

# Towards Smooth Monotonicity in Fuzzy Inference System based on Gradual Generalized Modus Ponens

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## Abstract

Monotonic relationship between input and output often is an inherent property of fuzzy systems. For example, for a cart-pole system, there is a gradual relationship between the pole angle and the cart velocity of the form: “The more the pole deviates from the upright position, the higher velocity the cart must gain”.

Thus, we propose an approach, that we call Gradual GMP, which is able to integrate the graduality, when this underlying hypothesis exists, into the generalized modus ponens.

In this paper, we proposed to study in a simple case (crisp observations) the monotonic response of what could be the adaptation to this case of GGMP-based FIS. We showed that for a single rule FIS, the GGMP induces strict monotonicity, to be compared with static response in the standard case (here Mamdani controller). Further experimental results, on single and double input FIS, show that the proposed method implies a smooth monotone behavior even when in presence of several linguistic values for each variable.

Finally, this work reveals that the only way to guarantee strict monotonicity is to have an inference that has a gradual behavior.

**Keywords:** Fuzzy Inference System, Fuzzy Control, Generalized Modus Ponens, Monotonicity

## 1. Introduction

In most of the applications of fuzzy control, monotonicity is an inherent property of the system output (control action) with respect to the input. For example, consider a simple cart-pole system [1] in which the pole angle and its angular velocity can be set as two state variables and the cart velocity as the system output. An appropriate controller for the cart-pole system must have the monotonicity property between the given pole angle (as well as the angular velocity) and the desired cart velocity: “The more the pole deviates from the upright position, the higher velocity the cart must gain.” In the same way, in case of controlling an automated guided ve-

hicle to avoid obstacles [2], there is a monotonic relationship between the distance (from the vehicle to the obstacle) and the turning degree of the vehicle: “The nearer the obstacle is, the more the vehicle should be turned.”

In recent years, several research works have focused on designing monotonic fuzzy inference systems (FIS). In [3] and [4], K. M. Tay et al. propose an approach to build a fuzzy inference system that preserves the monotonicity property. The authors introduce a fuzzy re-labeling technique to re-order the consequences of fuzzy rules in the database and a monotonicity index. This approach is able to overcome several restrictions; in particular it uses a similarity-based reasoning scheme to design monotonic multi-input FIS models. There are also some studies ([5], [6]) on the monotonicity of single input rule modules (SIRMs) connected fuzzy inference method where the output of the rule is generally simple, i.e., a precise value. The authors study the properties of the inference method and especially the conditions for a monotonicity of the inference results.

In the case of controllers, especially Mamdani-like models [7], Broekhoven et al. ([8], [9]) show that, ordered linguistic values for all input variables and for all output variables, plus a set of rules describing a monotone system are not enough to guarantee a monotone input-output behavior. They state that the choice of the mathematical operators used when calculating the model output and the properties of the membership functions are also of crucial importance. These constraints have been observed as well as in another work by M. Štěpnička et al. [10].

All these works focused on the post factum analysis of the inference system and on the interaction of the rules. In this paper, we propose to modify the underlying inference mechanism by using, what we called, the Gradual GMP that allows to integrate the graduality - in a sense of monotonicity - in a GMP framework. Based on the GGMP, described in next section, an inference system may be built, as described in section 3. Moreover, we present a set of experiments showing how, the introduction of the GGMP in the inference step, can lead to a smooth monotone behavior, compared to a partially

static classical fuzzy inference system (here Mamdani controller).

## 2. Gradual Generalized Modus Ponens

Generalized modus ponens (GMP) is a key inferring mechanism in inference systems. As defined in [11], for  $X$  and  $Y$ , two variables in the universes of discourse  $U$  and  $V$ ,  $A$  and  $A'$  two fuzzy subsets of  $U$ , and  $B$  a fuzzy subset of  $V$ . The most general form of GMP is given by:

$$\frac{\text{If } X \text{ is } A \text{ then } Y \text{ is } B}{X \text{ is } A'} \quad Y \text{ is } B'$$

where the membership function of  $B'$  is defined by, for each  $y \in V$ :

$$\mu_{B'}(y) = \sup_{x \in U} \mathcal{T}(\mu_{A'}(x), \mathcal{I}(\mu_A(x), \mu_B(y))) \quad (1)$$

with  $\mathcal{T}$  is a triangular norm, and  $\mathcal{I}$  is a fuzzy implication operator. Figure 1 shows some examples of inferences obtained using GMP.

