(\{x_{i,1} \geq t\}, \dots, \{x_{i,h} \geq t\}) \in \mathcal{Q}$.

We aim to define a tool allowing for the assignment of a "weight" to such elements of \mathcal{Q} .

Definition 1 An h -interval-capacity on \mathcal{Q} is a function $\mu_h : \mathcal{Q} \rightarrow [0, 1]$ such that

- $\mu_r(\emptyset, \dots, \emptyset) = 0$, and $\mu_h(N, \dots, N) = 1$; and
- for all $(A_j)_1^h, (B_j)_1^h \in \mathcal{Q}$ such that $A_j \subseteq B_j$ for all $j = 1, \dots, h$, $\mu_h[(A_j)_1^h] \leq \mu_h[(B_j)_1^h]$.

Definition 2 An $h-k$ -interval capacity is a vector $(\mu_{h_1}, \dots, \mu_{h_k}) = (\mu_{h_r})_{r=1}^k$ such that

- for every $r = 1, \dots, k$, $\mu_{h_r} : \mathcal{Q} \rightarrow [0, 1]$ is an h -interval capacity; and
- for all $(A_j)_1^h \in \mathcal{Q}$ and for all $r = 1, \dots, k-1$, $\mu_{h_r}[(A_j)_1^h] \leq \mu_{h_{r+1}}[(A_j)_1^h]$.

Definition 3 An h -interval-capacity μ_h is an additive h -interval-capacity on \mathcal{Q} if for all $(A_j)_1^h \in \mathcal{Q}$, for any $j = 1, \dots, h$, for any $B \subseteq N$ such that $A_h \cap B = \emptyset$,

$$\mu_h(A_1, \dots, A_{k-1}, A_k \cup B, \dots, A_h \cup B) = \mu_h(A_1, \dots, A_h) + \mu_h(\emptyset, \emptyset, \dots, \overbrace{B, B, \dots, B}^{h-k+1}).$$

An $h-k$ -interval capacity $(\mu_{h_r})_{r=1}^k$ is additive if μ_{h_r} is additive for all $r = 1, \dots, k$.

Let us provide a simple example of an additive 2-interval capacity. Let us consider $N = \{1, 2\}$ and suppose that $h = 2$, i.e. on each of the two

criteria an alternative is evaluated by means of an interval. In this case

$$\mathcal{Q} = \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1\}), (\{1\}, \{1, 2\}), (\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\},$$

and we can set, e.g.,

$$\begin{cases} \mu_2(\emptyset, \emptyset) = 0 \\ \mu_2(\emptyset, \{1\}) = 0.2 \\ \mu_2(\emptyset, \{2\}) = 0.2 \\ \mu_2(\{1\}, \{1\}) = 0.4 \\ \mu_2(N, N) = 1. \end{cases}$$

The hypothesis that μ_2 is additive constrains the other values of μ_2 , indeed

$$\begin{cases} \mu_2(\{2\}, \{2\}) = \mu_2(N, N) - \mu_2(\{1\}, \{1\}) = 0.6 \\ \mu_2(\emptyset, N) = \mu_2(\emptyset, \{1\}) + \mu_2(\emptyset, \{2\}) = 0.4 \\ \mu_2(\{1\}, N) = \mu_2(\emptyset, \{2\}) + \mu_2(\{1\}, \{1\}) = 0.6 \\ \mu_2(\{2\}, N) = \mu_2(\emptyset, \{1\}) + \mu_2(\{2\}, \{2\}) = 0.8. \end{cases}$$

Definition 4 *The h -Choquet integral of*

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}])$$

with respect to the h -interval capacity μ_h is

$$Ch_h(\mathbf{x}, \mu_h) = \min_{i \in N} x_{i,1} + \int_{\min_{i \in N} x_{i,1}}^{\max_{i \in N} x_{i,h}} \mu_{h_r}(\{x_{i,1} \geq t\}, \dots, \{x_{i,h} \geq t\}) dt. \quad (4)$$

The $h-k$ -Choquet integral of \mathbf{x} with respect to the $h-k$ -interval capacity $(\mu_{h_r})_{r=1}^k$ is given by

$$Ch_{h-k}(\mathbf{x}, (\mu_{h_r})_{r=1}^k) = (Ch_h(\mathbf{x}, \mu_{h_r}))_{r=1}^k \quad (5)$$

