

The Structure of the Type-Reduced Set of a Continuous Type-2 Fuzzy Set

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Abstract

This paper is concerned with the structure of the type-reduced set (TRS) of the continuous type-2 fuzzy set, in both its interval and generalised forms. In each case the TRS structure is approached by first investigating the discretised set. The TRS of a continuous interval type-2 fuzzy set is shown to be a continuous straight line, and that of a generalised type-2 fuzzy set, a continuous, convex curve.

Keywords: Type-2 Fuzzy Set, Type-Reduced Set, Type-Reduction, Defuzzification

1. Introduction

Type-2 fuzzy sets are an extension of type-1 fuzzy sets in which the sets' membership grades are themselves type-1 fuzzy sets. The concept dates back to Zadeh's seminal paper of 1975 [1]. They take two forms, the interval, for which all secondary membership grades are 1, and the generalised, where the secondary membership grade may take any value between 0 and 1. For the computationally simpler interval type-2 Fuzzy Inferencing Systems (FISs) [2] applications have been developed in areas such as control, simulation and optimisation [3, 4].

There are relatively few generalised type-2 fuzzy applications [5, 2, 6]. Strategies have been developed that reduce the computational complexity of all stages of the generalised type-2 FIS [7, 8, 9, 10]. We believe that the research presented in this paper will lead to further complexity reducing techniques. Hopefully in the future there will be an increasing number of generalised type-2 FIS applications.

Defuzzification is the crucial final stage of an FIS. For discretised type-1 fuzzy sets, defuzzification is a straightforward matter. There are several defuzzification techniques available, including the centroid, centre of maxima and mean of maxima [11]. In contrast, defuzzification of a discretised type-2 fuzzy set is a process consisting of two stages [12]:

1. Type-reduction, which converts a type-2 fuzzy set to a type-1 fuzzy set known as the Type-Reduced Set (TRS), and
2. defuzzification of the type-1 TRS.

Mathematically, the type-reduction algorithm depends upon the *Extension Principle* [1], which generalises operations defined for crisp numbers to

type-1 fuzzy sets. Type-2 defuzzification techniques therefore derive from and incorporate type-1 defuzzification methods.

This paper is structured as follows: Section 2 deals with type-reduction and the formation of the TRS. Section 3 concerns the structure of the TRS of an interval type-2 fuzzy set, and Section 4 the TRS structure of a generalised type-2 fuzzy set. Lastly, conclusions are presented in Section 5.

1.1. Preliminaries

1.1.1. Assumptions

1. All primary and secondary membership functions are convex.
2. The type-2 fuzzy set is contained within a unit cube and may be viewed as a surface represented by (x, u, z) co-ordinates.
3. The centroid method of defuzzification for type-1 fuzzy sets is used.
4. The minimum t-norm is employed.
5. The *grid method of discretisation* for generalised type-2 fuzzy sets [13, 8] is employed.

1.1.2. Definitions

Let X be a universe of discourse. A type-1 fuzzy set A on X is characterised by a membership function $\mu_A : X \rightarrow [0, 1]$ and can be expressed as follows [14]:

$$A = \{(x, \mu_A(x)) \mid \mu_A(x) \in [0, 1] \forall x \in X\}. \quad (1)$$

Note that the membership grades of A are crisp numbers. In the following we will use the notation $U = [0, 1]$.

Let $\tilde{P}(U)$ be the set of fuzzy sets in U . A type-2 fuzzy set \tilde{A} in X is a fuzzy set whose membership grades are themselves fuzzy [1, 15, 16]. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in U for all x , i.e. $\mu_{\tilde{A}} : X \rightarrow \tilde{P}(U)$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid \mu_{\tilde{A}}(x) \in \tilde{P}(U) \forall x \in X\}. \quad (2)$$

It follows that $\forall x \in X \exists J_x \subseteq U$ such that $\mu_{\tilde{A}}(x) : J_x \rightarrow U$. Applying (1), we obtain:

$$\mu_{\tilde{A}}(x) = \{(u, \mu_{\tilde{A}}(x)(u)) \mid \mu_{\tilde{A}}(x)(u) \in U \forall u \in J_x \subseteq U\}. \quad (3)$$

X is called the primary domain and J_x the primary membership of x while U is known as the secondary domain and $\mu_{\tilde{A}}(x)$ the secondary membership of x .

