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First Integrals Generated by Pseudosymmetries in Nambu-Poisson Mechanics

Mircea CRÂŞMĂREANU

Faculty of Mathematics, University "Al. I. Cuza", Iaşi, 6600, Romania
E-mail: mcrasm@uaic.ro
and
Institute of Mathematics "Octav Mayer", Iaşi Branch of Romanian Academy, Iaşi, 6600, Romania

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Abstract

Some types of first integrals for Hamiltonian Nambu-Poisson vector fields are obtained by using the notions of pseudosymmetries. In this theory, the homogeneous Hamiltonian vector fields play a special role and we point out this fact. The differential system which describe the $SU(2)$-monopoles is given as example. The paper ends with two appendices.

1 Introduction

Some physical systems can be described by using multibrackets instead of Poisson brackets; for example: the Euler equations of the free rigid body, the Maxwell-Bloch equations of laser-matter dynamics, the Toda lattice equation and the heavy top. Beginning with Nambu [12] and following with Takhtajan [16] these multibrackets, usually called *Nambu-Poisson structures*(NP structures, on short) led to some interesting mathematical developments [2], [8], [17], [18].

However, to the best of our knowledge, one of the main subjects of dynamics, namely *the first integrals*, was not intensively studied except [2]. The aim of the present paper is to make an attempt of studying this subject. More precisely, two types of first integrals of Hamiltonian NP vector fields are obtained by using the notions of pseudosymmetry and adjoint pseudosymmetry. Also, the fact that the bracket of n-first integrals is again a first integral is generalized.

The paper is structured as follows: the first section covers the main result of the author's paper [4], the second section gives the basics of NP theory and the third section presents applications to homogeneous Hamiltonian vector fields on $\mathbb{R}^n$. The next section contains the adjoint pseudosymmetries approach to find first integrals in NP dynamics. Two appendices are inserted at the end of paper.

2 First integrals generated by pseudosymmetries

Since evolution in NP mechanics is given by a vector field, called *Hamiltonian vector field* like in symplectic mechanics, we will be interested in methods to find first integrals for vector fields. In [4] we obtained such a method by using pseudosymmetries and this section recall the results of the cited paper.

Let M be a smooth, m-dimensional manifold, $C^{\infty}(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^p(M)$ the $C^{\infty}(M)$-module of p-differential forms, $1 \leq p \leq m$. For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ one considers the system of differential equations which gives the flow of X:

$$\dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i\left(x^1(t), \ldots, x^m(t)\right). \tag{2.1}$$

A solution of (2.1) is called an *integral curve* of X.

Definition 2.1. A function $\mathcal{F} \in C^{\infty}(M)$ is called a *conservation law* (or *first integral,* or *constant of motion,* or *invariant function*) of X or of (2.1) if $\mathcal{F}$ is constant along the solutions of (2.1) that is

$$\frac{d(\mathcal{F} \circ c)}{dt}(t) = 0$$

for every integral curve $c(t)$ of X.

The following characterization is useful:

Proposition 2.2. *$\mathcal{F} \in C^{\infty}(M)$ is a first integral of (1.1) if and only if*

$$\mathcal{L}_X \mathcal{F} = 0$$

where the right-hand side means the Lie derivative of $\mathcal{F}$ with respect to X.

For our approach we need the following:

Definition 2.3. (i) $Y \in \mathcal{X}(M)$ is called a *symmetry of X* if

$$\mathcal{L}_X Y = 0.$$

(ii) If $Y \in \mathcal{X}(M)$ is fixed then $Z \in \mathcal{X}(M)$ is called a *Y-pseudosymmetry of X* if there exists $f \in C^{\infty}(M)$ such that

$$\mathcal{L}_X Z = fY.$$

(iii) $\omega \in \Omega^p(M)$ is called an *invariant p-form of X* if

$$\mathcal{L}_X \omega = 0.$$

If $p = 1$ then ω is called an *adjoint symmetry* in [5, p. 36].