Note that the $2-1$ -Choquet integral is the *robust Choquet integral* presented in [8]. Now we give some additional information about the h -Choquet integral. Let us consider $\mathbf{x} \in \mathcal{I}_h^n$ and a fixed evaluation level $t \in \mathbb{R}$. We define

$$A_j(\mathbf{x}, t) = \{i \in N \mid x_{i,j} \geq t\} \quad \text{for all } j = 1, \dots, h.$$

Thus, $A_j(\mathbf{x}, t)$ aggregates the criteria whose j th evaluation of \mathbf{x} is at least t , and $A_j(\mathbf{x}, t) \subseteq A_{j+1}(\mathbf{x}, t)$, $j = 1, \dots, h-1$ and then

$$A(\mathbf{x}, t) := ((A_1(\mathbf{x}, t), \dots, A_h(\mathbf{x}, t))) \in \mathcal{Q}$$

for all $t \in \mathbb{R}$ and for all $\mathbf{x} \in \mathcal{I}_h^n$. An alternative formulation of the h -Choquet integral (4) implies some additional notations. We identify every vector $\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}]) \in \mathcal{I}_h^n$ with the vector $\mathbf{x}^* = (x_1, \dots, x_{nh}) \in \mathbb{R}^{nh}$ defined by setting for all $i = 1, \dots, nh$

$$x_i = \begin{cases} x_{i,1} & \text{if } i \leq n \\ x_{i,2} & \text{if } n < i \leq 2n \\ \vdots & \\ x_{i,h} & \text{if } n(h-1) < i \leq nh. \end{cases} \quad (6)$$

This corresponds to identify $\mathbf{x} \in \mathcal{I}_h^n$ with

$$\mathbf{x}^* = (x_{1,1}, \dots, x_{n,1}, \dots, x_{1,h}, \dots, x_{n,h}) \in \mathbb{R}^{nh}.$$

If $(\cdot) : \{1, \dots, nh\} \rightarrow \{1, \dots, nh\}$ is a permutation of indices such that $x_{(1)} \leq \dots \leq x_{(nh)}$, then two alternative formulations of the h -Choquet integral (4) computed with respect to the h -interval capacity μ_h are:

$$Ch_h(\mathbf{x}, \mu_h) = \sum_{i=1}^{nh} (x_{(i)} - x_{(i-1)}) \mu_h(A(\mathbf{x}, x_{(i)})) = \sum_{i=1}^{nh} x_{(i)} [\mu_h(A(\mathbf{x}, x_{(i)})) - \mu_h(A(\mathbf{x}, x_{(i+1)}))].$$

3.1. Interpretation and characterization

The indicator function of a set $A \subseteq N$ is the function $1_A : N \rightarrow \{0, 1\}$ which takes the value of 1 on A and 0 elsewhere. Such a function can be identified with the vector $\mathbf{1}_A \in \mathbb{R}^n$ whose i th component equals 1 if $i \in A$ and equals 0 if $i \notin A$. For all $(A_j)_1^h \in \mathcal{Q}$ the generalized indicator function $1_{(A_j)_1^h} : N \rightarrow \mathcal{I}_h$ is defined by

$$1_{(A_j)_1^h}(i) = \begin{cases} [\overbrace{0, \dots, 0}^{t-1}, \overbrace{1, \dots, 1}^{h-t+1}] & i \in A_t \setminus A_{t-1}, \\ & t = 1, \dots, h \\ [0, \dots, 0] & i \in N \setminus A_h \end{cases}$$

with $A_0 = \emptyset$. The function $1_{(A_j)_1^h}$ can be identified with the correspondent vectors of \mathcal{I}_h^n , $\mathbf{1}_{(A_j)_1^h}$, whose i th component equals $[0, \dots, 0, 1, \dots, 1]$ with $t-1$ zeros and $h-t+1$ ones if $i \in A_t \setminus A_{t-1}$ for some $t = 1, \dots, h$ and equals $[0, \dots, 0]$ if $i \in N \setminus A_h$. It follows by the definition of the h -Choquet integral (4) that for any h -interval capacity μ_h , $Ch_h(\mathbf{1}_{(A_j)_1^h}, \mu_h) = \mu_h[\mathbf{1}_{(A_j)_1^h}]$. This relation offers an appropriate definition of the weights $\mu_h[(A_j)_1^h]$.