Putting (2) and (3) together we obtain

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u))) \mid \mu_{\tilde{A}}(x)(u) \in U, \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \quad (4)$$

This *vertical representation* of a type-2 fuzzy set is used to define the concept of an *embedded set* of a type-2 fuzzy set (Definition 6), which is fundamental to the definition of the *centroid* of a type-2 fuzzy set (Definition 7). Alternative notations may be found in [17].

Definition 1 (Interval Type-2 Fuzzy Set) An interval type-2 fuzzy set is a type-2 fuzzy set whose secondary membership grades are all 1.

In the interval case, Equation 4 reduces to:

$$\tilde{A} = \{(x, (u, 1)), \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \quad (5)$$

Definition 2 (Slice) A slice of a type-2 fuzzy set is a plane either

1. through the x -axis, parallel to the $u - z$ plane, or
2. through the u -axis, parallel to the $x - z$ plane.

Definition 3 (Vertical Slice [2]) A vertical slice of a type-2 fuzzy set is a plane through the x -axis, parallel to the $u - z$ plane.

Definition 4 (Degree of Discretisation) The degree of discretisation is the separation of the slices.

For a type-2 fuzzy set, both the primary and secondary domains are discretised, the former into *vertical slices*. The primary and secondary domains may have different degrees of discretisation. Furthermore the secondary domain's degree of discretisation may vary from one vertical slice to another.

Definition 5 (Scalar Cardinality [18]) The scalar cardinality of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A . Thus,

$$||A|| = \sum_{x \in X} \mu_A(x).$$

2. Type-reduction of the type-2 fuzzy set

In this section, type-reduction, the initial stage of type-2 defuzzification, is discussed.

2.1. The Wavy-Slice Representation Theorem

The concept of an *embedded type-2 fuzzy set* (embedded set) or *wavy-slice* [2] is crucial to type-reduction. An embedded set is a special kind of type-2 fuzzy set. It relates to the type-2 fuzzy set in which it is embedded in this way: For every primary domain

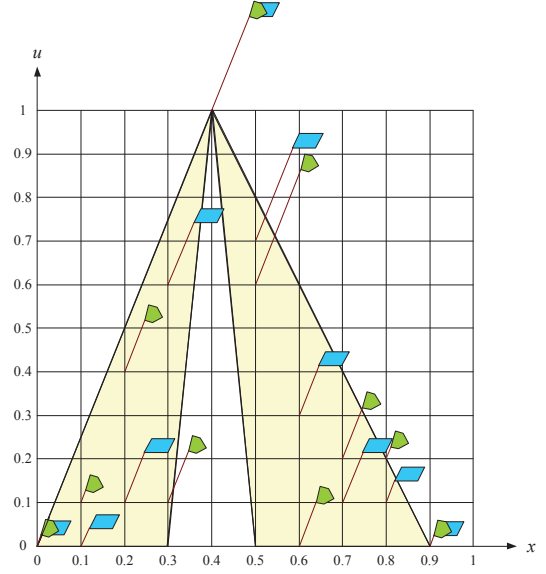


Figure 1: Two embedded sets, indicated by different flag styles. The flag height reflects the secondary membership grade. Degree of discretisation of primary and secondary domains is 0.1. The shaded region is the FOU.

value, x , there is a unique secondary domain value, u , plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)$.

Example 1 In Figure 1 we have identified two embedded sets of a type-2 fuzzy set with primary and secondary domain degree of discretisation of 0.1. The embedded set \tilde{P} is represented by pentagonal, pointed flags, and embedded set \tilde{Q} is symbolised by quadrilateral shaped flags. We can represent these embedded sets as sets of points, thus:

$$\tilde{P} = \{[0.1/0]/0 + [0.1/0.1]/0.1 + [0.5/0.4]/0.2 + [0.5/0.1]/0.3 + [1/1]/0.4 + [0.9/0.6]/0.5 + [0.4/0]/0.6 + [0.4/0.2]/0.7 + [0.2/0.2]/0.8 + [0.1/0]/0.9\}.$$

$$\tilde{Q} = \{[0.1/0]/0 + [0.2/0]/0.1 + [0.5/0.1]/0.2 + [0.5/0.6]/0.3 + [1/1]/0.4 + [0.8/0.7]/0.5 + [0.5/0.3]/0.6 + [0.5/0.1]/0.7 + [0.3/0.1]/0.8 + [0.1/0]/0.9\}.$$