To integrate the graduality into GMP, the basic idea of our approach is to separate the universe of the premise  $A$  (respectively the consequence  $B$ ) into three parts, named *Smaller*, *Greater* and *Indistinguishable*. By doing so, the model focuses its action in the corresponding parts of  $A$  and  $B$ . In other words, only the *Smaller* (respectively *Greater*, *Indistinguishable*) part of  $A$  is used to infer the *Smaller* (respectively *Greater*, *Indistinguishable*) part of  $B$ .

### Membership requirements

In this study, the membership functions are assumed to be convex, normalized and continuous, and their supports are required to be bounded. Moreover, the convexity notion is restricted to *strict convexity* as discussed in [10], i.e., the fuzzy sets have no partially constant membership functions. In this sense, fully continuous fuzzy numbers can be used with the proposed method. Examples of viable membership functions are shown in Figure 2.

### Universe partitioning

We define membership functions of three parts *Smaller*, *Greater* and *Indistinguishable* of fuzzy set  $A$ , using the *kernel* and *complement* of  $A$  (Figure 3).

Let us refer to the kernel of a fuzzy set  $A$  by  $[A_L, A_R]$ , i.e., the interval so that  $\forall x \in [A_L, A_R] \mu_A(x) = 1$ .

The membership function of the *Smaller* part of  $A$  is defined as follows:

$$\varphi_{\text{Smaller}_A}(x) = \begin{cases} \overline{\mu_A(x)} = 1 - \mu_A(x) & x < A_L \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

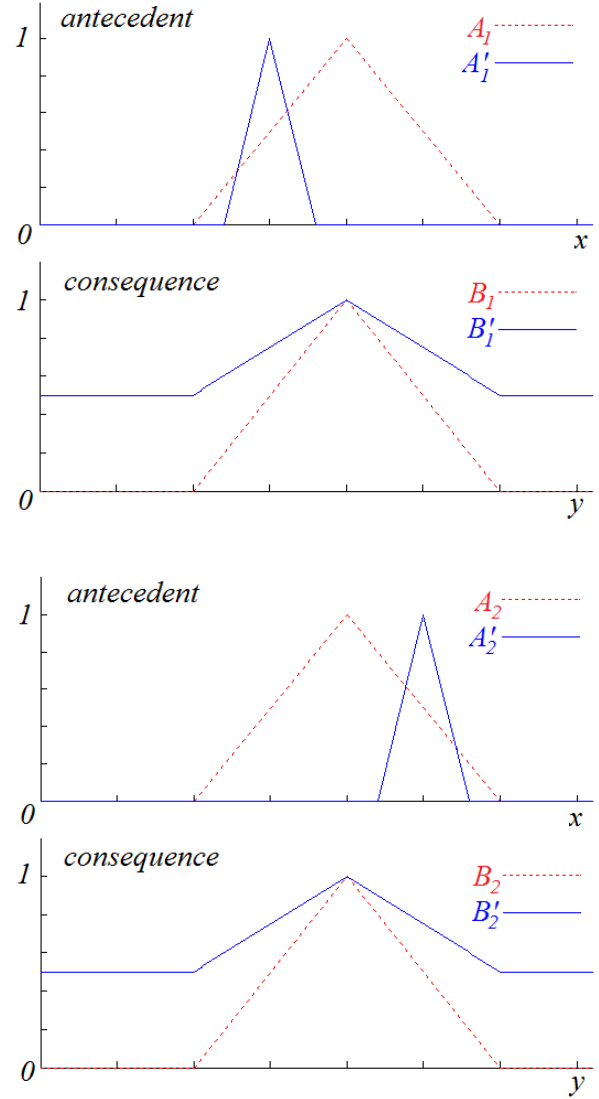


Figure 1: Examples of inferences using the Generalized Modus Ponens (t-norm Lukasiewicz, implication Reichenbach). We observe that if there is gradual relationship between premise and conclusion (what we call the gradual hypothesis) then it is not modeled by the GMP: we obtain the same conclusion for observations clearly smaller or larger than the premise. If the gradual hypothesis is handled we expect to obtain an inferred conclusion smaller (or larger) than the rule's conclusion.