Definition 5 *Given $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{I}_h^n$ with*

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}]),$$

$$\mathbf{y} = ([y_{1,1}, \dots, y_{1,h}], \dots, [y_{n,1}, \dots, y_{n,h}])$$

we define $\alpha\mathbf{x} + \beta\mathbf{y}$ as the vector of \mathcal{I}_h^n whose i th component is $[\alpha x_{i,1} + \beta y_{i,1}, \dots, \alpha x_{i,h} + \beta y_{i,h}]$, $i = 1, \dots, n$.

Definition 6 *The two vectors of \mathcal{I}_h^n*

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}]),$$

$$\mathbf{y} = ([y_{1,1}, \dots, y_{1,h}], \dots, [y_{n,1}, \dots, y_{n,h}])$$

are comonotone if the two vectors of \mathbb{R}^{nh} (defined according to 6)

$$\mathbf{x}^* = (x_{1,1}, \dots, x_{1,h}, \dots, x_{n,1}, \dots, x_{n,h}),$$

$$\mathbf{y}^* = (y_{1,1}, \dots, y_{1,h}, \dots, y_{n,1}, \dots, y_{n,h})$$

are comonotone.

An $h-k$ -aggregation function G_{h-k} is comonotone additive if it is additive for comonotone vectors, i.e. $G_{h-k}(\mathbf{x}+\mathbf{y}) = G_{h-k}(\mathbf{x})+G_{h-k}(\mathbf{y})$ whenever \mathbf{x} and \mathbf{y} are comonotone.

Theorem 2 *The $h-k$ -Choquet integral is the only $h-k$ -aggregation function which is comonotone additive and idempotent.*

Theorem 3 *$Ch_{h-k}(\cdot, (\mu_{h_1}, \dots, \mu_{h_k}))$ is the $h-k$ -weighted average if and only if the $h-k$ -interval capacity $(\mu_{h_1}, \dots, \mu_{h_k})$ is additive.*

4. Other non-additive $h-k$ -aggregation functions

In [8] the robust Shilkret and Sugeno integrals have been presented. These are 2-1-aggregation functions which can be generalized to the case of $h-k$ -aggregation functions.

Definition 7 *The h -Shilkret integral of*

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}])$$

with respect to the h -interval capacity μ_h is

$$Sh_h(\mathbf{x}, \mu_h) = \bigvee_{(A_j)_1^h \in \mathcal{Q}} \left\{ \bigwedge_{(A_j)_1^h} \mathbf{x} \cdot \mu_h [(A_j)_1^h] \right\} \quad (7)$$

where

$$\bigwedge_{(A_j)_1^h} \mathbf{x} = \bigwedge \left\{ \bigwedge_{i \in A_1} x_{i,1}, \dots, \bigwedge_{i \in A_h} x_{i,h} \right\}.$$

The $h-k$ -Shilkret integral of \mathbf{x} with respect to the $h-k$ -interval capacity $(\mu_{h_r})_{r=1}^k$ is given by

$$Sh_{h-k}(\mathbf{x}, (\mu_{h_r})_{r=1}^k) = (Sh_h(\mathbf{x}, \mu_{h_r}))_{r=1}^k \quad (8)$$

Definition 8 *The h -Sugeno integral of*

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}])$$

with respect to the h -interval capacity μ_h is

$$Su_h(\mathbf{x}, \mu_h) = \bigvee_{(A_j)_1^h \in \mathcal{Q}} \left\{ \bigwedge_{(A_j)_1^h} \mathbf{x}, \mu_h [(A_j)_1^h] \right\} \quad (9)$$

where

$$\bigwedge_{(A_j)_1^h} \mathbf{x} = \bigwedge \left\{ \bigwedge_{i \in A_1} x_{i,1}, \dots, \bigwedge_{i \in A_h} x_{i,h} \right\}.$$

The $h-k$ -Sugeno integral of \mathbf{x} with respect to the $h-k$ -interval capacity $(\mu_{h_r})_{r=1}^k$ is given by

$$Su_{h-k}(\mathbf{x}, (\mu_{h_r})_{r=1}^k) = (Su_h(\mathbf{x}, \mu_{h_r}))_{r=1}^k \quad (10)$$

In [8] several non-additive 2-1-aggregation functions have been presented, i.e. the robust Choquet integral with respect to a bipolar interval-capacity, the robust Choquet integral with respect to an interval capacity level dependent, the robust concave integral and the robust universal integral. All these integrals admit a natural generalization to the case of $h-k$ -aggregation functions presented here.

