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Existence and Homogenization of the Rayleigh-Bénard Problem

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Abstract

The Navier-Stokes equation driven by heat conduction is studied. As a prototype we consider Rayleigh-Bénard convection, in the Boussinesq approximation. Under a large aspect ratio assumption, which is the case in Rayleigh-Bénard experiments with Prandtl number close to one, we prove the existence of a global strong solution to the 3D Navier-Stokes equation coupled with a heat equation, and the existence of a maximal B-attractor. A rigorous two-scale limit is obtained by homogenization theory. The mean velocity field is obtained by averaging the two-scale limit over the unit torus in the local variable.

1 Introduction

In this paper we study the Navier-Stokes equation driven by heat conduction. Under a large aspect ratio assumption (the spatial domain being a thin layer) we prove the existence of a global strong solution. The aspect ratio Γ is the width/height ratio of the experimental apparatus and $\Gamma \geq 40$ is considered large, see Hu et al. [1, 2]. For existence results for the Navier-Stokes equation in thin domains we particularly refer to Raugel [3] and the references therein. Our contribution is the addition of a heat equation where the heat conduction is driving the fluid. The prototype we have in mind is the classical Bénard problem and this work is motivated by the instabilities of roll-patterns observed in experiments and simulations. Rayleigh-Bénard convection is a model for pattern formation and has been extensively studied, Busse and Clever [4, 5] established the stability of straight parallel convection rolls and used them to explain many experimental observations, for large Prandtl numbers $P = \frac{\nu}{\kappa}$ (ν is the kinematic viscosity and κ is the thermal diffusitivity). For low Prandtl numbers, the situation is much more complicated and it has long been recognized [6, 7, 8, 9] that mean flows are crucial in understanding the complex pattern dynamics observed in experiments. In this paper we establish that this complexity is not caused by singularity formation but our ultimate goal is a theoretical understanding and a quantitative capture of the mean flow. It is known that wave-number distortion, roll curvature and the mean flow make straight convection rolls become unstable [10, 11, 12, 13]. These effects have been successfully modelled by Decker and Pesch [8] and simulated. However, the resulting equations contain non-local terms due to the mean-flow and are theoretically intractable. It is our hope that our results will put the study of the contribution of the mean flow on a rigorous mathematical footing, both simplifying and reaching a better theoretical understanding in the process. The experimental difficulties of measuring weak global flow in the presence of dominant local roll circulations are formidable, [2], and a theoretical insight may be crucial in the case of dynamical patterns.

As a preparation, we prove a new statement of the classical, see Leray [14], existence of a global strong solution and a global attractor for the three-dimensional Navier-Stokes equation under smallness assumptions on the data, see Ladyshenskaya [15]. For different results with large forcing compare Foias and Temam [16] and Sell [17]. This new statement and new proof of the theorem are crucial in the statement and the proof of the existence of a global solutions of the Rayleigh-Bénard problem with a large aspect ratio. The existence is proven after an initial time-interval, corresponding to a settling-down period in experiments. In experiments in gases (CO_2) this settling-down time is a few hours for experiments that take a few days, [2].

We will also prove that the Rayleigh-Bénard problem has a global attractor. Then theorems of Milnor [18], Birnir and Grauer [19], and Birnir [20] are used to prove that the Rayleigh-Bénard problem has a unique maximal B-attractor. This attractor has the property that every point attracts a set (of functions) that is not shy [21], or of positive infinite-dimensional measure. The discovery of a spiral-defect chaotic attractor [22, 11, 23], in a parameter region were previously only straight rolls were known to be stable [24, 25], was one of the more startling results in recent pattern formation theory. The experimentally observed attractors are B-attractors, and the spiral-defect chaotic attractor and the straight roll attractor may be two minimal B-attractors of the unique maximal B-attractor whose existence we prove.

1.1 The Boussinesq equations

We start with the Boussinesq equations which are two coupled equations for the fluid velocity u, the pressure p and the temperature T,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = g\alpha(T - T_2), \\ \dfrac{\partial T}{\partial t} + (u \cdot \nabla)T - \kappa\Delta T = 0, \\ \operatorname{div} u = 0, \end{cases} \qquad x \in \Omega,\ t \in \mathbf{R}^+. \tag{1.1}$$

Here g is the gravitational acceleration and α is the volume expansion coefficient of the fluid. Moreover ν and κ are the viscosity and conductivity coefficients which determine the dimensionless Rayleigh $R = \frac{\alpha g(T_1 - T_2)h^3}{\nu\kappa}$ and Prandtl numbers. We assume that Ω is a rectangular box and fix the temperatures at the bottom T_1 and top T_2 of the box. The box is heated from below so $T_1 > T_2$. We can impose periodic boundary conditions for u on the lateral sides of the box. However, as the temperature T_1 increases these have to be relaxed due to the presence of a boundary layer. The velocity is assumed to vanish on the horizontal surfaces of the box. Finally, we must supply the appropriate initial data.

1.2 The homogenization

The mathematical framework for the homogenization starts by the introduction of a (small) parameter $\epsilon > 0$ and a scaling of the Navier-Stokes system above

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^{3/2} \nu \Delta u_\epsilon + \nabla p_\epsilon = g\alpha(T_2 - T_\epsilon), \\ \dfrac{\partial T_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) T_\epsilon - \epsilon^{3/2} \kappa \Delta T_\epsilon = 0, & x \in \Omega,\ t \in \mathbf{R}^+. \\ \operatorname{div} u_\epsilon = 0, \end{cases} \tag{1.2}$$

The bulk of the paper will be devoted to proving that there exist unique functions u_0, p_0 and T_0 such that

$$\epsilon^{-1/2} u_\epsilon \to u_0, \;\; p_\epsilon \to p_0, \;\; T_\epsilon \to T_0,$$

as $\epsilon \to 0$ in the appropriate Sobolev spaces. At a first glance, sending ϵ to zero seems to give the Euler equation, but this is wrong. The limit obtained is viscid and the above equation only makes sense for $\epsilon > 0$. However, for any fixed $\epsilon > 0$ the solutions of the scaled equations have global existence in two dimensions and the equations possess a smooth global attractor in dimensions two or three, see Foias et al. [26], Ladyshenskaya [15, 27] and Sell [17]. The equations for the leading order coefficients, in a power series in ϵ, are still evaluated at $\epsilon = 0$ and turn out to be the Navier-Stokes system

$$\begin{cases} \dfrac{\partial u_0}{\partial \tau} + (u_0 \cdot \nabla_y) u_0 - \nu \Delta_y u_0 + \nabla_y p_1 = g\alpha(T_2 - T_0) - \nabla_x p_0, \\ \dfrac{\partial T_0}{\partial \tau} + (u_0 \cdot \nabla_y) T_0 - \kappa \Delta_y T_0 = 0, \\ \mathrm{div_y}\, u_0 = 0, \;\; \mathrm{div_x}\Big(\displaystyle\int_{T^n} u_0 dy\Big) = 0, \end{cases} \tag{1.3}$$

where $x \in \Omega$, $y \in T^n$ and $\tau \in \mathbf{R}^+$. Here $y = x/\epsilon$ is the local spatial variable and $\tau = t/\sqrt{\epsilon}$ is the scaled fast time variable. T^n, the unit torus in y, is what is referred to as the unit cell in the terminology of homogenization. The initial data is so highly oscillatory that we can assume that the boundary conditions are periodic in the local variable y. The Navier-Stokes system (1.3) differs from the original system (1.2) in that it has an additional forcing term $-\nabla_x p_0$. This is the obvious influence of the global pressure on the local flow. The solutions of the system (1.3) enjoy global (in τ) existence in two dimensions and the equations possess a smooth (in y) global attractor in dimensions two or three. This, possibly high-dimensional, attractor of the local flow is the physically relevant quantity for the local flow, except in the strong turbulence limit, when long transients may play a role.

We will show that the solution of (1.3) is the unique two-scales limit of a sequence of solutions to the scaled system (1.2). This uses the weak sequential compactness property of reflexive Banach spaces and says that any bounded sequence $\{u_\epsilon\}$ in say $L^2(\Omega)$ contains a subsequence, still denoted by $\{u_\epsilon\}$, such that for smooth test functions $\varphi(x,y)$, periodic in y

$$\int_\Omega u_\epsilon(x)\varphi(x,\frac{x}{\epsilon})dx \to \int_\Omega\int_{T^n} u_0(x,y)\varphi(x,y)dydx.$$

The main result (Theorem 7.1) is that if $\{u_\epsilon\}$ is a sequence of solutions to the Navier-Stokes system (1.2), then the so called two-scales limit u_0 is the solution to the local Navier-Stokes system (1.3). Our proof is based upon a compactness result which was first proved by Nguetseng [28] and then further developed by Allaire [29, 30, 31]. Moreover, if u_0 is a globally defined unique solution of the system (1.3), which is the case if u_0 lies on the attractor of (1.3), then, by uniqueness, the whole sequence $\{u_\epsilon\}$ two-scale converges to u_0.

The mean field turns out to be

$$\overline{u}_0(x,\frac{t}{\sqrt{\epsilon}}) = \int_0^{t/\sqrt{\epsilon}} \left(\pi(e_n\overline{\theta}_0)\right)(x,s)ds,$$

where e_n, $n=2,3$, is the unit vector in the vertical direction and $\pi(e_n\overline{\theta}_0)$ denotes the projection onto the divergence free part of $e_n\overline{\theta}_0$,

$$\pi(e_n\overline{\theta}_0) = -\nabla\times(\Delta^{-1}(\nabla\times e_n\overline{\theta}_0)).$$

The mean field is derived from the local Navier-Stokes system (1.3). It gives the contribution of the conduction to the small scale flow. We have averaged (in y), denoted by overbar, over the unit cell T^n, $n=2,3$. Once we have the local problem (4.3) the boundary conditions on the local cell can also be relaxed to capture the contribution of (global) convection to the mean field. The mean field with the influence of the convection taken into account, turns out not surprisingly to satisfy a forced Euler's equation

$$\frac{\partial\overline{u}_0}{\partial\tau} + \overline{u}_0\cdot\nabla\overline{u}_0 + \nabla\overline{p}_1 = \pi(e_n\overline{\theta}_0),$$

where $\tau = t/\sqrt{\epsilon}$ and $\nabla=\nabla_y$, with $y=x/\epsilon$.

1.3 Problem setting

We let Ω be a rectangular box, of thickness h and Lebesgue measure $m(\Omega)$, in $\mathbf{R}^n$, $n=2$ or 3. By (e_i), $i=1,2$ or $i=1,2,3$, we denote the canonical basis in $\mathbf{R}^2$ or $\mathbf{R}^3$, respectively. The system (1.2) is equipped with the following initial data:

$$u(x,0)=u_0(x) \text{ and } T(x,0)=T_0(x),$$

which are assumed to belong to $L^2(\Omega)$, and boundary data:

$$u=0 \text{ at } x_n=0 \text{ and at } x_n=h.$$

As above

$$T=T_1 \text{ at } x_n=0 \text{ and } T=T_2 \text{ at } x_n=h.$$

Moreover we assume that

$$u, \ \nabla u, \ T, \ \nabla T \ \text{ and } \ p$$

are periodic with period l in the horizontal x_1-direction, in the two-dimensional case and periodic with period l in the horizontal x_1-direction and period L in the x_2-direction in the three-dimensional case. In fact we will without loss of generality assume that $l = L$ throughout the paper. This determines the pressure p up to a constant that can be fixed by normalization, see Remark 2.

Since we are interested in the fluctuations in the temperature we follow [26] and put

$$\theta = (T - T_1 - \frac{x_n}{h}(T_2 - T_1))$$

and

$$p = p - g\alpha(x_n + \frac{x_n^2}{2h})(T_1 - T_2).$$

We get the following system which is equivalent to (1.1).