Definition 6 (Embedded Set) Let \tilde{A} be a type-2 fuzzy set in X . For discrete universes of discourse X and U , an embedded type-2 set \tilde{A}_e of \tilde{A} is defined as the following type-2 fuzzy set

$$\tilde{A}_e = \{(x_i, (u_i, \mu_{\tilde{A}}(x_i)(u_i))) \mid \forall i \in \{1, \dots, N\} : \quad (6) \\ x_i \in X \quad u_i \in J_{x_i} \subseteq U\}.$$

\tilde{A}_e contains exactly one element from $J_{x_1}, J_{x_2}, \dots, J_{x_N}$, namely u_1, u_2, \dots, u_N , each with

its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_1)$, $\mu_{\tilde{A}}(x_2)(u_2)$, ..., $\mu_{\tilde{A}}(x_N)(u_N)$.

Mendel and John have shown that a type-2 fuzzy set can be represented as the union of its type-2 embedded sets [2, Page 121]. This powerful result is known as the type-2 fuzzy set *Representation Theorem* or *Wavy-Slice Representation Theorem*; in [2] it was derived without reference to the Extension Principle. Bringing a conceptual simplicity to the manipulation of type-2 fuzzy sets, it is applied to give simpler derivations of results previously obtained through the Extension Principle [2].

The Representation Theorem is formally stated thus [2, Page 121]:

Let \tilde{A}_e^j denote the j th type-2 embedded set for type-2 fuzzy set \tilde{A} , i.e.,

$$\tilde{A}_e^j \equiv \left\{ \left(u_i^j, \mu_{\tilde{A}}(x_i)(u_i^j) \right), i = 1, \dots, N \right\}$$

where $\{u_i^j, \dots, u_N^j\} \in J_{x_i}$. Then \tilde{A} can be represented as the union of its type-2 embedded sets, i.e.,

$$\tilde{A} = \sum_{j=1}^n \tilde{A}_e^j$$

where

$$n \equiv \prod_{i=1}^N M_i.$$

2.2. TRS of a generalised type-2 fuzzy set

The first stage of type-2 defuzzification is type-reduction, the creation of the TRS. Assuming that the primary domain X has been discretised, the TRS of a type-2 fuzzy set may be defined through the application of Zadeh's Extension Principle [1]. Alternatively the TRS may be defined via the Representation Theorem [2, Page 121].

Definition 7 The TRS associated with a type-2 fuzzy set \tilde{A} with primary domain X discretised into N points $X = \{x_1, x_2, \dots, x_N\}$, is

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^N x_i \cdot u_{k_i}}{\sum_{i=1}^N u_{k_i}}, \mu_{\tilde{A}}(x_1)(u_{k_1}) * \dots \right. \right. \\ \left. \left. \dots * \mu_{\tilde{A}}(x_N)(u_{k_N}) \right) \middle| \forall (u_{k_1}, u_{k_2}, \dots, u_{k_N}) \right. \\ \left. \in J_{x_1} \times J_{x_2} \times \dots \times J_{x_N} \subseteq U^N \right\}. \quad (7)$$

The type reduction stage requires the application of a t-norm (*) to the secondary membership grades. Algorithm 1 (adapted from Mendel [12]) is used to compute the TRS of a type-2 fuzzy set.

2.2.1. Exhaustive type-reduction

Mendel and John's Representation Theorem (Subsection 2.1) provides a precise, straightforward method for type-2 defuzzification. Though Definition 7 does not explicitly mention embedded sets, they appear implicitly in Equation 7. When this equation is presented in algorithmic form (Algorithm 1), explicit mention is made of embedded sets. As *every* embedded set is processed, this stratagem has become known as the *exhaustive method* [19]. Discretisation inevitably brings with it an element of approximation. However the exhaustive method does not introduce further inaccuracies subsequent to discretisation.

Exhaustive type-reduction processes every embedded set in turn. Each embedded set is defuzzified as a type-1 fuzzy set. The defuzzified value is paired with the minimum secondary membership grade of the embedded set. The set of ordered pairs constitutes the TRS.