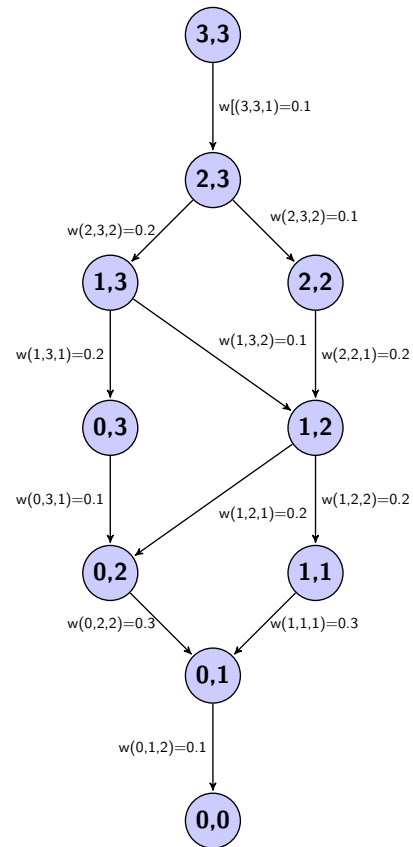


Figure 1: The lattice $Q_{\#}$, 2-intervals and 3 criteria

5. h -OWA operators

An h -interval capacity $\mu_h : \mathcal{Q} \rightarrow [0,1]$ only depends on the cardinality of the sets in its arguments if for all $(A_j)_1^h, (B_j)_1^h \in \mathcal{Q}$, such that $|A_j| = |B_j|, j = 1, \dots, h$ it holds that $\mu_h [(A_j)_1^h] = \mu_h [(B_j)_1^h]$. Let us define the following sets. The set of nodes

$$\mathcal{Q}_{\#} = \{(r_1, \dots, r_h) \in \{0, 1, \dots, n\}^h \mid r_1 \leq \dots \leq r_h\}$$

and the set of edges

$$\mathcal{A} = \{(\mathbf{r}, \mathbf{r}') \in \mathcal{Q}_{\#}^2 \mid r_t \geq r'_t, t = 1, \dots, h \text{ and}$$

$$\sum_{t=1}^h (r_t - r'_t) = 1\}. \quad (11)$$

Obviously the set $\mathcal{Q}_{\#}$ inherits the structure of lattice from the set \mathcal{Q} , see, e.g., the tree-diagram of figure 1 where the nodes represent the lattice $\mathcal{Q}_{\#}$ corresponding to the situation of 2-intervals and 3 criteria. We identify any h -interval capacity $\mu_h : \mathcal{Q} \rightarrow [0,1]$ depending only on the cardinality of the sets in its arguments with the corresponding function $\mu_h : \mathcal{Q}_{\#} \rightarrow [0,1]$ defined by $\mu_h(r_1, \dots, r_h) = \mu_h(A_1, \dots, A_h)$ for all $(r_1, \dots, r_h) \in \mathcal{Q}_{\#}$ and $(A_1, \dots, A_h) \in \mathcal{Q}$ such that $|A_i| = r_i, i = 1, \dots, h$.

Definition 9 *The class of h -OWA operators is the*

class of h -Choquet integrals computed with respect to the h -interval capacities $\mu_h : \mathcal{Q}_\# \rightarrow [0, 1]$.

Definition 10 The class of $h-k$ -OWA operators is the class of $h-k$ -Choquet integrals computed with respect to the $h-k$ -interval capacities $(\mu_{h_r})_{r=1}^h$ with $\mu_{h_r} : \mathcal{Q}_\# \rightarrow [0, 1]$ $r = 1, \dots, k$.

We define an nh -path in $\mathcal{Q}_\#$ as a sequence of nh consecutive edges

$$P_{nh} = ((\mathbf{r}^1, \mathbf{r}^2), (\mathbf{r}^2, \mathbf{r}^3), \dots, (\mathbf{r}^{nh-1}, \mathbf{r}^{nh})) \in \mathcal{A}^{nh}$$

For example in figure 1 a 6-path is

$$((3, 3), (2, 3)), [(2, 3), (1, 3)], [(1, 3), (1, 2)], \\ [(1, 2), (1, 1)], [(1, 1), (0, 1)], [(0, 1), (0, 0)].$$

Note that in any nh -path we have $nh + 1$ nodes and nh edges.