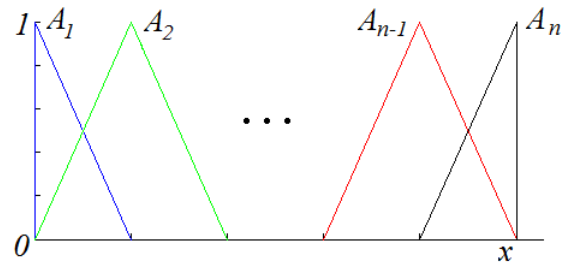


Figure 2: Examples of strictly convex, normalized and continuous membership functions

The membership function of the *Greater* part of  $A$  is:

$$\varphi_{Greater_A}(x) = \begin{cases} \overline{\mu_A(x)} = 1 - \mu_A(x) & A_R < x \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The membership function of the *Indistinguishable* part of  $A$  is:

$$\varphi_{Indistinguishable_A}(x) = \begin{cases} 1 & x \in [A_L, A_R] \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

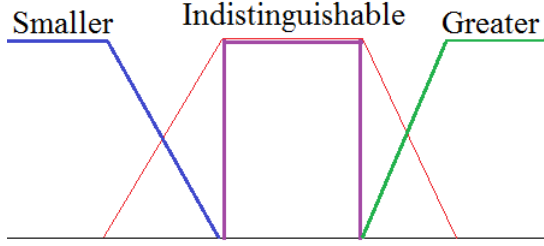


Figure 3: Partitioning of the universe of discourse based on the complement and kernel of the fuzzy set

### Formal definition of the GGMP

In order to induce a *gradual*<sup>1</sup> behavior in the GMP, we propose to compute conclusion's membership function  $B'$  as follows, for each  $y \in V$ :

$$\mu_{B'}(y) = \sup_{x \in \psi(y)} \mathcal{T}(\mu_{A'}(x), \mathcal{I}(\mu_A(x), \mu_B(y))) \quad (5)$$

where

$$\begin{aligned} \psi(y) = \{ & x \in U \mid \\ & \varphi_{P_A}(x) = \varphi_{P_B}(y) \text{ and } \varphi_{P_B}(y) > 0, \\ & P \in \{Smaller, Greater, \\ & Indistinguishable\} \} \end{aligned}$$

Figure 4 shows the conclusion  $B'$  inferred using GGMP, that should be compared with what would have been obtained using normal GMP, as shown on Figure 1.

### Crisp observation

In practice, input values of fuzzy inference systems often are crisp (without any fuzziness). The following theorem shows that when the observation is a crisp value, the proposed inference method, GGMP, infers a crisp conclusion.

**Theorem 1:** If the observation is a crisp value, the conclusion of GGMP is a crisp value.

*Proof:*

For each  $y$ , from the definition of  $\psi(y)$  in formula (5),  $\mathcal{I}(\mu_A(x), \mu_B(y)) = 1$  for all  $x \in \psi(y)$ .

Because  $A'$  is crisp, for any  $x \in U$ ,  $\mu_{A'}(x)$  takes one of two values 0 or 1:

<sup>1</sup>This is not the "gradual" meant by some authors, who focus on the graduality of the truth-values as in [12]