Input: a discretised generalised type-2 fuzzy set

Output: a discrete type-1 fuzzy set (the TRS)

```

1 forall the embedded sets do
2   find the minimum secondary membership
   grade ( $z$ ) ;
3   calculate the primary domain value ( $x$ ) of
   the type-1 centroid of the type-2 embedded
   set ;
4   pair the secondary grade ( $z$ ) with the
   primary domain value ( $x$ ) to give set of
   ordered pairs ( $x, z$ ) {some values of  $x$  may
   correspond to more than one value of  $z$ } ;
5 end
6 forall the primary domain ( $x$ ) values do
7   select the maximum secondary grade {make
   each  $x$  correspond to a unique value} ;
8 end

```

Algorithm 1: Type-reduction of a discretised type-2 fuzzy set to a type-1 fuzzy set, adapted from Mendel [12].

Stage 3 of Algorithm 1 requires the calculation of the embedded set's centroid. Example 2 relates to the embedded sets introduced in Example 1.

Example 2 Embedded set \tilde{P} has minimum secondary grade $z_{\tilde{P}} = 0.1$ and primary domain value of its type-1 centroid $x_{\tilde{P}} = 0.4308$:

$$x_{\tilde{P}} = \frac{\sum_{i=1}^N x_i \cdot u_i}{\sum_{i=1}^N u_i} = \frac{1.12}{2.6} = 0.4308.$$

Similarly embedded set \tilde{Q} has minimum secondary grade $z_{\tilde{Q}} = 0.1$ and primary domain value of its type-1 centroid $x_{\tilde{Q}} = 0.4414$:

$$x_{\tilde{Q}} = \frac{\sum_{i=1}^N x_i \cdot u_i}{\sum_{i=1}^N u_i} = \frac{1.28}{2.9} = 0.4414.$$

2.3. TRS of an interval type-2 fuzzy set

For the TRS of an interval type-2 fuzzy set, Definition 7 reduces to:

Definition 8 (TRS of an Interval Type-2 Set)

The TRS associated with an interval type-2 fuzzy set \tilde{A} with primary domain X discretised into N points $X = \{x_1, x_2, \dots, x_N\}$, is

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^N x_i \cdot u_{k_i}}{\sum_{i=1}^N u_{k_i}}, 1 \right) \middle| \forall (u_{k_1}, u_{k_2}, \dots, u_{k_N}) \in J_{x_1} \times J_{x_2} \times \dots \times J_{x_N} \subseteq U^N \right\}. \quad (8)$$

3. Structure of the TRS of an interval type-2 fuzzy set

Any non-continuous type-1 fuzzy set A may be thought of as a crisp, finite, set of co-ordinate points, i.e. $A \equiv \{(x, \mu_A(x))\}$. The TRS of a discretised interval type-2 fuzzy set may be regarded as a *crisp* set of tuples $(x, 1)$ which lie on the horizontal line $u = 1$. In this section we investigate how the TRS tuples are positioned along the interval of the TRS.

Figure 2 shows an interval type-2 fuzzy set. The domain is discretised into M vertical slices with degree of discretisation d_x such that $M = \frac{1}{d_x} + 1$. The codomain degree of discretisation is d_u . E is an embedded set whose codomain value at the I^{th} vertical slice $x = x_I$ is U . E_I^1 is another embedded set, identical to E apart from the codomain value on vertical slice x_I , which is $U + d_u$. For embedded set E_I^2 the codomain value at x_I is $U + 2d_u$, and for embedded set E_I^n the codomain value at x_I is $U + nd_u$. Let X_E be the (centroid) defuzzified value of embedded set E . $X_{E_I^1}, X_{E_I^2}, \dots, X_{E_I^n}$ are similarly defined.

In Appendix A we derive a formula for the difference between two consecutive defuzzified values:

$$X_{E_I^n} - X_{E_I^{n-1}} = \frac{\|E\|d_u(x_I - X_E)}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)}. \quad (9)$$

But

$$\frac{\|E\|d_u(x_I - X_E)}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)} < \frac{\|E\|d_u(x_I - X_E)}{\|E\|^2} = \frac{d_u(x_I - X_E)}{\|E\|}. \quad (10)$$

We have shown that

$$X_{E_I^n} - X_{E_I^{n-1}} < \frac{d_u(x_I - X_E)}{\|E\|}. \quad (11)$$

In Inequation 11, as n increases, so does the number of points on the vertical slice x_I . Consequently the distance between these points, d_u , decreases, and $\frac{d_u(x_I - X_E)}{\|E\|}$ decreases with it. $\forall \varepsilon \in$