Definition 11 An OWA-weighting function is a function $w : \mathcal{A} \rightarrow [0, 1]$ such that for any nh -path,

$$P_{nh} = ((\mathbf{r}^1, \mathbf{r}^2), (\mathbf{r}^2, \mathbf{r}^3), \dots, (\mathbf{r}^{nh-1}, \mathbf{r}^{nh})) \in (\mathcal{A}')^{nh}$$

it holds that $\sum_{i=1}^{nh-1} w((\mathbf{r}^i, \mathbf{r}^{i+1})) = 1$.

In words, an OWA-weighting function is a function which assigns a weight in $[0, 1]$ to each edge in such a way that the sum of the weights along each path is 1. Now we show that to define an h -OWA operator through the capacity $\mu_h : \mathcal{Q}_\# \rightarrow [0, 1]$ is equivalent to define an OWA-weighting function and vice-versa. This will be initially cleared with a tree-diagram, where the nodes are the elements of $\mathcal{Q}_\#$, while on the edges we represents the weights assigned by the OWA-function $w : \mathcal{A} \rightarrow [0, 1]$. In figure 1 we have plotted the lattice $\mathcal{Q}_\#$ corresponding to the situation of 2-intervals and 3 criteria. The elements of $\mathcal{Q}_\#$ are represented by the nodes, while the values of the OWA-function $w : \mathcal{A} \rightarrow [0, 1]$ are represented on the edges. Note that $w(i, j, 1)$ stands for $w[(i, j), (i-1, j)]$ while $w(i, j, 2)$ stands for $w[(i, j), (i, j-1)]$. The capacity $\mu : \mathcal{Q}_\# \rightarrow [0, 1]$ is elicited by computing on each node the difference between 1 and the sum of all the values on the previous nodes along any nh -path, and then, w.r.t. figure 1

$$\left\{ \begin{array}{l} \mu(3, 3) = 1 \\ \mu(2, 3) = 1 - 0.1 = 0.9 \\ \mu(1, 3) = 1 - (0.1 + 0.2) = 0.7 \\ \mu(2, 2) = 1 - (0.1 + 0.1) = 0.8 \\ \mu(0, 3) = 1 - (0.1 + 0.2 + 0.2) = 0.5 \\ \mu(1, 2) = 1 - (0.1 + 0.1 + 0.2) = 0.5 \\ \mu(0, 2) = 1 - (0.1 + 0.1 + 0.2 + 0.2) = 0.4 \\ \mu(1, 1) = 1 - (0.1 + 0.1 + 0.2 + 0.2) = 0.4 \\ \mu(0, 1) = 1 - (0.1 + 0.1 + 0.2 + 0.2 + 0.3) = 0.1 \\ \mu(0, 0) = 0 \end{array} \right.$$

Conversely, from the values of the capacity on the nodes we can elicit the values of the weights on the edges (see figure 1) by means of

$$w(\mathbf{r}^i, \mathbf{r}^{i+1}) = \mu_h(\mathbf{r}^i) - \mu_h(\mathbf{r}^{i+1}). \quad (12)$$

Finally, we wish to note that the h -OWA could be defined also in the following manner. For a given $\mathbf{x} \in \mathcal{I}_h^n$ let us consider the permutation (\cdot) of values $x_{i,j}, i = 1, \dots, n, j = 1, \dots, h$, such that $x_{(1)} \leq x_{(2)} \leq \dots x_{(nh)}$. In case $x_{(p)} < x_{(p+1)}$ for all $p = 1, \dots, nh$, the h -OWA of \mathbf{x} with respect to OWA weights w is given by

$$OWA_w(\mathbf{x}) = \sum_{p=1}^{nh} x_{(p)} w(|\{\mathbf{x} \geq x_{(p)}\}|, |\{\mathbf{x} \geq x_{(p+1)}\}|)$$

with

$$|\{\mathbf{x} \geq t\}| = (|A_1(\mathbf{x}, t)|, \dots, |A_h(\mathbf{x}, t)|)$$

and $x_{(nh+1)} \in \mathbb{R}$ is some value such that $x_{(nh+1)} \geq x_{(nh)}$.