The result which give the association between pseudosymmetries and first integrals is:

Proposition 2.4. *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in \Omega^p(M)$ be an invariant $p-$form of X. If $Y \in \mathcal{X}(M)$ is a symmetry of X and $S_1, \dots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetries of X then:*

$$\mathcal{F} = \omega(S_1, \dots, S_{p-1}, Y) \tag{2.2}$$

is a first integral for X. In particular, if $Y, S_1, \dots, S_{p-1}$ are symmetries of X then $\mathcal{F}$ given by (2.2) *is a first integral.*

Remark 2.5. (i) For $Y = X$ one obtain the main result of G. L. Jones [10, p. 1056].
(ii) If $p = 1$ one obtain Theorem 2.5.10 of ten Eikelder [5, p. 48].
(iii) Some applications to Lagrangian and Hamiltonian systems can be found in [4]. In these cases we have an invariant 2-form, the so-called *Cartan 2-form*.
(iv) In [19] it is solved a somewhat inverse problem: more precisely is given a way to obtain a symmetry of a vector field which admits $n-1$ first integrals. Let us note that Ünal's method has tangency with the Bayen-Flato generalization of Nambu mechanics.

Example 2.6 (Static SU(2)-monopoles). The Nahm's system in the theory of static SU(2)-monopoles is presented in [8]:

$$\frac{dx^1}{dt} = x^2x^3, \quad \frac{dx^2}{dt} = x^3x^1, \quad \frac{dx^3}{dt} = x^1x^2. \tag{2.3}$$

The vector field $X = x^2x^3\frac{\partial}{\partial x^1} + x^3x^1\frac{\partial}{\partial x^2} + x^1x^2\frac{\partial}{\partial x^3}$ is homogeneous of order two, that is :

$$[\Upsilon, X] = X \tag{2.4}$$

where :

$$\Upsilon = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}.$$

But (2.4) means that Υ is X-pseudosymmetry for (2.3).

3 Nambu-Poisson revisited

In this section we review the necessary facts from NP structures. For more details see [2], [8], [14], [15], [16], [17], [18] and the references therein.

Definition 3.1. A *Nambu-Poisson bracket or structure* of order n, $2 \le n \le m$ is an internal n-ary operation on $C^\infty(M)$, denoted by $\{ \}$, which satisfies the following axioms:
(i) $\{ \}$ is $I\!R$-multilinear and skew-symmetric
(ii) the *Leibniz rule:*

$$\{f_1, \dots, f_{n-1}, gh\} = \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\}$$

(iii) the *fundamental identity:*

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{k=1}^{n}\{g_1, \dots, \{f_1, \dots, f_{n-1}, g_k\}, \dots, g_n\}.$$

Remember that if we use the same definition for $n = 2$, we get a *Poisson bracket.*

By (ii), { } acts on each factor as a vector field, whence it must be of the form:

$$\{f_1, \ldots, f_n\} = \Lambda(df_1, \ldots, df_n)$$

where Λ is a field of n-vectors on M. If such a field defines a NP bracket, it is called a *NP tensor (field)*. Λ defines a bundle mapping

$$\sharp_\Lambda : \underbrace{T^*M \times \ldots \times T^*M}_{(n-1)\ \text{times}} \longrightarrow TM$$

given by:

$$< \beta, \sharp_\Lambda(\alpha_1, \ldots, \alpha_{n-1}) >= \Lambda(\alpha_1, \ldots, \alpha_{n-1}, \beta)$$

where all the arguments are covectors.

The next basic notion is that of the Λ-*Hamiltonian vector field* of $(n-1)$ functions defined by:

$$X_{F_1 \ldots F_{n-1}} = \sharp_\Lambda(dF_1, \ldots, dF_{n-1}).$$

Related to our subject of interest, namely first integrals, there are three important properties:

Property 3.2. A function $f \in C^\infty(M)$ is a first integral of $X_{F_1 \ldots F_{n-1}}$ if and only if

$$\{F_1, \ldots, F_{n-1}, f\} = 0.$$

Property 3.3. The NP bracket of n first integrals is again a first integral.