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = g\alpha e_n \theta, \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta - \kappa\Delta\theta = \dfrac{(T_1 - T_2)}{h}(u)_n, \qquad x \in \Omega, \ t \in \mathbf{R}^+. \\ \operatorname{div} u = 0, \end{cases} \tag{1.4}$$

For the temperature θ we get initial data $\theta(x, 0) = \theta_0(x)$ and the new boundary data $\theta = 0$ at $x_n = 0$ and at $x_n = h$. The initial and boundary data for u remain unchanged. Moreover, u and θ and their gradients and p are periodic as above. We will find it useful to work with the system in the form (1.1) in some situations, whereas the form (1.4) is more suitable in other situations.

2 The Navier-Stokes equation

We will denote by $|\cdot|$ and $\|\cdot\|$ the usual norms in $L^2(\Omega)$ and $H^1(\Omega)$. We denote by $\|\cdot\|_2$ the $H^2(\Omega)$-norm and by $|\cdot|_\infty$ the $L^\infty(\Omega)$-norm. Further, $|u|_{2,\infty} = \text{ess sup}\,|u(t)|$, where supremum is taken over all $t \geq 0$.

Let us consider the Navier-Stokes equation for incompressible fluids

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f, \qquad x \in \Omega, \ t \in \mathbf{R}^+, \\ \operatorname{div} u = 0, \end{cases} \tag{2.1}$$

where u is the velocity, p is the pressure and ν is the viscosity, with initial condition

$$u(x, 0) = u_0(x)$$

and vanishing boundary conditions on $\partial\Omega$ (periodic boundary conditions with mean zero, if $\Omega = T^3$). f denotes the forcing and λ_1 is the smallest eigenvalue of $-\Delta$ on Ω, with vanishing boundary conditions on $\partial\Omega$. We start by an estimate which goes back to Leray [14]. For the readers convenience we present the (old) proof since the arguments therein will be used repeatedly both in the proof of Theorem 2.2 and Theorem 3.3 below.

Lemma 2.1. *Every weak solution u to the Navier-Stokes equation (2.1) satisfies the estimate*

$$|u(t)| \leq |u_0|e^{-\lambda_1 \nu t} + \frac{|f|_{2,\infty}}{\lambda_1 \nu}(1 - e^{-\lambda_1 \nu t}).$$

Moreover, there exists a sequence $t_j \to \infty$ such that

$$\|u(t_j)\|^2 \leq 3\frac{|f|_{2,\infty}^2}{\lambda_1^2 \nu^3}.$$

Proof. We take the inner product of (2.1) with u and integrate over Ω. By the divergence theorem and the incompressibility, we are left with

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|\nabla u(t)|^2 = \int_\Omega f(t) \cdot u(t)\, dx.$$

The Schwarz and Poincaré inequalities give

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \lambda_1 \nu |u(t)|^2 \leq |f(t)||u(t)|,$$

so by cancellation of $|u(t)|$,

$$\frac{d}{dt}|u(t)| + \lambda_1 \nu |u(t)| \leq |f(t)|.$$

An integration over $(0, t)$, taking sup in t of f, gives

$$|u(t)| \leq |u_0|e^{-\lambda_1 \nu t} + \frac{|f|_{2,\infty}}{\lambda_1 \nu}(1 - e^{-\lambda_1 \nu t}). \tag{2.2}$$

Next we integrate the inequality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|\nabla u(t)|^2 \leq |f(t)||u(t)|$$

over the interval $[t_1, t_2]$,

$$\nu \int_{t_1}^{t_2} |\nabla u(t)|^2\, dt \leq \frac{1}{2}(|u(t_1)|^2 - |u(t_2)|^2) + \int_{t_1}^{t_2} |f(t)||u(t)|\, dt.$$

By (2.2) we get

$$\nu \int_{t_1}^{t_2} |\nabla u(t)|^2\, dt \leq |u_0|(\frac{|u_0|}{2} + |f|_{2,\infty})(e^{-\lambda_1 \nu t_1} + e^{-\lambda_1 \nu t_2}) + \frac{|f|_{2,\infty}^2}{(\lambda_1 \nu)^2}(1 + (t_2 - t_1)).$$

Finally, we let $t_2 = t_1 + 1$, and choose t sufficiently large to get

$$\int_{t_1}^{t_1+1} \|u(t)\|^2\, dt \leq \frac{3|f|_{2,\infty}^2}{\lambda_1^2 \nu^3}.$$

This implies that there is a set of positive Lebesgue measure in every interval $[t_1, t_1 + 1]$ such that

$$\|u(t)\|^2 \leq \frac{3|f|_{2,\infty}^2}{\lambda_1^2 \nu^3},$$

for t in this set. ∎

We continue by stating a local existence theorem.

Theorem 2.1. *Suppose that the pressure is normalized, i.e.,*

$$\frac{1}{m(\Omega)}\int_\Omega p\,dx = 0,$$

then there exists a unique local solution $(u,\,p)$ *of (2.1) in* $C([0,t];(H^1(\Omega))^{n+1})$, $n=2,\,3$, *with initial data* $u_0 = u(x,0)$ *in* $(H^1(\Omega))^n$. *The local existence time* t *depends only on the* L^2*-norms of* $\operatorname{curl} u_0$.

Kreiss and Lorentz [32] can be consulted for details of the proof.

It is well-known that global solutions of the Navier-Stokes equations exist, $u \in C(\mathbf{R}^+;(H^1(\Omega))^2)$, in two dimensions. In three dimensions that analogous statement is open. However, if transients are allowed to settle for a sufficiently long time, then global solutions exist in three dimensions, after this settling of the initial velocity.

We now state and prove a result saying that every weak global solution to the three-dimensional Navier-Stokes equation becomes a strong solution after some finite time $t_0 > 0$. This also goes back to Leray [14]. The existence of a global attractor is due to Ladyshenskaya [15, 27] and more recently Temam [33], Sell [17] and P. L. Lions [34]. Both the statement and the proof of Theorem 2.2 below are new and they are crucial in the statement and proof of the new existence Theorem 3.3 below, for the Rayleigh-Bénard problem in the large aspect ratio.

Theorem 2.2. *Consider the three-dimensional Navier-Stokes equation (2.1). For every weak solution* $u_w \in C_w(\mathbf{R}^+;(L^2(\Omega))^3)$ *to (2.1) there exists a* $t_0 < \infty$, *such that if the forcing* $|f|_{2,\infty}$ *is sufficiently small there exists a unique strong solution* $u \in C([t_0,\infty);(H^1(\Omega))^3)$ *to (2.1), with data* $u(t_0) = u_w(t_0)$. *Moreover, (2.1) possesses a global attractor. If the initial data* $|u_0|$ *is small, then the solutions have global existence, i.e.* $t_0 = 0$.

Proof. The subscript w indicates that the solutions are only weakly continuos in t. We take the inner product of (2.1) with Δu and integrate over Ω. Using Schwarz's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 &= \int_\Omega f(t)\Delta u(t)\,dx - \int_\Omega ((u(t)\cdot\nabla)u(t))\cdot\Delta u(t)\,dx \\ &\le |f(t)||\Delta u(t)| + |u(t)\cdot\nabla u(t)||\Delta u(t)|.\end{aligned}$$

Now

$$|u(t)\cdot\nabla u(t)| \le |u(t)|_\infty|\nabla u(t)| \le K|u(t)|^{1/4}\|u(t)\|_2^{3/4}|\nabla u(t)|$$

by the Gagliardo-Nirenberg inequalities, where K is a constant. Thus,

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 \le (|f(t)| + K|u(t)|^{1/4}\|u(t)\|_2^{3/4}|\nabla u(t)|)|\Delta u(t)|.$$

An application of the inequality $ab \le a^2/2\nu + \nu b^2/2$, on the right hand side, gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \frac{\nu}{2}|\Delta u(t)|^2 &\le \frac{1}{\nu}(|f(t)| + K|u(t)|^{1/4}\|u(t)\|_2^{3/4}|\nabla u(t)|)^2 \\ &\le \frac{1}{\nu}(|f(t)|^2 + K^2|u(t)|^{1/2}\|u(t)\|_2^{3/2}|\nabla u(t)|^2)\end{aligned}$$

by another application of the inequality above with $\nu = 1$. From Lemma 2.1 we get, again by the same inequality,

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|\nabla u(t)|^2 \leq \frac{1}{2}(|f(t)|^2 + |u(t)|^2).$$

Adding these inequalities, applying the Sobolev inequality and repeating the use of the inequality above results in

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\nu}{4}\|u(t)\|_2^2 \leq C_1|u(t)|^2\|u(t)\|^8 + C_2(|f(t)|^2 + |u(t)|^2),$$

where C_1 and C_2 are constants. Using Poincaré's inequality and Lemma 2.1 we find that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\lambda_1\nu}{4}\|u(t)\|^2 - 2C_1(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t})\|u(t)\|^8$$
$$\leq 3C_2(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t}),$$

since

$$(|f|_{2,\infty} + |u_0|e^{-\lambda_1\nu t})^2 \leq 2(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t}).$$

Now the point is that if the coefficient $|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t}$ is small, or the forcing is small and we have waited a sufficiently long time, to let the initial data $|u_0|^2e^{-2\lambda_1\nu t}$ decay, then the inequality gives us a bound on $\|u(t)\|$. The argument is as follows. We integrate the inequality above over $[t_0, t]$ to get

$$\|u(t)\|^2 - 4C_1\int_{t_0}^t(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})e^{-\beta(t-s)}\|u(s)\|^8\,ds$$
$$\leq \frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})(1 - e^{-\beta(t-t_0)}),$$

where $\beta = \lambda_1\nu/2$. Now assume that $\|u(s)\|$ assumes its maximum in the interval $[t_0, t]$ at $s = t$. Then

$$\|u(t)\|^2 - \frac{4C_1}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})\|u(t)\|^8\,ds$$
$$\leq \frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})(1 - e^{-\beta(t-t_0)}).$$

Now put

$$v(t) = \|u(t)\|^2, \quad a = \frac{4C_1}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})$$

and

$$M = \frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0}).$$

Then the inequality above can be written as

$$v(t) - av^4(t) \leq M.$$

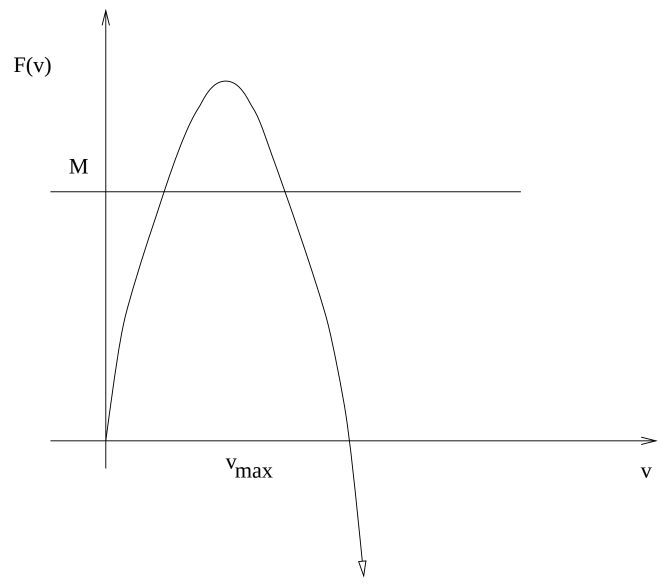

Figure 1. The bound on the norm.