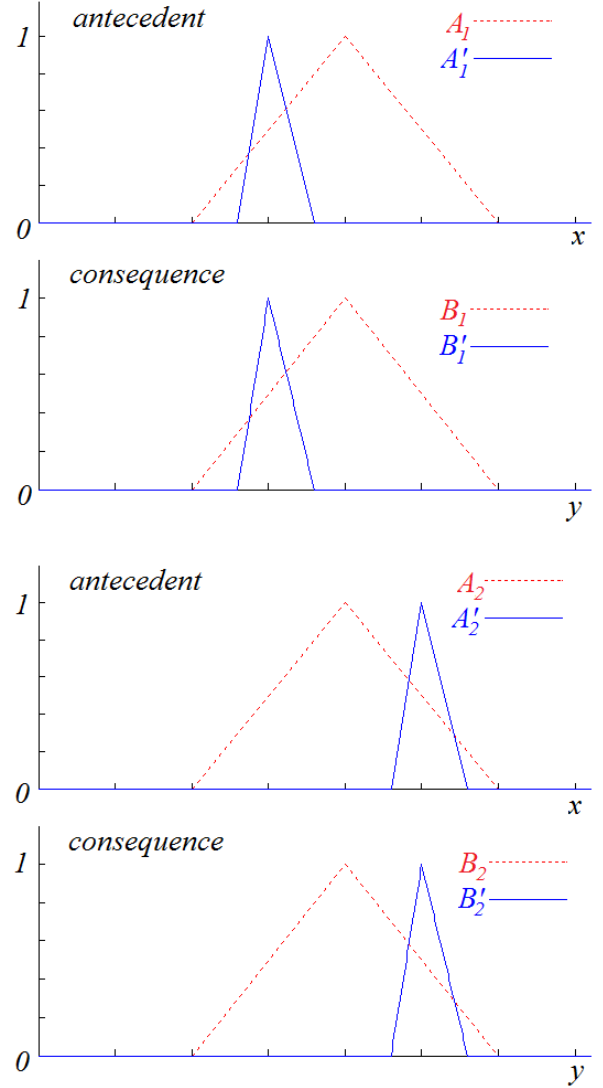


Figure 4: Gradual Generalized Modus Ponens improves the specificity of the inferred conclusion by taking advantage of the gradual relationship (t-norm product, implication Brouwer-Godel)

- If  $\mu_{A'}(x) = 0$   
then  $\mathcal{T}(\mu_{A'}(x), \mathcal{I}(\mu_A(x), \mu_B(y))) = 0$
- If  $\mu_{A'}(x) = 1$   
then  $\mathcal{T}(\mu_{A'}(x), \mathcal{I}(\mu_A(x), \mu_B(y))) = 1$

That means, the conclusion of GGMP is crisp. ■

### Compatible t-norms and implications

The compatibility [13] between a t-norm and implication for the generalized modus ponens translates the requirement that if the observation is identical with the premise ( $A' \equiv A$ ), the inferred conclusion should also be identical with the consequence of the rule ( $B' \equiv B$ ). Table 1 lists some implications and t-norms which are compatible for the Gradual GMP.

T-norm	Definition
Lukasiewicz	$\mathcal{T}_L(u, v) = \max(0, u + v - 1)$
Minimum	$\mathcal{T}_M(u, v) = \min(u, v)$
Product	$\mathcal{T}_P(u, v) = uv$

Implication	T-norm
Rescher-Gaines $\mathcal{I}_{RG}(x, y) = \begin{cases} 1 & x \leq y \\ 0 & x > y \end{cases}$	$\mathcal{T}_L, \mathcal{T}_M, \mathcal{T}_P$
Brouwer-Godel $\mathcal{I}_{BG}(x, y) = \begin{cases} 1 & x \leq y \\ y & x > y \end{cases}$	$\mathcal{T}_L, \mathcal{T}_M, \mathcal{T}_P$
Goguen $\mathcal{I}_G(x, y) = \begin{cases} 1 & x \leq y \\ y/x & x > y \end{cases}$	$\mathcal{T}_L, \mathcal{T}_M, \mathcal{T}_P$
Lukasiewicz $\mathcal{I}_L(x, y) = \min(1 - x + y, 1)$	$\mathcal{T}_L, \mathcal{T}_M, \mathcal{T}_P$

Table 1: T-norms and Implications compatible for the Gradual GMP

### Preserving the ordering by Gradual GMP

Ordering is a notion which is closely related to monotonicity. Preserving the ordering of the conclusion with regard to the observation is a property of GGMP. This property plays a key role when inducing smooth monotonicity of fuzzy systems, which are discussed in the sections below.

For the purpose of this paper, we focus, here, on the *fuzzy max order* introduced by J. Ramík and J. Římánek in [14]:

**Definition:** (*fuzzy max order*) Fuzzy set  $A$  is said to be *below or equal to* fuzzy set  $B$  according to the fuzzy max order ( $A \preceq_m B$ ) if  $\forall \alpha \in (0, 1]$ :

$$\inf\{A_\alpha\} \leq \inf\{B_\alpha\} \text{ and } \sup\{A_\alpha\} \leq \sup\{B_\alpha\} \quad (6)$$

in which  $A_\alpha$  is the  $\alpha$ -cut of  $A$  defined by

$$A_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\}$$

**Theorem 2:** Let  $A$  and  $B$  be the antecedent and consequence of a fuzzy rule;  $A'$  and  $B'$  be the observation and conclusion of GGMP.

$$\text{If } A \preceq_m A' \text{ then } B \preceq_m B'$$

*Proof:*

From the conditions of the membership functions mentioned above, there exist both the minimal and maximal elements of the  $\alpha$ -cut of a fuzzy set for all  $\alpha \in (0, 1]$ , i.e.,

$$\inf\{A_\alpha\} = \min\{A_\alpha\} \text{ and } \sup\{A_\alpha\} = \max\{A_\alpha\}$$