6. $h-k$ -order statistics

Another noticeable case of $h-k$ -aggregation function is given by the $h-k$ -order statistics. The lattice $\mathcal{Q}_\#$ is partial ordered with respect to the dominance relation $\succ_\#$ on it defined as follows: for all $(r_1, \dots, r_h), (r'_1, \dots, r'_h) \in \mathcal{Q}_\#$

$$(r_1, \dots, r_h) \succ_\# (r'_1, \dots, r'_h) \text{ iff } r_1 \geq r'_1, \dots, r_h \geq r'_h.$$

Definition 12 For any $\mathbf{r} = (r_1, \dots, r_h) \in \mathcal{Q}_\#$, the h -order statistic $OS_{\mathbf{r}}$ of

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}]) \in \mathcal{I}_h^n$$

associated to \mathbf{r} is given by

$$OS_{\mathbf{r}}(\mathbf{x}) = \max\{t \in \mathbb{R} : |\{\mathbf{x} \geq t\}| \succ_\# \mathbf{r}\}$$

where

$$|\{\mathbf{x} \geq t\}| = (|A_1(\mathbf{x}, t)|, \dots, |A_h(\mathbf{x}, t)|).$$

Definition 13 For any k -uple of profiles

$$(\mathbf{r}^{(l)})_1^k = (\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(k)})$$

such that $\mathbf{r}^{(l)} = (r_1^{(l)}, \dots, r_h^{(l)}) \in \mathcal{Q}_\#, l = 1, \dots, k$ and

$$(r_1^l, \dots, r_h^l) \succ_\# (r_1^{l+1}, \dots, r_h^{l+1}), l = 1, \dots, k-1,$$

the $h-k$ -order statistic $OS_{(\mathbf{r}^{(l)})_1^k}$ of

$$\mathbf{x} = ([x_{1,1}, \dots, x_{1,h}], \dots, [x_{n,1}, \dots, x_{n,h}]) \in \mathcal{I}_h^n$$

associated to $(\mathbf{r}^{(l)})_1^k$ is given by

$$OS_{(\mathbf{r}^{(l)})_1^k}(\mathbf{x}) = (OS_{\mathbf{r}^{(1)}}(\mathbf{x}), \dots, OS_{\mathbf{r}^{(k)}}(\mathbf{x})).$$

	mathematics	literature	language
S_1	6	[5, 7]	[7, 8]
S_2	7	[6, 7]	9
S_3	[6, 8]	7	7

Table 1: Students evaluation

The h -order statistics can be characterized in terms of OWA and in terms of h -Choquet integral.

Theorem 4 For any $\mathbf{r} = (r_1, \dots, r_h) \in \mathcal{Q}_\#$, the h -order statistic $OS_{\mathbf{r}}$ is an h -OWA such that for any edge

$$((r_1^1, \dots, r_h^1), (r_1^2, \dots, r_h^2))$$

in the graph $\mathcal{G}_{\mathcal{Q}_\#}$ we have

$$w((r_1^1, \dots, r_h^1), (r_1^2, \dots, r_h^2)) = 1$$

if $(r_1^2, \dots, r_h^2) = (r_1, \dots, r_h) = \mathbf{r}$ and

$$w((r_1^1, \dots, r_h^1), (r_1^2, \dots, r_h^2)) = 0$$

otherwise.

Theorem 5 For any $\mathbf{r} = (r_1, \dots, r_h) \in \mathcal{Q}_\#$, the h -order statistic $OS_{\mathbf{r}}$ is an h -Choquet integral with respect to a h -capacity μ_h such that

$$\mu(A_1, \dots, A_h) = 1 \text{ if } (|A_1|, \dots, |A_h|) \succ_{\#} \mathbf{r}$$

and

$$\mu(A_1, \dots, A_h) = 1 \text{ otherwise.}$$

7. A motivating example

Let us provide an example where 2-interval numbers need to be aggregated into a triangular number. The director of a university decides on students who are applying for graduate studies in management. Since some prerequisites from school are required, three students, S_1, S_2 and S_3 , are indeed evaluated according to mathematics (Mat), literature (Lit) and language (Lang) skills. All the marks with respect to the scores are given on the scale from 0 to 10. The director receives the candidates evaluations serving as a basis for the selection. He notes that some judgments are expressed as intervals (corresponding to some evaluators doubts, see Table 1).