The graph of the function $F(v) = v - av^4$ is shown in Figure 1. It is concave and attains its maximum at $v_{max} = 1/(4a)^{1/3}$. By Lemma 2.1 there exists a $t_0 > 0$ such that

$$v(t_0) \le \|u(t_0)\|^2 \le \frac{3|f|_{2,\infty}}{\lambda_1^2 \nu^3}$$

and we can choose $|f|_{2,\infty}$ so small that $v(t_0)$ lies between zero and v_{max}, see Figure 1. Moreover, $v(t)$ can never reach v_{max}, because

$$F(v(t)) = v(t) - av^4(t) \le M < \frac{3}{4}\frac{1}{(4a)^{1/3}} = F(v_{max}).$$

It is clear that we can choose t_0 so large that

$$\begin{aligned} M &= \frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0}) \\ &\le \frac{3}{4}\frac{1}{(4a)^{1/3}} = \frac{3}{4}\frac{1}{(\frac{16C_1}{\beta}|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0})^{1/3}}. \end{aligned}$$

Moreover, notice that the derivative of F in Figure 1 is positive at the point, where F first reaches M, i.e.,

$$F'(v) = 1 - 4av^3 = 1 + \frac{4(v - av^4)}{v} - 4 = \frac{4M}{v} - 3 \ge 0,$$

which gives the bound

$$v(t) \le \frac{4M}{3} + \delta$$

where δ is arbitrarily small. The only question left is whether the initial data makes sense as a function in $H^1(\Omega)$. But we have already used above that, by Lemma 2.1, there exists a sequence $t_j \to \infty$ such that $\|u(t_j)\| < \infty$. We now choose the initial time to be the smallest $t_j \geq t_0$. Then we let this t_j be the new t_0.

Now we apply the local existence Theorem 2.1 and the above bound to get global existence. We recall the definition of an absorbing set from Coddington and Levinson [35]. A set $D \subset L^2$ is an absorbing set if for every bounded set M there exists a time $t(M)$ such that $t > t(M)$ implies that $u(t) \in D$, if $u(0) \in M$. The estimate in Lemma 2.1 shows that the weak-flow of the weak solution of the Navier-Stokes equations has an absorbing set in L^2. But we have shown in addition that there exists a $t_1(M)$ such that the flow is a continuos flow of strong solutions for $t > t_1(M)$ and has the absorbing set $D \subset H^1$. Moreover, since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ it follows that the Navier-Stokes equation has a global attractor in $L^2(\Omega)$. In fact one can now show, see [32], that the solutions are spatially smooth and thus $D \subset C^\infty$. It follows, see Hale [36], Babin and Vishik [37], Temam [33] and Birnir and Grauer [38], that the Navier-Stokes equation has a global attractor consisting of spatially smooth solutions. It is also clear that if $|u_0|$ is small then $v(t)$ satisfies the above bound and the solutions have global existence for $t_0 = 0$. ■

Remark 1. The statement of Theorem 2.2 is really just a restatement of Leray's classical result of global existence for small initial data. The duality in the smallness condition is that one can either take small initial data or, with arbitrary initial data, just wait for a sufficiently long time. This statement is what most physicist and engineers are interested in, at least for low Reynolds numbers. Namely, transients die out and the flow settles down to the flow on the attractor in at most a few hours, in experiments, see for instance [2].

Remark 2. The existence of the pressure term follows from a standard orthogonality argument which we present below. Consider the Navier-Stokes equation (1.1) and the solution given by Theorem 2.2. We take the inner product by u, integrate over Ω, apply the divergence theorem, use the incompressibility and collect all terms on one side. This gives

$$\int_\Omega (f - \frac{\partial u}{\partial t} - u \cdot \nabla u + \nu \Delta u) \cdot u \, dx = 0.$$

Now we recall that the orthogonal to the divergence free elements in L^2 are gradients in L^2. Therefore, there exists a unique gradient ∇p in $L^2(\Omega)$ given by

$$\nabla p = f - \frac{\partial u}{\partial t} - u \cdot \nabla u + \nu \Delta u.$$

We say that a set M is *invariant* under the flow, defined by a nonlinear semi-group $S(t)$, if $S(t)M = M$, i.e. M is both positively and negatively invariant. An *attractor* $\mathcal{A}$ is an invariant set which attracts a neighbourhood U or for all $x_o \in U$, $S(t)x_o$ converges to $\mathcal{A}$ as $t \to \infty$. The largest such set U is called the *basin of attraction* of the attractor $\mathcal{A}$. If the basin of attraction of an attractor $\mathcal{A}$ contains all bounded sets of X and $\mathcal{A}$ is compact, then we say that $\mathcal{A}$ is the *global attractor*.

We have shown above that the Navier-Stokes equations have a global attractor. But we have not explained what the semi-group is that shrinks bounded sets onto the attractor. It is possible to define a semi-group on the space of weak-solutions, see Sell [17], but in light of Theorem 2.2 we can do much better. Namely, a bounded subset of a Banach space is a complete metric space and we have shown that for every bounded set $M \subset L^2$ eventually the solution starting in M will lie in an absorbing set $D \subset C^\infty$. It is the semi-group defined on this complete metric space $D \subset H^1$ that has the global attractor $\mathcal{A} = \omega(D)$, where ω denotes the ω-limit set, see [38]. $\mathcal{A}$ is invariant, so on it we can solve the Navier-Stokes equation in backward time, it is non-empty and compact and attracts a neighbourhood of itself, see [38], moreover, it also attracts every bounded set in L^2. This attractor consists of spatially smooth solutions and has finite Hausdorff and fractal dimensions, see [39, 40, 41, 42]. The dimension estimates are however large, see [33], and not necessarily a good indication of the size of the attractor.

We focus on the core of the attractor $\mathcal{A}$ which is called the basic attractor $\mathcal{B}$. For now, let us assume that the Banach space X is finite dimensional. An attractor $\mathcal{B}$ is called a *basic attractor* if

- The basin of attraction of $\mathcal{B}$ has positive measure.
- There exists no strictly smaller $\mathcal{B}' \subset \mathcal{B}$, such that up to sets of measure zero, *basin*$(\mathcal{B})$ $\subset$*basin*$(\mathcal{B}')$.

A *global basic attractor* is a basic attractor whose basin of attraction contains all bounded sets of X up to sets of measure zero.

A theorem by Milnor [18] states that in finite dimensions there exists a unique decomposition of the attractor $\mathcal{A}$,

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$$

where $\mathcal{B}$ is a basic attractor and $\mathcal{C}$ is a remainder such that $m(basin(\mathcal{C})\backslash basin(\mathcal{B})) = 0$ where m is the Lebesque measure.

The notion of a basic attractor can be extended to infinite dimensions and Milnor's Theorem was first proven in an infinite-dimensional setting by Birnir and Grauer [19]. They used cumbersome projections to a finite dimensional space where the $\mathcal{A}$-attractor resides. A more elegant notion of a $\mathcal{B}$-attractor requires an extension of the concepts of measure zero and almost everywhere. Their counterparts in infinite dimensions are *shy* and *prevalent* sets respectively, see Hunt *et al.* [21]. These are defined in the following way. Let X denote a Banach space. We denote by $S+v$ the translate of the set $S \subset X$ by a vector $v \in X$. A measure μ is said to be *transverse* to a Borel set $S \subset X$ if the following two conditions hold:

- There exists a compact set $U \subset X$ for which $0 < \mu(U) < \infty$.
- $\mu(S+v) = 0$ for every $v \in X$.

A Borel set $S \subset X$ is called *shy* if there exists a compactly supported measure transverse to S. More generally, a subset of X is called shy if it is contained in a shy Borel set. The complement of a shy set is called a *prevalent* set.

The infinite-dimensional analogue of Milnor's Theorem can now be stated,

Theorem 2.3. *Let $\mathcal{A}$ be the compact attractor of a continuous map $S(t)$ on a separable Banach space X. Then $\mathcal{A}$ can be decomposed into a maximal basic attractor $\mathcal{B}$ and a remainder $\mathcal{C}$,*

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$$

such that the realm of attraction of $\mathcal{B}$ *is prevalent but the realm of attraction of* $\mathcal{C}$*, excluding points that are attracted to* $\mathcal{B}$*, is shy.*

For a proof of this Theorem see Birnir [20] and [43]. It implies that the Navier-Stokes equation has a maximal $\mathcal{B}$-attractor in two and three dimensions. If $\mathcal{B}$ can be decomposed into finitely many (disjoint) minimal $\mathcal{B}$-attractors, then the union of the realms of attraction of these minimal $\mathcal{B}$-attractors is the whole space $(L^2(\Omega \subset \mathbf{R}))^n, n = 2, 3$. The realm of attraction is a slight generalization of the basin of attraction, as the basin is an open set, but the realm can be either open or closed.

For systems that are simple enough, the basic attractor contains only the stable trajectories of the global attractor. When transients are ignored, one will only see the basic attractor in physical experiments and numerical simulations, not the remainder $\mathcal{C} = \mathcal{A} \backslash \mathcal{B}$ of the attractor. All the relevant dynamics are therefore contained in the basic attractor. Ladyshenskaya [44] gives more examples of $\mathcal{B}$-attractors for the Navier-Stokes equation with nonlinear viscosity.

3 The Rayleigh-Bénard convection

3.1 Existence results

Now consider the Boussinesq system (1.1) equipped with initial data

$$u(x, 0) = u_0(x) \text{ and } T(x, 0) = T_0(x).$$

The boundary conditions are periodic on the vertical surfaces $x_k = 0, l$ for $k < n$. Furthermore, on the horizontal surfaces $x_n = 0$ and $x_n = h$, the velocity u vanishes (no-slip condition) and the temperature T is kept constant, i.e.,

$$T = T_1 \text{ on } x_n = 0, \ \ T = T_2 \text{ on } x_n = h.$$

We want to state a global existence theorem for the solution to the system (1.1). For that purpose we start with the following maximum principle, c.f. [26]:

Lemma 3.1. *Suppose that* u *and* T *solve (1.1). If*

$$T_2 \leq T(x, 0) \leq T_1,$$

for a.e. $x \in \Omega$*, then*

$$T_2 \leq T(x, t) \leq T_1,$$

for a.e. $x \in \Omega$ *and all* $t \geq 0$*.*

Proof. We consider

$$(T - T_1)_+(x, t) = \operatorname{ess} \sup_{x \in \Omega} (T - T_1)(x, t).$$

The second equation in (3.1) gives

$$\frac{\partial}{\partial t}(T - T_1)_+ + (u \cdot \nabla)(T - T_1)_+ - \kappa \Delta (T - T_1)_+ = 0.$$

We multiply this equation by $(T-T_1)_+$, integrate over Ω and use the divergence theorem, to get

$$\frac{1}{2}\frac{d}{dt}|(T-T_1)_+(t)|^2 + \kappa|\nabla(T-T_1)_+(t)|^2 = 0.$$

Thus, by Poincaré's inequality

$$\frac{1}{2}\frac{d}{dt}|(T-T_1)_+(t)|^2 + \lambda_1\kappa|(T-T_1)_+(t)|^2 \le 0,$$

where again λ_1 is the first eigenvalue of the negative Laplacian with vanishing boundary conditions on Ω. Consequently

$$|(T-T_1)_+(t)| \le |(T-T_1)_+(0)|e^{-\lambda_1\kappa t},$$

which shows that

$$(T-T_1)_+(\cdot,t) = 0,$$

for all $t \ge 0$, if

$$(T-T_1)_+(\cdot,0) = 0.$$

Similarly we get that

$$(T-T_2)_-(x,t) = \operatorname{ess}\sup_{x\in\Omega}(-(T-T_2)(x,t)) = 0,$$

if $(T-T_2)_-(x,0) = 0$ and we conclude that

$$T_2 \le T(x,t) \le T_1. \qquad \blacksquare$$

Remark 3. Lemma 3.1 actually yields a uniform bound on T in $L^2([0,s]\times\Omega)$ for any $s>0$. This also gives a bound for θ, see below.