If  $A \preceq_m A'$ , for each  $\alpha \in (0, 1]$ , we determine  $y_0 \in V$  such that  $y_0 = \inf\{B'_\alpha\}$ . There exists  $x_0 \in$

$U$  which is the chosen point to infer  $\mu_{B'}(y_0)$ , i.e.,

$$x_0 = \arg \max_{x \in \psi(y_0)} \mathcal{T}(\mu_{A'}(x), \mathcal{I}(\mu_A(x), \mu_B(y_0)))$$

First, we will show that  $\inf\{B_\alpha\} \leq \inf\{B'_\alpha\}$ .

- i) If  $x_0$  is in the *Indistinguishable* or *Greater* part of  $A$  then  $y_0$  is also in the corresponding *Indistinguishable* or *Greater* part of  $B$ , respectively.

Thus we have  $\inf\{B_\alpha\} \leq y_0 = \inf\{B'_\alpha\}$  (since  $\inf\{B_\alpha\}$  belongs to the *Smaller* part of  $B$ ).

- ii) If  $x_0$  is in the *Smaller* part of  $A$  then  $y_0$  is in the *Smaller* part of  $B$ .

In case of using implications  $\mathcal{I}_{RG}, \mathcal{I}_{BG}, \mathcal{I}_G, \mathcal{I}_L$ , we have  $\mu_{A'}(x_0) = \mu_{B'}(y_0) = \alpha$

then  $\inf\{A_\alpha\} \leq x_0$

(from the hypothesis that  $\inf\{A_\alpha\} \leq \inf\{A'_\alpha\}$ ).

Therefore,  $\mu_B(y_0) = \mu_A(x_0) \geq \alpha$

(by non-decreasing property of convex fuzzy set in the *Smaller* part),

this implies  $\inf\{B_\alpha\} \leq y_0 = \inf\{B'_\alpha\}$ .

In case of using implication  $\mathcal{I}_M$  with t-norm minimum, from the definition of  $x_0$ ,

$$\mu_{B'}(y_0) = \min(\mu_{A'}(x_0), \mu_A(x_0)) = \alpha.$$

Here, we have  $\mu_{A'}(x_0) = \alpha$  and  $\mu_A(x_0) \geq \alpha$

(otherwise, if  $\mu_A(x_0) = \alpha$  and  $\mu_{A'}(x_0) = \alpha_1 > \alpha$ ,

then  $\inf\{A_{\alpha_1}\} > x_0$ , this is inconsistent with the hypothesis that  $\inf\{A_{\alpha_1}\} \leq \inf\{A'_{\alpha_1}\}$ ).

Therefore  $\mu_B(y_0) = \mu_A(x_0) \geq \alpha$ ,

it implies  $\inf\{B_\alpha\} \leq \inf\{B'_\alpha\}$ .

Similarly, we have  $\sup\{B_\alpha\} \leq \sup\{B'_\alpha\}$ , i.e.,  $B \preceq_m B'$ .

In the same way, we could prove that:

$$\text{If } A' \preceq_m A \text{ then } B' \preceq_m B$$

■

Order preservation of the GGMP is even stronger than what is presented by this property, which focuses on comparing observation with the premise and obtained conclusion with rule's consequence. In fact, for GGMP, if one observation is *smaller* than another observation, then the corresponding conclusions also fulfill this order, i.e., if  $A'_1$  is *smaller* than  $A'_2$  then  $B'_1$  is *smaller* than  $B'_2$  (Figure 5). From the definition of GGMP and above theorem, it can be deducted the following corollary about the ordering of the conclusions according to the ordering of the observations.

**Corollary:** Let  $A$  and  $B$  be the antecedent and consequence of a fuzzy rule;  $B'_1$  and  $B'_2$  respectively be the conclusions according to the observations  $A'_1$  and  $A'_2$  using GGMP.