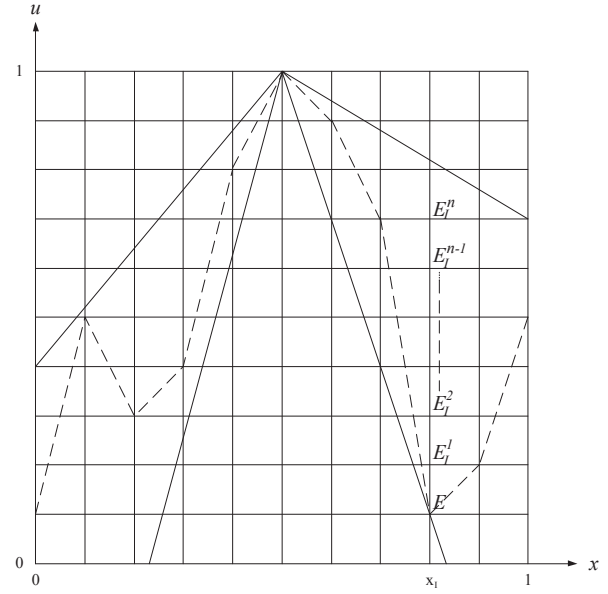


Figure 2: Embedded set E (dashed line), with related embedded sets.

$[0, 1], \exists N \in \mathbb{N}$ such that $\frac{d_u(x_I - X_E)}{\|E\|} < \varepsilon$. It follows that $d(X_{E_I^n}, X_{E_I^{n-1}}) < \varepsilon \forall n \in \{1, 2, \dots, N\}$, which shows that the TRS of an interval type-2 fuzzy set with a non-discretised codomain is a continuous line. Each vertical slice defines its own continuous line in the TRS; these lines overlap. As d_x , the domain degree of discretisation, decreases, the number of overlapping lines tends to infinity, but they all have least domain value ≥ 0 and greatest domain value ≤ 1 . This shows that the TRS of an interval type-2 fuzzy set with continuous domain and codomain is a continuous line.

Theorem 1 *The TRS of a continuous interval type-2 fuzzy set is an interval of the line $u = 1$ with least domain value ≥ 0 and greatest domain value ≤ 1 .*

In Subsection 4.1 below, Figures 3 to 5 show how the TRS from a generalised type-2 fuzzy set is built up point by point. The number of embedded sets of the originating type-2 set runs into many millions; the centroids of each embedded set, paired with the associated minimum secondary membership grade, have been plotted as dots for 50 embedded sets (Figure 3), 500 embedded sets (Figure 4), and 5000 embedded sets (Figure 5). Each horizontal line is analogous to the TRS of an interval set. As the number of TRS tuples increases, it can be clearly seen that the points merge into a continuous line¹.

¹In these figures, the lines appear continuous because the relatively large size of the dots eliminates the spaces between

4. TRS of a generalised type-2 fuzzy set

4.1. Stratification in the discretised TRS

In 2008 Greenfield and John reported on the stratified structure exhibited by the TRS [20]. This structure was observed during investigations into the sampling method of type-2 defuzzification [21, 8]. The membership function of the TRS of a discretised type-2 fuzzy set may be thought of as a set of tuples (Section 3). Figures 3 to 5 show typical TRSs derived from different sized samples of randomly generated embedded sets, originating from the same discretised type-2 fuzzy set. Each tuple is shown as a dot; the dots clearly align themselves into strata.

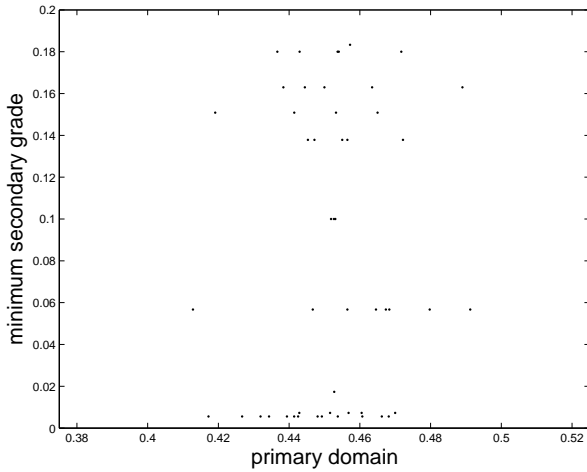


Figure 3: The TRS strata. A sample of 50 TRS tuples is shown.