Lemma 3.2. *Every weak solution u to the Boussinesq equations (1.4) satisfies the estimate*

$$|u(t)| \le |u_0|e^{-\lambda_1\nu t} + \frac{Kh^{1/2}}{\lambda_1\nu}(1-e^{-\lambda_1\nu t}),$$

$$|\theta(t)| \le \frac{Kh^{1/2}}{g\alpha},$$

where $K = g\alpha(T_1-T_2)L/3^{1/2}$. The equations possess an absorbing set in $(L^2(\Omega))^4$, defined by

$$|u(t)| + |\theta(t)| \le \left(\frac{1}{g\alpha} + \frac{1}{\lambda_1\nu}\right)Kh^{1/2} + \delta,$$

where δ is arbitrarily small. Moreover, there exists a sequence $t_j\to\infty$ such that

$$\|u(t_j)\|^2 + \|\theta(t_j)\|^2 \le K_1,$$

where

$$K_1 = 3\frac{K^2}{\lambda_1^2\nu^2}\left(\frac{h}{\nu} + \frac{(T_1-T_2)^2}{\lambda_1^2\kappa^3 h}\right).$$

Proof. We recall the relationship between T and θ

$$T = \theta + T_1 - \frac{x_n}{h}(T_1 - T_2).$$

The maximum principle in Lemma 3.1, for T, $T_2 \leq T \leq T_1$, implies that

$$-(1 - \frac{x_n}{h})(T_1 - T_2) \leq \theta \leq \frac{x_n}{h}(T_1 - T_2).$$

Thus

$$|\theta|_2^2 \leq (T_1 - T_2)^2 L^2 h/3.$$

Now consider the system (1.4). We multiply the first equation by u and integrate over Ω. Integration by parts and Schwarz's inequality give

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \lambda_1 \nu |u(t)|^2 \leq g\alpha|\theta(t)||u_n(t)|.$$

Thus by the use of the bound for θ above, and Poincaré's inequality, we get

$$\frac{d}{dt}|u(t)| + \lambda_1 \nu |u(t)| \leq g\alpha|\theta(t)| \leq g\alpha(T_1 - T_2)L(h/3)^{1/2}.$$

Integration in t then gives

$$|u(t)| \leq |u_0|e^{-\lambda_1 \nu t} + \frac{Kh^{1/2}}{\lambda_1 \nu}(1 - e^{-\lambda_1 \nu t}),$$

where $K = g\alpha(T_1 - T_2)^{1/2} L/(3)^{1/2}$.

We combine the bounds for θ and u to get the absorbing set

$$|\theta| + |u| \leq Kh^{1/2}\left(\frac{1}{g\alpha} + \frac{1}{\lambda_1 \nu}\right) + \delta$$

in $(L^2(\Omega))^4$, where

$$K = g\alpha \frac{(T_1 - T_2)}{3^{1/2}} L$$

and δ is arbitrarily small.

The last statement of the lemma is proven by a straightforward application of Lemma 2.1, to the equations (1.4), if we recall that the nonlinear term did not play a role in the proof of Lemma 2.1. Namely, for the first equation in (1.4),

$$|f|_{2,\infty}^2 = g^2\alpha^2|\theta|^2 = K^2 h,$$

and for the second equation

$$|f|_{2,\infty}^2 = \frac{(T_1 - T_2)^2}{h^2}|u|^2 = \frac{(T_1 - T_2)^2 K^2}{\lambda_1^2 \nu h}.$$

We divide these by the decay coefficients $\lambda_1^2 \nu^3$ and $\lambda_1^2 \kappa^3$, respectively and add them. This produces the bound on the H^1 norm for the sequence t_j. ■

Theorem 3.1. *Suppose that the pressure is normalized, i.e.,*

$$\frac{1}{m(\Omega)}\int_\Omega p\,dx = 0,$$

then there exists a unique local solution (u,θ,p) *of (1.4) in* $C([0,t];(H^1(\Omega))^{n+2})$, $n=2,\ 3$, *with initial data* $(u_0,\theta_0)(x)$ *in* $(H^1(\Omega))^{n+1}$.

Proof. The maximum principle in Lemma 3.1, for T, $T_2 \le T \le T_1$, implies a maximum principle for θ,

$$-(1-\frac{x_n}{h})(T_1-T_2) \le \theta \le \frac{x_n}{h}(T_1-T_2),$$

by the relationship between T and θ

$$T = \theta + T_1 - \frac{x_n}{h}(T_1-T_2).$$

This implies that

$$|\theta|_2^2 \le (T_1-T_2)^2 L^2 h/3.$$

The rest of the proof, using this bound on θ, is similar to the proof of the local existence of the Navier Stokes equation. Kreiss and Lorentz [32] can be consulted for details. Then given the local solution $(u,\theta)(x,t)$ the pressure is recovered as in Remark 2. ■

In two dimensions we have the following global existence result.

Theorem 3.2. *The Boussinesq system (1.4) has a unique global solution* $(u,\ \theta)$ *in* $C(\mathbf{R}^+;(H^1(\Omega))^3)$, *where* Ω *is a bounded open set in* $\mathbf{R}^2$. *Moreover, the system (1.4) possesses a global attractor in* $(L^2(\Omega))^3$.

Proof. A proof of Theorem 3.2 can be found in Foias et. al. [26].

In three dimensions the following global existence result holds true:

Theorem 3.3. *Consider the Boussinesq system (1.4). For every weak solution* u_w, θ_w *in* $C_w(\mathbf{R}^+;(L^2(\Omega))^4)$, Ω *a bounded open set in* $\mathbf{R}^3$ *with a large aspect ratio,* $\frac{L}{h} >> \frac{g\alpha^2(T_1-T_2)^2L^3}{\nu\kappa}$, *where* L *is the width (radius) and* h *is the height of* Ω; *there exist a time* t_0, *such that there exists a unique strong solution* $(u,\ \theta)$ *in* $C([t_0,\infty);(H^1(\Omega))^4)$, $t \ge t_0$, *with initial data* $u(t_0) = u_w(t_0)$ *and* $\theta(t_0) = \theta_w(t_0)$. *Moreover, the system (1.4) possesses a global attractor in* $(L^2(\Omega))^4$.

Proof. The subscript w indicates that the weak solutions are only weakly continuos in t. We multiply the second equation in (1.4) above by θ and treat it in the same way as u in the proof of Lemma 2.1 to get

$$\frac{1}{2}\frac{d}{dt}|\theta(t)|^2 + \kappa|\nabla\theta(t)|^2 \le \frac{(T_1-T_2)}{h}|\theta(t)||u_n(t)|,$$

and by adding and subtracting $\beta|\theta|$ and applying Poincaré's inequality, we get

$$\frac{d}{dt}|\theta(t)| + \beta|\theta(t)| + (\lambda_1^{1/2}\kappa - \frac{\beta}{\lambda_1^{1/2}})|\nabla\theta(t)| \le \frac{(T_1-T_2)}{h}|u_n(t)|$$

$$\leq \frac{(T_1 - T_2)}{h}(|u_0|e^{-\lambda_1 \nu t} + \frac{K h^{1/2}}{\lambda_1 \nu}(1 - e^{-\lambda_1 \nu t})) \leq c_1 + c_2 e^{-\lambda_1 \nu t}.$$

Then integrating with respect to t, we get

$$(\lambda_1^{1/2}\kappa - \frac{\beta}{\lambda_1^{1/2}}) \int_0^t e^{-\beta(t-s)}|\nabla\theta(s)|ds$$

$$\leq |\theta(0)|e^{-\beta t} + \frac{c_1}{\beta}(1 - e^{-\beta t}) + c_2 \frac{(e^{-\beta t} - e^{-\lambda_1 \kappa t})}{(\lambda_1 \kappa - \beta)}.$$

Next we multiply the u equation in (1.4) by Δu and integrate over Ω. By integration by parts and Schwarz's inequality

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 \leq g\alpha|\nabla\theta(t)||\nabla u_n(t)| + |u(t)\cdot\nabla u(t)||\Delta u(t)|$$

$$\leq g\alpha|\nabla\theta(t)||\nabla u_n(t)| + C(1 + \lambda_1^{-1} + \lambda_1^{-2})^{3/8}|u(t)|^{1/4}|\nabla u(t)||\Delta u(t)|^{7/4},$$

by the Gagliardo-Nirenberg inequalities, where we have used that the H^2 Sobolev norm is bounded by

$$\|u\|_2 \leq C(|u(t)|^2 + |\nabla u(t)|^2 + |\Delta u(t)|^2)^{1/2} \leq C(1 + \lambda_1^{-1} + \lambda_1^{-2})^{1/2}|\Delta u(t)|,$$

by Poincaré's inequality. We use Young's inequality to eliminate $|\Delta u|^{7/4}$, namely

$$((\frac{\nu}{2})^7 C_0)^{1/8}|u|^{1/4}|\nabla u||\Delta u|^{7/4} \leq C_0|u|^2|\nabla u|^8 + \frac{\nu}{2}|\Delta u|^2$$

so

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \frac{\nu}{2}|\Delta u(t)|^2 \leq g\alpha|\nabla\theta(t)||\nabla u(t)| + C_0|u(t)|^2|\nabla u(t)|^8$$

and by Poincaré's inequality

$$\frac{d}{dt}|\nabla u(t)| + \frac{\lambda_1 \nu}{2}|\nabla u(t)| \leq g\alpha|\nabla\theta(t)| + C_0|u(t)|^2|\nabla u(t)|^7.$$

Then we integrate the equation

$$\frac{d}{dt}|\nabla u(t)| + \beta|\nabla u(t)| - C_0|u(t)|^2|\nabla u(t)|^7 \leq g\alpha|\nabla\theta(t)|,$$

where $\beta = \lambda_1\nu/2$. We integrate from the initial time t_0, to get

$$|\nabla u(t)| - C_0 \int_{t_0}^t |u(s)|^2|\nabla u(s)|^7 e^{-\beta(t-s)}ds$$

$$\leq c_1 + c_2 e^{-\beta(t-t_0)} + c_3 e^{-\lambda_1\nu(t-t_0)},$$

by the above inequality for $\int |\nabla\theta|dt$. Now if $|\nabla u(s)|$ attains its maximum on the interval $[t_0, t]$ at $s = t$, then

$$\int_{t_0}^t |u(s)|^2|\nabla u(s)|^7 e^{-\beta(t-s)}ds \leq |\nabla u(t)|^7 \int_{t_0}^t |u(s)|^2 e^{-\beta(t-s)}ds$$

$$\leq \frac{|\nabla u(t)|^7}{2} \int_{t_0}^{t} \left(|u_0|^2 e^{-3\beta s} e^{-\beta t} + \frac{K^2 h}{\lambda_1^2 \nu^2} (1 - e^{-2\beta s})^2 e^{-\beta(t-s)} \right) ds$$

$$\leq \frac{|\nabla u(t)|^7}{6\beta} \left(|u_0|^2 (e^{-3\beta t_0} - e^{-3\beta t}) e^{-\beta t} + 3 \frac{K^2 h}{\lambda_1^2 \nu^2} (1 - e^{-\beta(t-t_0)}) \right),$$

where we have used the bound

$$|u(t)| \leq |u_0| e^{-\lambda_1 \nu t} + \frac{K h^{1/2}}{\lambda_1 \nu} (1 - e^{-\lambda_1 \nu t}),$$

from Lemma 3.2 and the inequality $ab \leq a^2/2 + b^2/2$. Thus, if we put $|\nabla u(t)| = v(t)$, we obtain the inequality

$$v(t) - a v^7(t) \leq M.$$

The constants a and M are,

$$a = \frac{K^2 h}{2\lambda_1^2 \nu^2 \beta} = g^2 \alpha^2 \frac{(T_1 - T_2)^2}{6\lambda_1^2 \nu^2 \beta} L^2 h$$

and

$$M = 2c_1 = 2 \frac{(T_1 - T_2) K}{h^{1/2} \lambda_1 \nu} = 2 g \alpha (T_1 - T_2)^2 L / 3^{1/2} \lambda_1 \nu h^{1/2}.$$