$$\text{If } A'_1 \preceq_m A'_2 \text{ then } B'_1 \preceq_m B'_2.$$

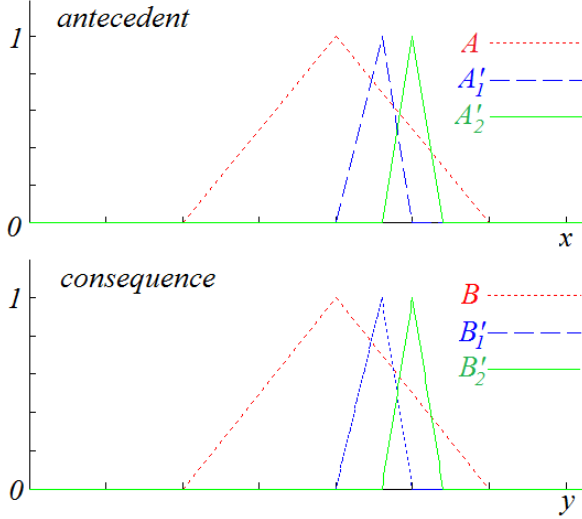


Figure 5: GGMP preserves the ordering of conclusions ( $B'_1 \preceq_m B'_2$ ) with regard to ordered observations ( $A'_1 \preceq_m A'_2$ )

### 3. Studying Fuzzy Inference Systems based on GGMP

Before introducing a GGMP-based FIS, we recall some fundamentals, while introducing notations, of Inference Models based on GMP.

Let us consider a system with a set of  $m$  fuzzy rules, each of them ( $R_i$ ) of the form:

**If**  $X_1$  is  $A_{i1,s}^1$  **AND**  $X_2$  is  $A_{i2,s}^2$  ...  $X_n$  is  $A_{in,s}^n$   
**then**  $Y$  is  $B_{is}$

where  $(X_1, X_2, \dots, X_n)$  are  $n$  input variables and  $(A_{i1,s}^1, A_{i2,s}^2, \dots, A_{in,s}^n)$  linguistic values of the antecedents in universes of discourse  $(U_1, U_2, \dots, U_n)$ ;  $B_{is}$  (with  $i \in \{1, \dots, m\}$ ) are linguistic values of the consequences in universe  $V$ .

The first stage of FIS is to infer the result of *each* rule. For rule  $R_i$ , the degree of fitness  $h_i$  of input vector  $X = (x_1, x_2, \dots, x_n)$  to the antecedent parts  $(A_{i1,s}^1, A_{i2,s}^2, \dots, A_{in,s}^n)$  is given by:

$$h_i = \mu_{A_{i1,s}^1}(x_1) \otimes \mu_{A_{i2,s}^2}(x_2) \otimes \dots \otimes \mu_{A_{in,s}^n}(x_n) \quad (7)$$

where  $\otimes$  stands for a conjunction operator (such as minimum or product operator). Then the inference result  $B'_i$  is deduced by using GMP.

Next, the overall consequence  $B'$  is aggregated from  $B'_1, B'_2, \dots, B'_m$ . Depending on the type of rules a t-norm (conjunction operator) or a t-conorm (disjunction operator) is used to aggregate  $B'$ .

The final stage is a defuzzification which concludes the control action. There are many methods to obtain the representative point  $y_0$  for the resulting fuzzy set  $B'$ , such as the Center-of-Gravity, Mean-of-Maxima and Center-of-Sums.

### 3.1. Towards GGMP-based FIS

Several difficulties have to be solved when replacing, in a classical FIS, the GMP by the introduced gradual version (equation 5). Most of these challenging questions are due to the high interaction of the different rules of the FIS. Thus, one of the adaptations we propose is to restrict the rules to be fired, to the ones concerned by the observation. In other words, if the observation is out of the support of the premise, the inference mechanism is not applied to deduce the conclusion. This is obtained with the following formulation: For the inference step, we propose that for given input vector  $X = (x_1, x_2, \dots, x_n)$ , for rule  $R_i$ , the result for *each* input value  $x_j$ , called  $B_i'^j$ , should be calculated separately by:

$$\mu_{B_i'^j}(y) = \sup_{x \in \psi_i^j(y)} \mathcal{T}(\mu_{x_j}(x), \mathcal{I}(\mu_{A_i^j}(x), \mu_{B_i}(y))) \quad (8)$$

where

$$\begin{aligned} \psi_i^j(y) &= \{x \in \text{supp}(A_i^j) \mid \\ &\varphi_{P_{A_i^j}}(x) = \varphi_{P_{B_i}}(y) \text{ and } \varphi_{P_{B_i}}(y) > 0, \\ &P \in \{\text{Smaller, Greater,} \\ &\text{Indistinguishable}\}\} \end{aligned}$$

Another set of challenges, this time related to the augmented precision induced by the gradual hypothesis has to be solved in order to be able to have a fully functional system (for any observed fuzzy set). In particular, this is the question of how to aggregate and defuzzify the conclusions.