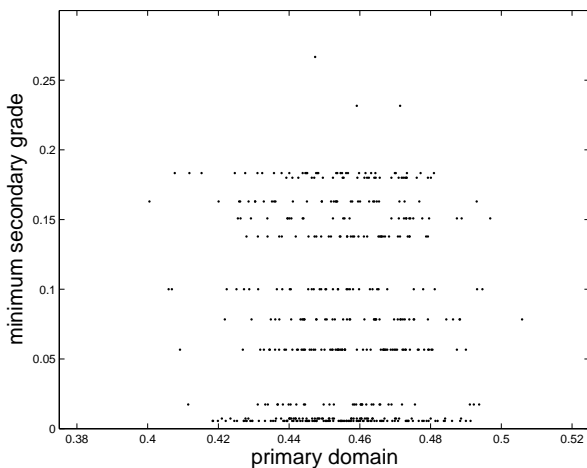


Figure 4: The TRS strata. A sample of 500 TRS tuples is shown.

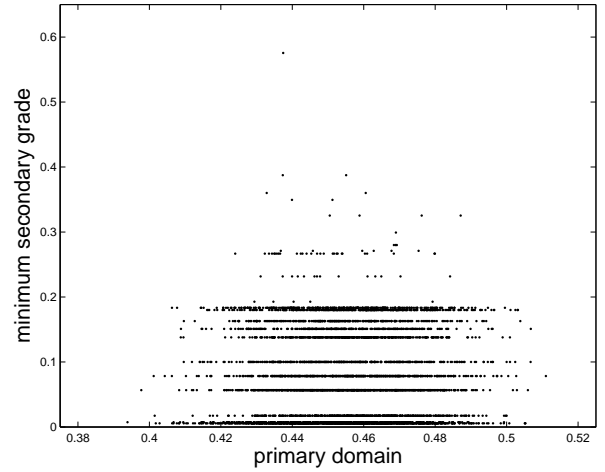


Figure 5: The TRS strata. A sample of 5000 TRS tuples is shown.

The explanation for the appearance of strata is that they derive directly from the originating type-2 fuzzy set, and are artifacts produced by the combination of the processes of discretisation and fuzzy inferencing. Since during type-reduction the minimum secondary grade of each embedded set is selected, unsurprisingly the same minimum values appear repeatedly, for different domain values.

Definition 9 (Stratum [20]) *Let T be the TRS of a discretised generalised type-2 fuzzy set. A stratum is a subset² of T for which every element has the same membership grade.*

$$S_\omega = \{(x, \mu_T(x)) \in T \mid \mu_T(x) = \omega\} \\ \text{for some } \omega \in [0, 1],$$

where S_ω is a stratum.

If the simplification stage (Lines 6–8) of Algorithm 1 is omitted, then Algorithm 2 results. We term the type-1 set resulting from this algorithm the *Unsimpilified Type-Reduced Set (UTRS)*.

We will now look at how the TRS strata relate to the originating type-2 fuzzy set.

Definition 10 (Min. Secondary Memb. Funct.)

A minimum secondary membership function of a type-2 fuzzy set is a secondary membership function whose maximum membership grade is the least of all the maximum membership grades of the vertical slices comprising the set.

Definition 11 (Truncated Type-2 Fuzzy Set)

A truncated type-2 fuzzy set is a type-2 fuzzy set for which all secondary membership grades greater than the maximum membership grade of the minimum secondary membership function have been reduced to the maximum membership grade of the minimum secondary membership function.

them. In the analysis presented above, the spaces between the tuples are eliminated as the codomain degree of discretisation tends to 0.

²In the commonly accepted ‘crisp’ sense of the word. Klir and Folger [18, Page 19] give a different definition of subset in the type-1 context.