This means that for a large aspect ratio $L/h >> \frac{g\alpha^2 (T_1 - T_2)^2 L^3}{\nu \kappa}$, a becomes small. Now we repeat the arguments from the proof of Theorem 2.2 and conclude that the function $F(v) = v - a v^7$ is concave and attains its maximum at $v_{max} = 1/(7a)^{1/6}$. This maximum is

$$F(v_{max}) = \frac{6}{7} \frac{1}{(7a)^{1/6}},$$

so that $v(t)$ can not escape beyond its maximum, see Figure 1. Moreover arguing as in Theorem 2.2 we conclude that the derivative of F is positive at the point where F first reaches M and this gives us the bound

$$v \leq \frac{7M}{6}.$$

The last step is to get a bound for $|\nabla \theta|$. We multiply the θ equation in (1.4) by $\Delta \theta$, to get

$$\frac{1}{2} \frac{d}{dt} |\nabla \theta(t)|^2 + \kappa |\Delta \theta(t)|^2 \leq \frac{(T_1 - T_2)}{h} |\nabla \theta(t)| |\nabla u(t)| + |u(t) \cdot \nabla \theta(t)| |\Delta \theta(t)|$$

by Schwarz's inequality

$$\leq \frac{(T_1 - T_2)}{h} |\nabla \theta(t)| |\nabla u(t)| + |u|_6 |\nabla \theta(t)|_3 |\Delta \theta(t)|,$$

by Hölder's inequality

$$\leq \frac{(T_1 - T_2)}{h} |\nabla \theta(t)| |\nabla u(t)| + C(|\nabla u(t)|^2 |\theta| / \delta^3 + \delta |\Delta \theta|) |\Delta \theta(t)|,$$

by Poincaré's and Sobolev's inequalities, because

$$|u|_6 \leq K \|u\| \leq C |\nabla u|,$$

as above, and because,

$$|\nabla\theta|_3 \le C\|\nabla\theta\|_{1/2} \le K(\frac{|\nabla\theta|}{\delta} + \delta|\Delta\theta|) \le C(\frac{|\theta|}{\delta^3} + \delta|\Delta\theta|)$$

by two applications of interpolation, where δ is small. Thus

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2 + \kappa|\Delta\theta(t)|^2 \le \frac{(T_1 - T_2)}{h}|\nabla\theta(t)||\nabla u(t)| + \frac{C^2|\nabla u(t)|^4|\theta|^2}{\delta^6} + 2\delta|\Delta\theta(t)|^2,$$

by Young's inequality. Now moving the $|\Delta\theta(t)|^2$ term over to the left hand side of the inequality and applying Poincaré's inequality, we get

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2 + \lambda_1(\kappa - 3\delta)|\nabla\theta(t)|^2 \le \frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2 + (\kappa - 3\delta)|\Delta\theta(t)|^2 \le c_3,$$

since

$$\frac{(T_1 - T_2)}{h}|\nabla\theta(t)||\nabla u(t)| \le \lambda_1\delta|\nabla\theta(t)|^2 + \frac{(T_1 - T_2)^2}{4h^2\lambda_1\delta}|\nabla u(t)|^2,$$

by the inequality $ab \le a^2 + b^2/4$. Thus

$$|\nabla\theta(t)|^2 \le |\nabla\theta(t_0)|^2 e^{-\gamma(t-t_0)} + (2c_3/\gamma)(e^{-\gamma t_0} - e^{-\gamma t}),$$

where $\gamma = 2(\kappa - 3\delta)$.

We can now put the estimates for $|\nabla u|$ and $|\nabla\theta|$ together to get an absorbing set

$$|\nabla u(t)| + |\nabla\theta(t)| \le constant + \epsilon,$$

where ϵ is arbitrarily small for t large enough. Namely, by Lemma 3.2, there exists a t_j, such that $(u, \theta)(t_j) \in H^1(\Omega)$ and the global bound on the H^1 norm holds for $t = t_0 = t_j$. Combined with the local existence Theorem 3.1, the a priori bound above now gives the existence of a global solution and an absorbing set in $H^1(\Omega)$, for $t \ge t_0$. The existence of a global attractor in $(L^2(\Omega))^4$ then follows from the compact embedding of $(H^1(\Omega))^4$ in $(L^2(\Omega))^4$, see Hale [36], Babin and Vishik [37], Temam [33] and Birnir and Grauer [38]. ■

Corollary 3.1. *The Boussinesq system (1.4) has a maximal $\mathcal{B}$-attractor whose basin is a prevalent set in* $(L^2(\mathbf{R}^n)^{n+1})$, $n = 2, 3$.

The proof is a straight-forward application of Theorem 2.3

Theorem 3.3 says that in experiments in pattern formation where one has a large aspect ratio, it is only necessary to wait a short time to have global spatially smooth solutions. The smoothness follows from the smoothness of the nonlinear semigroup, see Kreiss and Lorentz [32]. Corollary 3.1 says that what is observed in the experiments after the initial settling-down time, is a $\mathcal{B}$-attractor that is a component of the maximal $\mathcal{B}$-attractor. In other word this attractor is observed for an open set of initial conditions. (However, not necessarily an open set of parameter values.) It is then useful to know how long one has to wait and this time is easily estimated. It is the time it takes the exponentially decaying initial data to become comparable in size with the L^2 absorbing set. Namely,

$$t = \frac{1}{\lambda_1\nu}\ln\left[\frac{\lambda_1\nu|u_0|}{Kh^{1/2}}\right] = \frac{1}{\lambda_1\nu}\ln\left[\frac{3^{1/2}\lambda_1\nu|u_0|}{g\alpha(T_1 - T_2)Lh^{1/2}}\right].$$

In experiments for Prandtl numbers close to one, this time is measured in hours for experiments that take days [2].

4 Scaling and expansions

As a starting point we introduce a small "scaling" parameter $\epsilon > 0$ and consider the scaled system

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^\gamma \nu \Delta u_\epsilon + \nabla p_\epsilon = g\alpha(T_\epsilon - T_2), \\ \dfrac{\partial T_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) T_\epsilon - \epsilon^\gamma \kappa \Delta T_\epsilon = 0, \\ \operatorname{div} u_\epsilon = 0, \end{cases} \quad x \in \Omega,\ t \in \mathbf{R}^+, \tag{4.1}$$

equipped with initial data

$$u_\epsilon(x,0) = a(x) \ \text{ and } \ T_\epsilon(x,0) = b(x).$$

The boundary conditions are periodic on the vertical surfaces $x_k = 0, l$ for $k < n$. Furthermore, on the horizontal surfaces $x_n = 0$ and $x_n = h$, the velocity u_ϵ vanishes (no-slip condition) and the temperature T_ϵ is kept constant, i.e.,

$$T_\epsilon = T_1 \ \text{ on } \ x_n = 0, \ \ T_\epsilon = T_2 \ \text{ on } \ x_n = h.$$

A small value of ϵ corresponds to a high Reynolds number and the solutions will likely become turbulent. We also consider the scaled system corresponding to (1.4), i.e.,

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^\gamma \nu \Delta u_\epsilon + \nabla p_\epsilon = g\alpha e_n \theta_\epsilon, \\ \dfrac{\partial \theta_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) \theta_\epsilon - \epsilon^\gamma \kappa \Delta \theta_\epsilon = \dfrac{(T_1 - T_2)}{h}(u_\epsilon)_n, \\ \operatorname{div} u_\epsilon = 0, \end{cases} \quad x \in \Omega,\ t \in \mathbf{R}^+. \tag{4.2}$$

For the temperature θ_ϵ we get initial data $\theta_\epsilon(x,0) = \theta_\epsilon^0(x)$ and the new boundary data $\theta_\epsilon = 0$ at $x_n = 0$ and at $x_n = h$. The initial and boundary data for u_ϵ remain unchanged. Moreover, u_ϵ and θ_ϵ and their gradients and p_ϵ are periodic as above.

It turns out that the value $\gamma = 3/2$ is critical. To see that we perform a multiple scales expansion technique of the unknown quantities u_ϵ, p_ϵ and θ_ϵ and assume that

$$u_\epsilon(x,t) = \epsilon^\rho \sum_{i=0}^{\infty} \epsilon^i u_i(x, \frac{x}{\epsilon^\mu}, t, \frac{t}{\epsilon^\beta}),$$

$$p_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i p_i(x, \frac{x}{\epsilon^\mu}, t, \frac{t}{\epsilon^\beta}),$$

$$\theta_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i \theta_i(x, \frac{x}{\epsilon^\mu}, t, \frac{t}{\epsilon^\beta}),$$

where u_i, p_i and θ_i are all assumed to be T^n-periodic with respect to $y \in \mathbf{R}^n$, $n = 2, 3$, T^n being the usual unit torus in $\mathbf{R}^n$. If we put $y = x/\epsilon^\mu$ and $\tau = t/\epsilon^\beta$, the chain rule transforms the differential operators as

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{1}{\epsilon^\beta}\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{1}{\epsilon^\mu}\frac{\partial}{\partial y}.$$

The question is: Can we preserve the structure from the original problem in the leading order approximation? One sees that if the viscosity and conductivity terms scales like $\epsilon^{3/2}$, then the choices $\mu = 1$ and $\rho = \beta = 1/2$ preserve all quantities. By substituting all this into the system (4.2) we can equate the powers of ϵ. This formal manipulation shows that the functions u_i and θ_i, $i = 0, 1, ...$ are all independent of t and that p_0 is independent of y. Formally this means that there are two scales in space and in order to preserve the evolutionary behaviour, in the leading order approximation, a chance of time scale $t \mapsto \frac{t}{\epsilon^{1/2}} = \tau$ becomes necessary, c.f. Lions [45]. It also follows, from the equations, that the functions p_i are independent of t. The leading order system reads

$$\begin{cases} \dfrac{\partial u_0}{\partial \tau} + (u_0 \cdot \nabla_y) u_0 - \nu \Delta_y u_0 + \nabla_y p_1 = g\alpha e_n \theta_0 - \nabla_x p_0, \\ \dfrac{\partial \theta_0}{\partial \tau} + (u_0 \cdot \nabla_y)\theta_0 - \kappa \Delta_y \theta_0 = 0, \\ \mathrm{div_y} u_0 = 0, \ \ \mathrm{div_x}(\displaystyle\int_{T^n} u_0 dy) = 0, \end{cases} \tag{4.3}$$

on $\Omega \times T^n \times \mathbf{R}^+$ with T^n-periodicity in y.

We see that the system (4.3) differs from the system (1.4) only by the additional forcing term coming from the "global" pressure gradient. Thus the local problem (on the ϵ-scale) has *two pressure gradients*, one local $\nabla_y p_1$ and one global $\nabla_x p_0$.

Remark 4. The scaling discussed above is isotropic but the convection rolls in Rayleigh-Bénard convection have a preferred direction, say along the x_2-axis. This commands the use of anisotropic scaling for the amplitude equations and it turns out that the scaling by $\epsilon^{3/2}$ is the one pertaining to the other directions, perpendicular to the orientation of the rolls. The details of this affine scaling will be spelled out elsewhere.

4.1 Existence results

In three dimensions the following global existence result holds true:

Theorem 4.1. *Consider the Boussinesq system (4.2). For every fixed value of $\epsilon > 0$ the following holds true: For every weak solution $u_{w,\epsilon}$, $\theta_{w,\epsilon}$ in $C_w(\mathbf{R}^+; (L^2)^4(\Omega))$, Ω a bounded open set in $\mathbf{R}^3$ with a large aspect ratio, $\frac{L}{h} >> \frac{g\alpha^2(T_1-T_2)^2L^3}{\nu\kappa}$, where L is the length and h is the height of Ω; there exists a time t_0, such that there exists a unique strong solution u_ϵ, θ_ϵ in $C([t_0, \infty); (H^1)^4(\Omega))$, $t \geq t_0$, with initial data $u_\epsilon(t_0) = u_{w,\epsilon}(t_0)$ and $\theta_\epsilon(t_0) = \theta_{w,\epsilon}(t_0)$. Moreover, the system (4.2) possesses a global attractor in $(L^2)^4(\Omega)$.*

Proof. Theorem 4.1 follows from Theorem 3.3 if we set the viscosity and heat conductivity in Theorem 3.3 equal to $\epsilon^{3/2}\nu$ and $\epsilon^{3/2}\kappa$, respectively.