Before attempting to propose a general model, in this paper, we propose to study the induced gradual behavior, by comparing the FIS in the case of crisp observation. Notice that this is very often in the case of controllers, their observation are often precise physical readings. In this context we propose the following formula to obtain the aggregated and defuzzified conclusion  $y_0$ :

$$y_0 = \frac{\sum_{i=1}^m \sum_{j=1}^n \mu_{B_i}(y_i^j) \mu_{B_i'^j}(y_i^j) y_i^j}{\sum_{i=1}^m \sum_{j=1}^n \mu_{B_i}(y_i^j) \mu_{B_i'^j}(y_i^j)} \quad (9)$$

with  $m$  being the number of rules and  $n$  the number of input variables;  $y_i^j$  denotes the point at which  $B_i'^j$  has positive membership degree.

This formulation is analogous to a conjunctive aggregation and center of gravity defuzzification.

### Experimental Results and Discussion

In the following subsections, we study the proposed method based on some experiments. In particular, the behavior of the GGMP-based FIS will be compared with, what can be considered as a reference in terms of monotonic behavior ([8], [9] [10]), the Mamdani's inference system with Center-of-Gravity method as defuzzification.

The tests consist in, for each input universe, increase the precise observations gradually (notice

that there may be several inputs). The output of the system is plotted and compared with similar plot obtained using the reference system (Mamdani controller). In other words, for a system with one input variable, we compare two curves, while for one with two input variables, we compare two surfaces.

The linguistic values of the antecedents as well as of the consequences of the rules are ordered as illustrated in Figure 2, i.e., for universe  $U_i$ :

$$A_1^i \preceq_m A_2^i \preceq_m \dots \preceq_m A_n^i$$

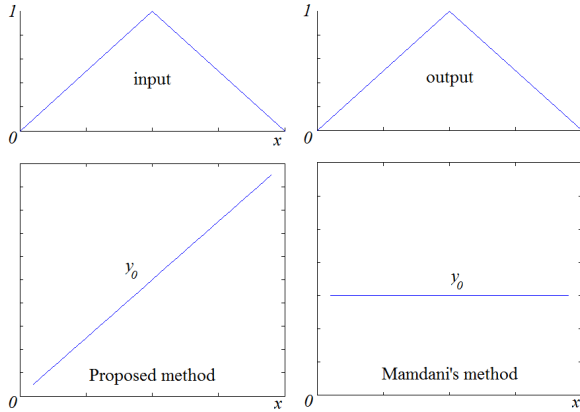


Figure 6: Smooth strict monotonic behavior induced by the GGMP, for a single rule, to be compared with the constant response of Mamdani's method

### 3.2. Single input FIS with one rule

The simplest possible FIS is a single input FIS with just one rule. There is no aggregation, it is just the inference and the defuzzification. Figure 6, compares the behavior of a Mamdani FIS versus that of a GGMP-base FIS. We observe that the latter has a smooth strict monotonic behavior. The consistently increasing curve translates the fact that when the observation value is increased (in the scale defined in universe  $U$ ) the inferred decision strictly increases (in the scale of universe  $V$ ). This is to be compared with what is obtained with Mamdani's method, where the increase of observations value has no influence on the conclusion.

### 3.3. Single input FIS with multiple rules

In a system with a single input variable with several linguistic values, the fuzzy rules  $R_i$  have the form:

$$\text{If } X \text{ is } A_i \text{ then } Y \text{ is } B_i$$

In Figure 7, we observe that GGMP improves the performance in terms of smoothness of monotonic behavior compared to Mamdani's inference. In fact, when the observation value evolves around the kernel of the premise of a rule there is no great impact in the conclusion. This can be furthermore

explained by the fact that *there is no gradual behavior in areas*, of the input universe, *where only one rule applies*. And since Mamdani inference (as well as others) does not integrate a gradual behavior (when this hypothesis can be retained), we observe plateaus in the response.