Input: a discretised generalised type-2 fuzzy set

Output: a discrete type-1 fuzzy set (the TRS)

```

1 forall the embedded sets do
2   find the minimum secondary membership
   grade ( $z$ ) ;
3   calculate the primary domain value ( $x$ ) of
   the type-1 centroid of the type-2 embedded
   set ;
4   pair the secondary grade ( $z$ ) with the
   primary domain value ( $x$ ) to give set of
   ordered pairs ( $x, z$ ) {some values of  $x$  may
   correspond to more than one value of  $z$ } ;
5 end

```

Algorithm 2: Algorithm 1 omitting the simplification stage (Lines 6–8), so creating the UTRS.

Consider a minimum secondary membership function whose minimum grade is l and maximum grade is h . As $d_u \rightarrow 0$ its curve becomes continuous, taking every value between l and h . Each secondary membership grade gives rise to a stratum. Therefore, in a continuous generalised type-2 fuzzy set, there is a stratum at every secondary membership function. We have already shown (Section 3) that the TRS of non-discretised interval type-2 fuzzy set is a continuous horizontal line. Taking these two observations together, we conclude that the UTRS in the continuous case is a continuous planar surface.

On simplification (at which the UTRS is converted into the TRS), all the z -values apart from the highest are eliminated, so forming a continuous type-1 membership function.

In 2008 Liu [9, 22] proposed the α -planes representation. Via this technique a generalised type-2 fuzzy set is decomposed into a set of α -planes, which are horizontal slices akin to interval type-2 fuzzy sets. As all the secondary membership functions of the originating type-2 fuzzy set are convex, any given α -plane must fit within the contours of a lower α -plane. It follows that the stratum corresponding to a lower α -plane must occupy an interval that includes the interval associated with a higher α -plane. The stratum originating from a higher α -plane cannot overhang that of a lower α -plane; the TRS of a continuous type-2 fuzzy set is convex, rising to a maximum and then decreasing.

Figure 6 depicts a minimum secondary membership function of a type-2 fuzzy set, behind which can be seen the rest of the set. Were the originating type-2 fuzzy set to be truncated horizontally at the level of the maximum grade of the minimum secondary membership function (Figure 7), the set's TRS would be unchanged, since no strata exist at grades higher than this level. It is interesting to note in this regard that Greenfield and John's analysis

of the mechanism by which type-2 fuzzy sets model uncertainty [23] postulates that the greater the volume under the surface of the type-2 fuzzy set, the higher the uncertainty it represents. On this view it is the higher secondary membership grades that affect the defuzzified value least, so it is unsurprising that they do not contribute strata to the TRS.

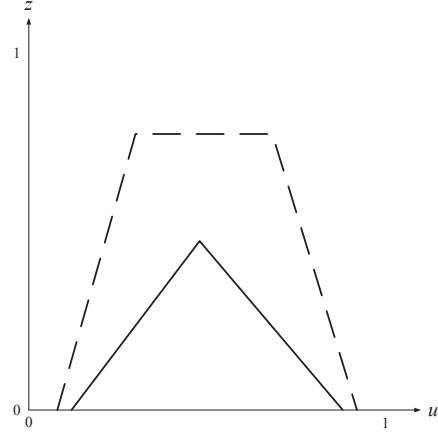


Figure 6: A generalised type-2 fuzzy set viewed from the $u - z$ plane. The minimum secondary membership function is shown in bold. The dashed line shows a non-minimum secondary membership function.

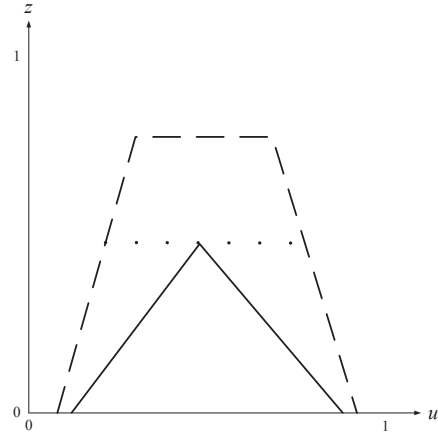


Figure 7: The generalised type-2 fuzzy set depicted in Figure 6. The dashed line shows a non-minimum secondary membership function; the dotted line indicates where it has been truncated to the height of the maximum grade of the minimum secondary membership function.

5. Conclusions

The structure of the type-reduced set of the continuous type-2 fuzzy set in its interval and generalised forms has been investigated by first looking into the structures of the discretised sets. The TRS of a continuous interval type-2 fuzzy set has been shown to

be a continuous straight line, specifically an interval of the line $u = 1$ with least domain value ≥ 0 and greatest domain value ≤ 1 .