Remark 5. The existence of a pressure $p_\epsilon \in C([t_0, \infty); (H^1)(\Omega)/\mathbf{R})$ follows by the same arguments as those in Remark 2 after the proof of Theorem 2.2.

We conclude with the existence result for the two-scale system (4.3).

Theorem 4.2. *Consider the system (4.3) with initial data $u_0(x, y, 0) = a(x, y)$ and $\theta_0(x, y, 0) = b(x, y)$, with zero mean over T^n. For almost every $x \in \Omega$ the system* *has a unique global strong solution* $u_0 \in C([t_0,\infty);(H^1(T^n))^n)$, $n = 2,3$, *and* $\theta_0 \in C([t_0,\infty);H^1(T^n))$. *Moreover,* $p_1 \in C([t_0,\infty);H^1(T^n)/\mathbf{R})$. *Integrated over* T^n, $(u_0)_i$, θ_0 *and* p_1 *all belong to* $L^2(\Omega)$. *Finally,* $p_0 \in C([t_0,\infty);H^1(\Omega))$. *In two dimensions* $t_0 = 0$ *and in three dimensions* $t_0 = t_0(|a|,|b|) > 0$, *in general. Moreover for almost every* $x \in \Omega$, *the system (4.3) possesses a global attractor in* $(L^2(T^n))^n \times L^2(T^n)$, $n = 2, 3$.

The proof in the two-dimensional case is relatively straightforward and details can be found in Foias et al. [26].

Proof. In the three-dimensional case the proof is similar to the proof of Theorem 3.3 but simpler. First we multiply the first equation in (4.3) by θ and integrate by parts to get

$$|\theta_0(t)| \leq |b|e^{-\gamma t}$$

after integration in t, where $\gamma = \lambda_1\kappa$. In [46] we show that

$$g\alpha e_3\theta_0 - \nabla_x p_0 = g\alpha\pi_2(e_3\theta_0),$$

where $\pi_2(e_3\theta_0)$ denotes the projection onto the divergence free part of $e_3\theta_0$. Hence,

$$|g\alpha e_3\theta_0 - \nabla_x p_0| \leq g\alpha|\theta_0| \leq g\alpha|b|e^{-\gamma t}.$$

This means that for t sufficiently large the forcing in the first equation of (4.3) becomes small. Thus we obtain the existence of unique global solution $u_0 \in C([t_0,\infty);(H^1(T^n))^3)$ to the first equation of (4.3), by Theorem 2.2, and an absorbing set in this space. Since $u_0(t) \in (H^1(T^n))^3$ we also obtain global existence for θ_0 in the second equation of (4.3), just as in the proof of Theorem 3.3 and the existence of an absorbing set in this space as well. The existence of the attractors follows from the fact that the equations possess an absorbing set in $(H^1(T^n))^3 \times H^1(T^n)$ which is compactly embedded in $(L^2(T^n))^3 \times L^2(T^n)$. ■

Remark 6. In Section 7, we prove, by using homogenization theory, that the solutions of the scaled system (4.2) converges in the two-scale sense, see [28], to the unique solution to the system (4.3).

Remark 7. By averaging the system (4.3) over the unit torus in local variable y we obtain the mean-field corresponding to the scaled system (4.2) in Section 8. The understanding of the effects of this field on the system is crucial in the theoretical understanding of the complex patterns in Rayleigh-Bénard convection.

5 A priori estimates

We are interested in the asymptotic behaviour of the system (4.2) as $\epsilon \to 0$. In order to accomplish this we need to establish uniform (in ϵ) bounds on u_ϵ, p_ϵ and θ_ϵ.

This is problematic, since the forcing term in the Navier-Stokes equation in (4.2) involves θ_ϵ which can be highly oscillatory for small values of ϵ. We begin with the following:

Lemma 5.1. *Let u, p and θ be the solution to the system*

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = e_n\theta, \\ \dfrac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta - \kappa\Delta\theta_\epsilon = (u)_n, \\ \operatorname{div} u = 0, \end{cases} \qquad x\in\Omega,\ t\in\mathbf{R}^+. \tag{5.1}$$

Suppose that u, p and θ are all T^n periodic. If

$$\int_{T^n} u_i(y,0)dy = \int_{T^n}\theta(y,0)dy = 0,$$

then

$$\int_{T^n} u_i(y,t)dy = \int_{T^n}\theta(y,t)dy = 0,$$

for all $t>0$.

Proof. By the equation (5.1) we have

$$\int_{T^n}(\frac{\partial u}{\partial t})_i dy = \int_{T^n}((e_n\theta)_i - ((u\cdot\nabla)u)_i + \nu(\Delta u)_i - (\nabla p)_i)dy$$

and

$$\int_{T^n}(\frac{\partial \theta}{\partial t})dy = \int_{T^n}((u)_n - (u\cdot\nabla)\theta + \kappa\Delta\theta)dy.$$

All terms involving spatial derivatives will vanish by the T^n-periodicity on the unit torus and by the incompressibility of u. Moreover, $(e_n\theta)_i = 0$, for $i<n$, so the system reduces to

$$\frac{d}{dt}\int_{T^n}(u)_n dy = \int_{T^n}\theta dy$$

and

$$\frac{d}{dt}\int_{T^n}\theta dy = \int_{T^n}(u)_n dy.$$

We solve this system of ODE's with the initial condition

$$\int_{T^n} u_i(y,0)dy = \int_{T^n}\theta(y,0)dy = 0,$$

to get the trivial solution,

$$\int_{T^n} u_i(y,t)dy = \int_{T^n}\theta(y,t)dy = 0,$$

for all $t>0$. ∎

We are clearly making a strong assumption assuming that the solution is periodic in the above Lemma. In the applications we have in mind it will not be exactly periodic but it will be sufficiently oscillatory so that periodicity on a small scale is a reasonable model. Flow in porous media is obviously one example of such a situation but others include, a mixture of hot and cold fluid, a two fluid mixture (oil and water), a mixture of snow and air, or water and sediment, and a turbulent fluid. This hypothesis is similar to the model one uses in homogenization of materials and the test of the hypothesis is how well one captures averaged quantities.

The next lemma is a Poincaré inequality, c.f. L. Tartar (Lemma 1 in the Appendix of [47]). Our version of the lemma differs from Tartar's in that we do not have a vanishing boundary condition on the local tori. Instead we benefit from the result of Lemma 5.1, that the mean value vanishes.

Lemma 5.2. *Let $u_i \in L^2([0,s];H^1(\Omega))$ and assume that u is periodic and that initially u has mean value zero over the unit torus T^n, then*

$$\int_0^s \int_\Omega |u(x,t)|^2 dxdt \leq \epsilon^2 C \int_0^s \int_\Omega |\nabla u(x,t)|^2 dxdt, \tag{5.2}$$

where C is a constant independent of ϵ.

Proof. We apply the Poincaré inequality on T^n which, by the result of Lemma 5.1, gives

$$\int_{T^n} |u(y,t)|^2 dy \leq C_1 \int_{T^n} |\nabla_y u(y,t)|^2 dy.$$

A change of variables $x = \epsilon y$ yields

$$\int_{\epsilon T^n} |u(x,t)|^2 dx \leq \epsilon^2 C_1 \int_{\epsilon T^n} |\nabla_x u(x,t)|^2 dx.$$

We note that the constant C_1 will be the same for all ϵT^n-cubes in the interior of Ω. For ϵT^n-cubes intersecting $\partial\Omega$, u will vanish at, at least, one point and the usual Poincaré inequality applies. Let C_2 denote the maximum of all the constants for these cubes. A summation over all ϵT^n-cubes gives

$$\int_\Omega |u(x,t)|^2 dx \leq C\epsilon^2 \int_\Omega |\nabla_x u(x,t)|^2 dx,$$

where $C = max\{C_1, C_2\}$. Finally, an integration with respect to t gives the desired inequality. ■

Lemma 5.3. *Let u_ϵ and θ_ϵ satisfy (2.1) and assume that the initial data in (4.2) is bounded independent of ϵ, then*

$$\epsilon^{-1/2} \|u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))} \leq C \tag{5.3}$$

and

$$\epsilon^{1/2} \|\nabla u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))} \leq C, \tag{5.4}$$

for any $s > 0$, where C is independent of ϵ.

In the two-dimensional case $t_0 = 0$ and in the three-dimensional case $t_0 > 0$ in general. This will hold true throughout the paper.

Proof. Let us consider the first equation in (4.2). We multiply by u_ϵ and integrate over $\Omega\times]t_0,s[$. By the incompressibility we get, by using the Schwarz's inequality

$$\frac{1}{2}|u_\epsilon(s)|^2+\epsilon^{3/2}\|\nabla u_\epsilon\|^2_{L^2([t_0,s];L^2(\Omega))}$$
$$\leq \|\theta_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}\|u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}+\frac{1}{2}|u_\epsilon(t_0)|^2.$$

By Lemma 3.1, see also the proof of Theorem 3.2, $\|\theta_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}$ is bounded, and assuming that t_0 is a time where $|\nabla u|$ is finite, see Lemma 2.1, we can absorb the term $\frac{1}{2}|u_\epsilon(t_0)|^2$ into the time integral using Lemma 5.2. This gives the estimate,

$$\frac{1}{2}|u_\epsilon(s)|^2+\epsilon^{3/2}\|\nabla u_\epsilon\|^2_{L^2([t_0,s];L^2(\Omega))}\leq C\|u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}$$

so that,

$$\epsilon^{3/2}\|\nabla u_\epsilon\|^2_{L^2([t_0,s];L^2(\Omega))}\leq C\|u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}.$$

Now we recall that $u_\epsilon\in L^2([t_0,s];H^1(\Omega))$ by Lemma 3.2 (or Lemma 2.1). Thus, by Lemma 5.2,

$$\epsilon^{1/2}\|\nabla u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}\leq C.$$

An application of Lemma 5.2 once again gives the desired result, i.e.,

$$\epsilon^{-1/2}\|u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}\leq C.$$

If the initial data $u_\epsilon(t_0)$ and $\theta_\epsilon(t_0)$ is bounded, in $L^2(\Omega)$, independent of ϵ then C will also be independent of ϵ. ∎

We continue with a few consequences of the above results:

Corollary 5.1. *Consider the first equation in (4.2). The convection term is bounded,*

$$\|(u_\epsilon\cdot\nabla)u_\epsilon\|_{L^2([t_0,s];L^1(\Omega))}\leq C, \tag{5.5}$$

where C is independent of ϵ.

Proof. By the Schwarz's inequality, Lemma 3.2 and Lemma 5.3 it immediately follows that

$$\|u_\epsilon\cdot\nabla u_\epsilon\|_{L^2([t_0,s];L^1(\Omega))}\leq\epsilon^{-1/2}\|u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}\epsilon^{1/2}\|\nabla u_\epsilon\|_{L^2([t_0,s];L^2(\Omega))}\leq C^2.$$ ∎

Corollary 5.2. *Consider the first equation in (4.2). The time derivative is bounded,*

$$\|\frac{\partial u_\epsilon}{\partial t}\|_{L^2([t_0,s];L^2(\Omega))}\leq C, \tag{5.6}$$

and the pressure is bounded,

$$\|p_\epsilon\|_{L^2([t_0,s];H^1(\Omega)/\mathbf{R})}\leq C, \tag{5.7}$$

where C is independent of ϵ.

Proof. (5.6) follows from duality by Corollary 5.1 and (5.7) follows by the Remark 2. ∎

6 Two-scale convergence

In this section we recall the technically useful concept of two-scale convergence ([31] and [28]), for the case when the functions also depend on a time variable. Let us consider the space $C^\infty(T^n)$ of smooth periodic functions, with unit period, in $\mathbf{R}^n$.