At the university the freshmen are initially divided into three groups, depending on the starting level. The assignment of a student to a group is not just decided on the basis of his average evaluation, but more properly, depends on the potentiality of the student. This means that the director prefers that every student is represented by a triangular number (E_p, E_a, E_o) , where E_p corresponds to a pessimistic evaluation, E_a corresponds to an

	Mathematics	Literature	Language
a_{ij1}	0.3, 0.1	0.2, 0.2	0.1, 0.1
a_{ij2}	0.25, 0.15	0.15, 0.25	0.05, 0.15
a_{ij3}	0.2, 0.2	0.1, 0.3	0.05, 0.15

Table 2: Weights for the $h - k$ -weighted average

	$W_{a,1}(\mathbf{x})$	$W_{a,2}(\mathbf{x})$	$W_{a,3}(\mathbf{x})$
S_1	6.3	6.45	6.55
S_2	7.2	7.25	7.3
S_3	6.8	6.9	7

Table 3: $h - k$ -weighted average of students' notes

average evaluation and E_o corresponds to an optimistic evaluation. On the basis of this triple information the director will decide, for each student, the pertinent group. This is a realistic example where 2-interval numbers need to be aggregated into a triangular number.

Let us aggregate the notes of students in the three subjects using different $h - k$ -aggregation functions presented in this paper.

7.1. Using the $h - k$ -weighted average

Let us first aggregate the notes of students in the three subjects using the $h - k$ -weighted average according to the weights in Table 2. The results are in Table 3.

7.2. Using the 2-OWA

We can after compute the 2-OWA of students' notes taking into consideration the weights in Figure 1 and obtaining the evaluations in table 4. Figure 2 shows the path corresponding to the evaluations of student S_1 on lattice $\mathcal{Q}_\#$.

7.3. Using a 2 - 3-order statistics

Finally we considered a 2 - 3-order statistics $OS_{(2,3)(1,2)(1,1)}(\mathbf{x})$ obtaining the results shown in Table 5.

8. Conclusions

In many decision-making problems, fuzzy numbers represent the evaluating values of alternatives. Thus methods to treat with these type of information have received increasing attention in literature, especially in recent years.

	2-OWA
S_1	6.6
S_2	7.7
S_3	7

Table 4: 2-OWA of students' notes

	$OS_{(1,1)}(\mathbf{x})$	$OS_{(1,2)}(\mathbf{x})$	$OS_{(2,3)}(\mathbf{x})$
S_1	6	7	7
S_2	7	7	9
S_3	7	7	7

Table 5: $h - k$ -order statistics of students' notes

Several researchers have proposed methods for ranking fuzzy numbers, see, e.g. [4] and the references therein.

Another relevant example is the ordered weighted averaging (OWA) operator introduced in [12], which has been studied in situations involving imprecise evaluations expressed by fuzzy numbers [9, 1, 2, 3, 5, 12].

Also in the context of multiple attribute group decision making problems it is assumed that the attribute values take the form of fuzzy numbers, see [11] and the references within.

However in the majority of cited papers it is faced the problem of ranking fuzzy numbers, while in this paper we have proposed innovative methods to aggregate imprecise information expressed by fuzzy numbers.

Finally let us note as in some context, like that of group decision making, it is often assumed that the more suitable form to express valuations is that of a generalized interval-valued trapezoidal fuzzy numbers [11]. These are more general form of fuzzy numbers and we hope that the aggregation of such a type of complex information will be the topic for future researches.

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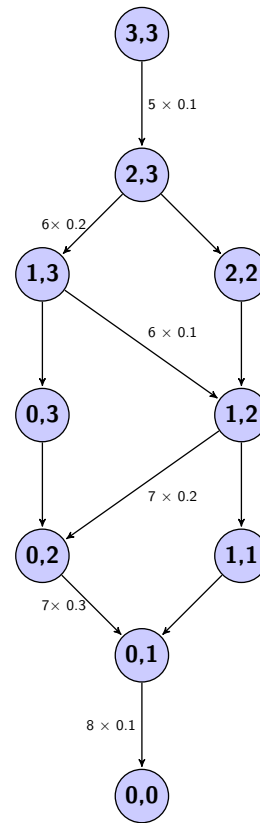


Figure 2: 2-OWA of student S_1 on the lattice $Q_{\#}$

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