The TRS of a continuous generalised type-2 fuzzy set has been shown to be a continuous, convex curve, rising to a maximum and then decreasing.

We believe these theoretical results to be interesting and significant.

Future Work It is anticipated that further research into the truncated type-2 fuzzy set will give rise to new, computationally simpler, techniques for type-2 fuzzy inferencing.

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Appendix A

$$\begin{aligned}
X_{E_I^n} &= \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i + \sum_{i=I+1}^M \mu_{i_E} x_i + (\mu_{I_E} + nd_u) x_I}{\sum_{i=1}^M \mu_{i_E} + nd_u} = \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i + \sum_{i=I+1}^M \mu_{i_E} x_i + \mu_{I_E} x_I + nd_u x_I}{\|E\| + nd_u}. \\
X_{E_I^{n-1}} &= \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i + \sum_{i=I+1}^M \mu_{i_E} x_i + (\mu_{I_E} + (n-1)d_u) x_I}{\sum_{i=1}^M \mu_{i_E} + (n-1)d_u} \\
&= \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i + \sum_{i=I+1}^M \mu_{i_E} x_i + \mu_{I_E} x_I + nd_u x_I - d_u x_I}{\|E\| + (n-1)d_u}.
\end{aligned}$$

Therefore

$$\begin{aligned}
X_{E_I^n} - X_{E_I^{n-1}} &= \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i + \sum_{i=I+1}^M \mu_{i_E} x_i + \mu_{I_E} x_I + nd_u x_I}{\|E\| + nd_u} \\
&\quad - \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i + \sum_{i=I+1}^M \mu_{i_E} x_i + \mu_{I_E} x_I + nd_u x_I - d_u x_I}{\|E\| + (n-1)d_u} \\
&= \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i (n-1)d_u + \sum_{i=I+1}^M \mu_{i_E} x_i (n-1)d_u + \mu_{I_E} x_I (n-1)d_u}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)} \\
&\quad + \frac{nd_u x_I (n-1)d_u - \sum_{i=1}^{I-1} \mu_{i_E} x_i nd_u - \sum_{i=I+1}^M \mu_{i_E} x_i nd_u}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)} \\
&\quad + \frac{-\mu_{I_E} x_I nd_u - x_I n^2 (d_u)^2 + d_u x_I \|E\| + x_I n (d_u)^2}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)}.
\end{aligned}$$

Since

$$\sum_{i=1}^M \mu_{i_E} x_i nd_u = \sum_{i=1}^{I-1} \mu_{i_E} x_i nd_u + \sum_{i=I+1}^M \mu_{i_E} x_i nd_u + \mu_{I_E} x_I nd_u,$$

the numerator may be simplified to obtain:

$$\begin{aligned}
X_{E_I^n} - X_{E_I^{n-1}} &= \frac{\sum_{i=1}^{I-1} \mu_{i_E} x_i (n-1)d_u + \sum_{i=I+1}^M \mu_{i_E} x_i (n-1)d_u + \mu_{I_E} x_I (n-1)d_u}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)} \\
&\quad + \frac{nd_u x_I (n-1)d_u - \sum_{i=1}^M \mu_{i_E} x_i nd_u - x_I n^2 (d_u)^2 + d_u x_I \|E\| + x_I n (d_u)^2}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)}.
\end{aligned}$$

Moreover, since

$$\sum_{i=1}^M \mu_{i_E} x_i (n-1)d_u = \sum_{i=1}^{I-1} \mu_{i_E} x_i (n-1)d_u + \sum_{i=I+1}^M \mu_{i_E} x_i (n-1)d_u + \mu_{I_E} x_I (n-1)d_u,$$

the numerator may be further simplified to give:

$$X_{E_I^n} - X_{E_I^{n-1}} = \frac{d_u x_I \|E\| - \sum_{i=1}^M \mu_{i_E} x_i d_u}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)}.$$

We know that

$$\sum_{i=1}^M \mu_{i_E} x_i = X_E \|E\|.$$

It follows that

$$X_{E_I^n} - X_{E_I^{n-1}} = \frac{d_u x_I \|E\| - X_E \|E\| d_u}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)} = \frac{\|E\| d_u (x_I - X_E)}{(\|E\| + nd_u)(\|E\| + (n-1)d_u)}.$$