Definition 6.1. A sequence $\{u_\epsilon\}$ in $L^2([0,s];L^2(\Omega))$ is said to two-scale converge to $u_0 = u_0(x,y,\tau)$ in $L^2([0,s];L^2(\Omega\times T^n))$ if, for any $\varphi \in C_0^\infty(\Omega\times[0,s];C^\infty(T^n))$,

$$\int_0^s\int_\Omega u_\epsilon(x,\tau)\varphi(x,\frac{x}{\epsilon},\tau)dxd\tau \to \int_0^s\int_\Omega\int_{T^n} u_0(x,y,\tau)\varphi(x,y,\tau)dydxd\tau, \tag{6.1}$$

as $\epsilon\to 0$.

We have the following extension of a compactness result first proved by Nguetseng [28] and then further developed by Allaire in [29, 30, 31] and by Holmbom in [48].

Theorem 6.1. *Suppose that $\{u_\epsilon\}$ is a uniformly bounded sequence in $L^2([0,s];L^2(\Omega))$. Then there exists a subsequence, still denoted by $\{u_\epsilon\}$, and a function $u_0 = u_0(x,y,\tau)$ in $L^2([0,s];L^2(\Omega\times T^n))$, such that $\{u_\epsilon\}$ two-scale converges to u_0.*

The relation between $\tilde{u}$ and u_0 is explained in the next theorem. In fact, by choosing test functions which do not depend on y we have the following (see e.g. [29]):

Theorem 6.2. *Suppose that $\{u_\epsilon\}$ two-scale converges to u_0, where $u_0 = u_0(x,y,\tau)$ in $L^2([0,s];L^2(\Omega\times T^n))$, then $\{u_\epsilon\}$ converges to $\overline{u_0}$ weakly in $L^2([0,s];L^2(\Omega))$, where*

$$\overline{u_0}(x,\tau) = \int_{T^n} u_0(x,y,\tau)dy.$$

In other words:

$$\tilde{u} = \overline{u_0}.$$

Remark 8. The results of Theorem 6.1 and Theorem 6.2 remain valid for the larger class of *admissible* test functions $\varphi \in L^2([0,s]\times\Omega;C(T^n))$, see e.g. [29].

7 Homogenization

With the help of the results from Section 4, Section 5 and the two-scale compactness result Theorem 6.1 from the previous section we can now state the main results of the paper.

Theorem 7.1. *Consider the Navier-Stokes system (4.2). Suppose that the initial data $u_\epsilon(x,0) = u_\epsilon^0(x)$ and $\theta_\epsilon(x,0) = \theta_\epsilon^0(x)$ two-scale converge to unique limits $u^0(x,y)$ and $\theta^0(x,y)$, respectively. Then, as $\epsilon\to 0$, the following quantities two-scale converge,*

$$\epsilon^{-1/2}u_\epsilon \to u_0,$$

$$p_\epsilon \to p_0,$$

$$\theta_\epsilon \to \theta_0,$$

where $p_0 = p_0(x,\tau)$. Moreover, there exists a function $p_1 = p_1(x,y,\tau)$ such that

$$\nabla p_\epsilon \to \nabla_x p_0 + \nabla_y p_1,$$

the functions u_0, p_0, p_1 and θ_0, being the unique solutions to the Navier-Stokes system (4.3) with initial data $u_0(x,y,0) = u^0(x,y)$, $\theta_0(x,y,0) = \theta^0(x,y)$ and boundary data T^n-periodic in the variable y.

Before we prove the theorem we state a corollary

Corollary 7.1. *Consider the Navier-Stokes system (4.2). The following quantities two-scale converge,*

$$\epsilon^{-1/2} u_\epsilon \to u_0,$$

$$p_\epsilon \to p_0,$$

$$\nabla p_\epsilon \to \nabla_x p_0 + \nabla_y p_1,$$

$$T_\epsilon \to T_0,$$

where u_0, p_0, p_1 and T_0 are the unique solution to the Navier-Stokes system (4.3).

Proof of Theorem 7.1. By assumption the initial data admit unique two-scale limits. Consider now the first equation of (4.2),

$$\frac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^{3/2} \nu \Delta u_\epsilon + \nabla p_\epsilon = e_2 \theta_\epsilon, \quad \text{in } \Omega \times \mathbf{R}^+.$$

Let $s > 0$ and choose test functions $\varphi \in C_0^\infty(\Omega \times [t_0,s]; C^\infty(T^n))$. By the results of Section 5, all the sequences $\{\epsilon^{-1/2} u_\epsilon\}$, $\{\frac{\partial u_\epsilon}{\partial t}\}$, $\{p_\epsilon\}$ and $\{\theta_\epsilon\}$ are uniformly bounded in $L^2([t_0,s]; L^2(\Omega))$. Therefore, according to Theorem 6.1, they admit two-scales limits. In order to identify these limits we multiply each of these terms by the smooth compactly supported test function $\varphi(x, \frac{x}{\epsilon}, \tau)$. For the time derivative we get (recall that $\tau = t/\sqrt{\epsilon}$)

$$\int_0^s \int_\Omega \frac{\partial u_\epsilon}{\partial t}(x,t) \varphi(x, \frac{x}{\epsilon}, \tau) dx d\tau = \epsilon^{-1/2} \int_0^s \int_\Omega u_\epsilon \frac{\partial \varphi}{\partial \tau} dx d\tau.$$

Sending $\epsilon \to 0$ and integrating by parts yield,

$$\int_0^s \int_\Omega \int_{T^n} \frac{\partial u_0}{\partial \tau} \varphi(x,y,\tau) dy dx d\tau.$$

For the second (inertial) term we consider

$$\int_0^s \int_\Omega [(u_\epsilon \cdot \nabla u_\epsilon(x,\tau) - u_0 \nabla_y u_0(x,y,\tau)] \varphi(x, \frac{x}{\epsilon}, \tau) dx d\tau$$

$$= \int_0^s \int_\Omega [\epsilon^{-1/2} u_\epsilon \cdot (\epsilon^{1/2} \nabla u_\epsilon - \nabla_y u_0) + (\epsilon^{-1/2} u_\epsilon - u_0) \nabla_y u_0] \varphi(x, \frac{x}{\epsilon}, \tau) dx d\tau.$$

By considering $\nabla_y u_0 \varphi(x, \frac{x}{\epsilon}, \tau)$ to be a test function in the second term on the right hand side this term immediately passes to zero in the two-scale sense. For the first term the Schwarz's inequality yields

$$\int_0^s \int_\Omega \epsilon^{-1/2} u_\epsilon \cdot (\epsilon^{1/2} \nabla u_\epsilon - \nabla_y u_0) \cdot \varphi(x, \frac{x}{\epsilon}, \tau) dx d\tau$$

$$\leq \|\epsilon^{-1/2}u_\epsilon\| \|(\epsilon^{1/2}\nabla u_\epsilon - \nabla_y u_0)\varphi(x, \frac{x}{\epsilon}, \tau)\|$$

$$\leq C\|(\epsilon^{1/2}\nabla u_\epsilon - \nabla_y u_0)\varphi(x, \frac{x}{\epsilon}, \tau)\|,$$

where all the norms are in $L^2([0,s];L^2(\Omega))$. In order to pass to the limit in the right hand side we consider the usual L^2-mollifications $\nabla u_{\epsilon,\mu}$ and $\nabla_y u_{0,\mu}$ of $\epsilon^{1/2}\nabla u_\epsilon$ and $\nabla_y u_0$, respectively. Since the mollified functions pass strongly to $\epsilon^{1/2}\nabla u_\epsilon$ and $\nabla_y u_0$, respectively, as $\mu \to 0$, we have, for μ sufficiently small, say $\mu \leq \mu_0$ and for every y in T^n

$$\|(\epsilon^{1/2}\nabla u_\epsilon - \nabla_y u_0)\varphi\| \leq \|(\nabla u_{\epsilon,\mu} - \nabla_y u_{0,\mu})\varphi\| + \delta,$$

where δ is arbitrarily small, independently of ϵ. This inequality still holds true if we take the supremum in y over T^n. Thus, for every $\mu \leq \mu_0$, the right hand side will tend to zero as ϵ tends to 0, by the uniqueness of the two-scale limit $\nabla_y u_0$ of $\epsilon^{1/2}\nabla u_\epsilon$. Consequently, we have proved that

$$u_\epsilon \nabla u_\epsilon \to u_0 \nabla_y u_0$$

in the two-scale sense. For the third term we get, by the divergence theorem,

$$-\epsilon^{3/2}\nu \int_0^s \int_\Omega \Delta u_\epsilon(x,\tau)\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau$$

$$= -\epsilon^{-1/2}\nu \int_0^s \int_\Omega u_\epsilon(x,\tau)\Delta_y\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau + \text{ terms tending to zero as } \epsilon \to 0.$$

Sending $\epsilon \to 0$ yields, after applying the divergence theorem again,

$$-\nu \int_0^S \int_\Omega \int_{T^n} \Delta_y u_0 \varphi(x,y,\tau)dydxd\tau.$$

For the right hand side we immediately get

$$\int_0^s \int_\Omega e_2\theta_\epsilon(x,\tau)\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau \to \int_0^s \int_\Omega \int_{T^n} e_2\theta_0\varphi(x,y,\tau)dydxd\tau.$$

For the fourth term (the pressure) we have to be a bit more careful. Let us multiply the first equation of (4.2) by $\epsilon\varphi(x, \frac{x}{\epsilon}, \tau)$. For the pressure term we get

$$\epsilon \int_0^s \int_\Omega \nabla p_\epsilon(x,\tau)\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau = \int_0^s \int_\Omega p_\epsilon(x,\tau)(\epsilon \mathrm{div}_x + \mathrm{div}_y)\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau.$$

A passage to the limit, and an application of the divergence theorem, using the fact that all other terms vanish, gives

$$\int_0^s \int_\Omega \int_{T^n} \nabla_y p_0(x,y,\tau)\varphi(x,y,\tau)dydxd\tau = 0,$$

which implies that p_0 does not depend on y. We now add the local incompressibility assumption on the test functions φ, i.e. $\mathrm{div}_y\varphi = 0$ and multiply the pressure term by $\varphi(x, \frac{x}{\epsilon}, \tau)$ and apply the divergence theorem,

$$\int_0^s \int_\Omega \nabla p_\epsilon(x,\tau)\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau = \int_0^s \int_\Omega p_\epsilon(x,\tau)\mathrm{div}_x\varphi(x, \frac{x}{\epsilon}, \tau)dxd\tau.$$

A passage to the limit, and an application of the divergence theorem, gives

$$\int_0^s \int_\Omega \int_{T^n} \nabla_x p_0(x,\tau)\varphi(x,y,\tau)dydxd\tau.$$

Collecting all two-scale limits on the right hand side gives

$$\int_0^s \int_\Omega \int_{T^n} (f - \frac{\partial u_0}{\partial \tau} - (u_0 \cdot \nabla_y)u_0 + \nu \Delta_y u_0 - \nabla_x p_0)\varphi dydxd\tau = 0.$$

Since $\mathrm{div}_y \varphi = 0$ we can argue as in Remark 2 and conclude that there exists a local pressure gradient $\nabla_y p_1(x,y,\tau)$ given by

$$\nabla_y p_1(x,y,\tau) = f - \frac{\partial u_0}{\partial \tau} - (u_0 \cdot \nabla_y)u_0 + \nu \Delta_y u_0 - \nabla_x p_0.$$

Let us now consider the second equation of (4.2). We already know that the sequence $\{\theta_\epsilon\}$ is uniformly bounded in $L^2([0,s];L^2(\Omega))$. We multiply by $\epsilon^{1/2}\varphi$ as above and for the time derivative we get

$$\epsilon^{1/2} \int_0^s \int_\Omega \frac{\partial \theta_\epsilon}{\partial t}\varphi dxd\tau = \int_0^s \int_\Omega \theta_\epsilon \frac{\partial \varphi}{\partial \tau} dxd\tau.$$