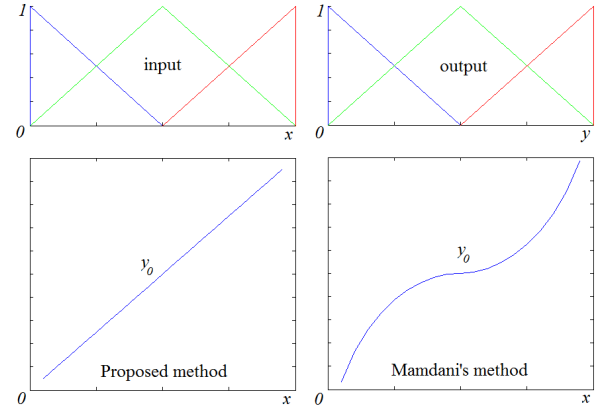


Figure 7: Effect of using GGMP, compared to Mamdani's method, in the case of an inference system with one input variable

### 3.4. Double input FIS

In order to study the influence that may have an input over another one, we set up several tests with FIS containing double inputs with several linguistic values each. Figure 8 shows the results of systems having two inputs and a rule base consisting of three rules of the form:

$$\text{If } X_1 \text{ is } A_i^1 \text{ AND } X_2 \text{ is } A_i^2 \text{ then } Y \text{ is } B_i$$

With Mamdani's method, there is a plateau on the surface of system output. Despite the increasing of input values, the output of Mamdani's method does not change while that of the proposed method consistently increases.

For systems with a more complex rule base, as shown on Figure 9 where there are 19 rules corresponding to the combinations of the underlying linguistic values, the proposed method still has smoother responses than those of Mamdani's method.

## 4. Conclusion

Monotonic relationship between input and output often is an inherent property of fuzzy systems. For example, for a cart-pole system, there is a gradual relationship between the pole angle and the cart velocity of the form: "The more the pole deviates from the upright position, the higher velocity the cart must gain".

Thus, we propose an approach, that we call Gradual GMP, which is able to integrate the graduality, when this underlying hypothesis exists, into the



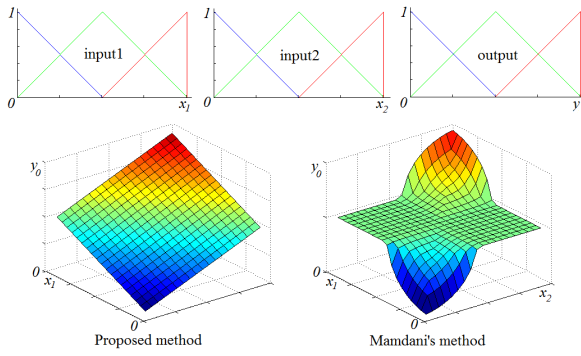


Figure 8: Smooth monotonic behavior of the proposed method in case of system with two inputs and the rule base consisting of three fuzzy rules

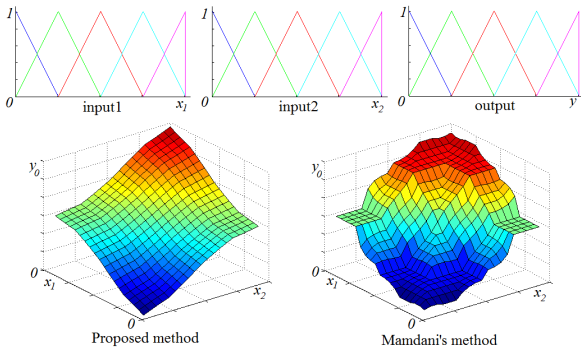


Figure 9: Smoother response of the fuzzy system based on GGMP in case of a complex system (2 inputs and 19 fuzzy rules)

generalized modus ponens. We show that the mechanism preserves the ordering property of the conclusion with respect to the observation, a property closely related to monotonicity.

In this paper, we studied in a simple case (crisp observations) the monotonic response of what could be the adaptation to this case of GGMP-based FIS. We showed that, for a single rule FIS, GGMP induces strict monotonicity, to be compared with static behavior in the reference case (Mamdani). Further experimental results, on single and double input FIS, show that the proposed method implies a smooth monotone behavior even when in presence of several linguistic values for each variable.

They also reveal that, for all classical FIS, monotonicity is due to rule overlap, and therefore when observations are handled by only one rule (even when inside a FIS), the monotone behavior can not be insured. The only way to guarantee strict monotonicity is to have an inference that has a gradual behavior.

In systems with multiple input variables, the analysis becomes more complex because of the interaction of system parameters, as has been studied by other authors. Nevertheless, our experiments point out that introducing graduality at inference level smoothes the monotone response of the system. This encouraging result opens up the way for

future works which will concentrate on a general formulation, able to handle any type of fuzzy observation, for GGMP-based FIS.

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