By letting $\epsilon \to 0$ we get

$$\int_0^s \int_\Omega \theta_\epsilon \frac{\partial \varphi}{\partial \tau} dxd\tau \to \int_0^s \int_\Omega \int_{T^n} \frac{\partial \theta_0}{\partial \tau}\varphi dydxd\tau.$$

For the non-linear term we have

$$\epsilon^{1/2} \int_0^s \int_\Omega (u_\epsilon \cdot \nabla)\theta_\epsilon \varphi dxd\tau$$

$$= \epsilon^{1/2} \int_0^s \int_\Omega (u_\epsilon \cdot \nabla_x)\varphi\theta_\epsilon dxd\tau + \epsilon^{-1/2} \int_0^s \int_\Omega (u_\epsilon \cdot \nabla_y)\varphi\theta_\epsilon dxd\tau.$$

The first term on the right hand side immediately passes to zero. For the second term we argue as above and consider the difference

$$\int_0^s \int_\Omega ((\epsilon^{-1/2}u_\epsilon \cdot \nabla_y)\varphi\theta_\epsilon - (u_0 \cdot \nabla_y)\varphi\theta_0)dxd\tau$$

$$= \int_0^s \int_\Omega ((\epsilon^{-1/2}u_\epsilon \cdot \nabla_y)\varphi\theta_\epsilon - (u_0 \cdot \nabla_y)\varphi\theta_\epsilon)dxd\tau$$

$$+ \int_0^s \int_\Omega ((u_0 \cdot \nabla_y)\varphi\theta_\epsilon - (u_0 \cdot \nabla_y)\varphi\theta_0)dxd\tau.$$

By considering $(u_0 \cdot \nabla_y)\varphi$ to be a test function in the second term, this term immediately passes to zero in the two-scale sense. For the first term we get, by the Schwarz's inequality,

$$\int_0^s \int_\Omega ((\epsilon^{-1/2}u_\epsilon \cdot \nabla_y)\varphi\theta_\epsilon - (u_0 \cdot \nabla_y)\varphi\theta_\epsilon)dxd\tau \le C\|(\epsilon^{-1/2}u_\epsilon - u_0) \cdot \nabla_y \varphi\|,$$

where the norm is in $L^2([0,s];L^2(\Omega))$. We introduce, as above, mollifiers and consider the sequence $\{u_{\epsilon,\mu}\}$ which converges to $\epsilon^{-1/2}u_\epsilon$ strongly as $\mu \to 0$. Arguing as above, we choose μ sufficiently small to get

$$\|(\epsilon^{-1/2}u_\epsilon - u_0) \cdot \nabla_y \varphi\| \le \|(u_{\epsilon,\mu} - u_0) \cdot \nabla_y \varphi\| + \delta,$$

where δ is arbitrarily small. Thus, by sending $\epsilon \to 0$,

$$u_\epsilon \nabla \theta_\epsilon \to u_0 \nabla_y \theta_0$$

in the two-scale sense. For the third term we get, by the divergence theorem,

$$-\epsilon^2 \kappa \int_0^s \int_\Omega \Delta \theta_\epsilon \varphi dx d\tau$$

$$= -\kappa \int_0^s \int_\Omega \theta_\epsilon \Delta_y \varphi dx d\tau + \text{ terms tending to zero as } \epsilon \to 0.$$

We let $\epsilon \to 0$ and get, by the divergence theorem,

$$-\kappa \int_0^s \int_\Omega \theta_\epsilon \Delta_y \varphi dx d\tau \to -\kappa \int_0^s \int_\Omega \int_{T^n} \Delta_y \theta_0 \varphi dy dx d\tau.$$

The right hand side of the second equation in (4.2) will vanish since u_ϵ is of order $\epsilon^{1/2}$. By Theorem 4.2 the system (4.3) has a unique solution $\{u_0, p_0, p_1, \theta_0\}$ and, thus, by uniqueness, (4.3) is two scales homogenized limit of the system (2.1). Also, by uniqueness, the whole sequence converges to its two-scale limit and the theorem is proven. ∎

8 The mean velocity field

In this section we derive the mean field $\overline{u}_0$ for the velocity. Let us consider the first equation of (4.3)

$$\begin{cases} \dfrac{\partial u_0}{\partial \tau} + (u_0 \cdot \nabla_y) u_0 - \nu \Delta_y u_0 + \nabla_y p_1 = e_n \theta_0 - \nabla_x p_0, \\ \mathrm{div_y} u_0 = 0, \ \ \mathrm{div_x}(\displaystyle\int_{T^n} u_0 \, dy) = 0, \end{cases}$$

in $\Omega \times T^n \times \mathbf{R}^+$ with T^n-periodicity as boundary data in y.

By letting $K = K(y, y', \tau)$ denote the heat kernel we can write the solution to the first equation of (4.3) as an integral,

$$u_0(x, y, \tau) = \int_{T^n} K(y, y', \tau) u_0(x, y', 0) dy'$$

$$+ \int_0^\tau \int_{T^n} K(y, y', \tau - s)(e_n \theta_0(x, y', s) - \nabla_x p_0(x, s) - \nabla_y p_1(x, y', s)$$

$$-(u_0 \cdot \nabla_y) u_0(x, y', s)) dy' ds.$$

Now, by the decay of the heat kernel, for τ sufficiently large, the first term becomes arbitrarily small. An averaging of the second term over T^n in y gives

$$\overline{u_0}(x, \tau) = \int_0^\tau (e_n \overline{\theta_0} - \nabla_x p_0 - \overline{(u_0 \cdot \nabla_y) u_0})(x, s) ds.$$

The divergence theorem gives

$$\int_{T^n} (u_0 \cdot \nabla_y) u_0 \, dy = -\int_{T^n} (\mathrm{div}_y u_0) u_0 dy + \int_{\partial T^n} (u_0 \cdot n_y) u_0 \, dS_y,$$

where n_y is the local unit normal and S_y the local surface element. Now, by the local incompressibility we get

$$\int_{T^n} (\mathrm{div}_y u_0) u_0 dy = 0.$$

Moreover, for the boundary integral we get, in the two-dimensional case,

$$\int_{\partial T^2} u_i n_i u_0 \, dS_y = \int_0^1 -u_2 u_0(y_1, 0) dy_1 + u_1 u_0(1, y_2) dy_2$$
$$+ \int_0^1 u_2 u_0(y_1, 1) dy_1 - u_1 u_0(0, y_2) dy_2 = 0,$$

by the T^2-periodicity of $u_0 = (u_1, u_2)$. Consequently

$$\overline{(u_0 \cdot \nabla_y) u_0} = 0.$$

The computation in the three-dimensional case is similar. Therefore the mean field reduces to

$$\overline{u_0}(x, \tau) = \int_0^\tau (e_n \overline{\theta_0} - \nabla_x p_0)(x, s) ds. \tag{8.1}$$

We can now use the incompressibility, which when applied to (8.1) gives

$$\mathrm{div}_x (e_n \overline{\theta}_0(x, \tau) - \nabla_x p_0(x, \tau)) = 0. \tag{8.2}$$

This says that

$$e_n \overline{\theta}_0(x, \tau) - \nabla_x p_0(x, \tau) = H(x, \tau) \tag{8.3}$$

where H is a divergence free (rotational) field.

The field H can be determined explicitly by solving the global equation (8.2), (in x), for the pressure p_0. For this purpose one needs to impose the appropriate boundary condition on the pressure in order to close the system. We impose the Neumann condition

$$n \cdot \nabla_x p_0 = 0$$

for the pressure. In fact Lions [45] and Sanchez-Palencia [47] impose the condition

$$n \cdot \overline{u_0} = 0$$

which when inserted in (8.1) gives

$$n \cdot \nabla_x p_0 = 0.$$

We consider

$$\begin{cases} \Delta_x p_0(\cdot, \tau) = \mathrm{div}_x \, e_n \overline{\theta}_0(\cdot, \tau), & \text{in } \Omega, \\ n \cdot \nabla_x p_0(\cdot, \tau) = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.4}$$

for every $\tau \geq 0$, and obtain

$$H = \pi(e_n \overline{\theta}_0) = -\nabla \times (\Delta^{-1}(\nabla \times e_n \overline{\theta}_0)),$$

where $\pi(e_n\overline{\theta}_0)$ denotes projection onto the divergence free part of the conduction term.

Collecting the results from the above discussion we can now express the mean field as

$$\overline{u}_0(x, \frac{t}{\sqrt{\epsilon}}) = \int_0^{t/\sqrt{\epsilon}} \pi(e_n\overline{\theta}_0)(x,s)ds.$$

This gives the contribution of the conduction and the global pressure to the small scale flow. We have let the local flow settle down and averaged (in y), denoted by overbar, over the unit cell T^n, $n = 2, 3$.

Remark 9. If we take boundary layer effects into account, we can, considering the simplest case of a boundary layer, specify the value of the global pressure gradient $\nabla_x p_0$ at the boundary, of the boundary layer. For instance we can put

$$n \cdot \nabla_x p_0 = f.$$

This will result in the additional term $G * f$ in the mean field, where G is the usual Neumann kernel for the Laplacian. In this case the mean field becomes

$$\overline{u}_0(x, \frac{t}{\sqrt{\epsilon}}) = \int_0^{t/\sqrt{\epsilon}} \left(\pi(e_n\overline{\theta}_0) + G * f\right)(x,s)ds.$$

Remark 10. The above formulas give the mean field flow with the global convection (rolls) taken out. This was done by imposing the periodic boundary conditions on the local cell, see the introduction. The two-scale convergence carries over to a larger class of test functions which are non-periodic , see [48]. This will not be repeated here, but to be able to compare the true (collective) mean field we compute the average over the convection term in the non-periodic case. By the Taylor and mean value theorems we get, in the two-dimensional case,

$$\int_{\partial T^2} u_i n_i u_0 \, dS_y = \int_0^1 (-u_2 u_0(y_1 + Y_1, Y_2) + u_2 u_0(y_1 + Y_1, 1 + Y_2))dy_1$$

$$+ \int_0^1 (u_1 u_0(1 + Y_1, y_2 + Y_2) - u_1 u_0(Y_1, y_2 + Y_2))dy_2$$

$$= \overline{u}_2 \int_0^1 (u_0(y_1 + Y_1, Y_2) - u_0(y_1 + Y_1, 1 + Y_2))dy_1$$

$$+ \overline{u}_1 \int_0^1 (u_0(1 + Y_1, y_2 + Y_2) - u_0(Y_1, y_2 + Y_2))dy_2 + O(\Delta)$$

$$= \overline{u}_2 \frac{\partial \overline{u}_0}{\partial Y_2} + \overline{u}_1 \frac{\partial \overline{u}_0}{\partial Y_1} + O(\Delta) = \overline{u}_0 \cdot \nabla \overline{u_0} + O(\Delta),$$

where $\Delta = (y_1 - \overline{y}_1, y_2 - \overline{y}_2)$, $(\overline{y}_1, \overline{y}_2)$ is the point in T^2 where u attains its mean value, and (Y_1, Y_2) is the location of the box. The error term $O(\Delta) \leq C\|u_0\||\Delta|$ is bounded by Theorem 4.2. In global coordinates we therefore get

$$\overline{u_0 \cdot \nabla u_0} = \overline{u}_0 \cdot \nabla \overline{u_0} + O(\epsilon),$$

and similarly

$$\overline{\nabla p_1} = \nabla \overline{p}_1.$$

If we insert this into the expression for the mean field and differentiate with respect to the fast time variable τ, we obtain, as $\epsilon \to 0$,

$$\frac{\partial \overline{u}_0}{\partial \tau} + \overline{u}_0 \cdot \nabla \overline{u_0} + \nabla \overline{p}_1 = \pi(e_2 \overline{\theta}_0),$$

i.e. a forced Euler equation, where u_0 is a function of the scaled variables $(\tau, Y) = (t/\sqrt{\epsilon}, x/\epsilon)$. The computation in the three-dimensional case is similar. Thus the local flow satisfies the Navier-Stokes equation, whereas the mean flow satisfies the Euler equation.

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