Property 3.4. The Hamiltonian vector fields are infinitesimal automorphisms of the NP tensor:

$$L_{X_{F_1 \ldots F_{n-1}}} \Lambda = 0.$$

4 The case $M \subseteq I\!R^n$

Let us suppose that $M \subseteq I\!R^n$(this imply $n = m$!) and introduce the following differential forms:

(i) $\Omega = dx^1 \wedge \ldots \wedge dx^n$, the volume n-form,

(ii) $\omega_{F_1 \ldots F_{n-1}} = i_{X_{F_1 \ldots F_{n-1}}} \Omega = dF_1 \wedge \ldots \wedge dF_{n-1}$, the $(n-1)$-form associated to $X_{F_1 \ldots F_{n-1}}$(cf. [11]).

The lemma 1 from [11] gives that Ω is an invariant n-form of $X_{F_1 \ldots F_{n-1}}$ or in other words $X_{F_1 \ldots F_{n-1}}$ is a *solenoidal* vector field. Also, from $i_X^2 = 0$ it results that $\omega_{F_1 \ldots F_{n-1}}$ is an invariant $(n-1)$-form of $X_{F_1 \ldots F_{n-1}}$. Therefore in the NP setting on $M \subseteq I\!R^n$ there are two invariant forms for a Hamiltonian vector field. Then we have the following form of Proposition 2.4:

Proposition 4.1. *If Y is a symmetry of $X_{F_1 \ldots F_{n-1}}$ and $S_1, \ldots, S_{n-1}$ are $n-1$ Y-pseudo-symmetries of $X_{F_1 \ldots F_{n-1}}$ then:*

$$\mathcal{F}_1 = \Omega(Y, S_1, \ldots, S_{n-1})$$

$$\mathcal{F}_2 = \omega_{F_1 \ldots F_{n-1}} (Y, S_1, \ldots, S_{n-2})$$

are first integrals of $X_{F_1 \ldots F_{n-1}}$. *In particular:*

(i) $n = 2$, $\mathcal{F}_2 = \omega_{F_1} (Y) = Y (F_1)$

(ii) $n = 3, \mathcal{F}_2 = dF_1 \wedge dF_2 (Y, S_1) = Y (F_1) S_1 (F_2) - Y (F_2) S_1 (F_1)$.

In the following let us suppose that $X_{F_1 \ldots F_{n-1}}$ is p-homogeneous, that is $\left[\Upsilon, X_{F_1 \ldots F_{n-1}}\right] = (p-1) X_{F_1 \ldots F_{n-1}}$ where $\Upsilon = x^1 \frac{\partial}{\partial x^1} + \ldots + x^n \frac{\partial}{\partial x^n}$ and p is an integer. In this case applying the previous result with $Y = X_{F_1 \ldots F_{n-1}}$ and $S_{n-1} = \Upsilon$ we get:

Proposition 4.2. *Let* $X_{F_1 \ldots F_{n-1}}$ *be a p-homogeneous Hamiltonian vector field on* $M \subseteq \mathbb{R}^n$. *If* $S_1, \ldots, S_{n-2}$ *are* $n-2$ $X_{F_1 \ldots F_{n-1}}$*-pseudosymmetries of* $X_{F_1 \ldots F_{n-1}}$ *then:*

$$\mathcal{F}_1 = \omega_{F_1 \ldots F_{n-1}} (\Upsilon, S_1, \ldots, S_{n-2})$$

is a first integral of $X_{F_1 \ldots F_{n-1}}$. *In particular, if* $n = 2$ *then* $\mathcal{F}_1 = \omega (\Upsilon)$ *is a first integral of* X_{F_1}.

Example 2.6 revisited. The system (2.3) can be written in NP form with([8]):

$$\{f_1, f_2, f_3\} = \frac{\partial (f_1, f_2, f_3)}{\partial (x^1, x^2, x^3)} \tag{4.1}$$

and:

$$F_1 = \frac{1}{2} \left(\left(x^1\right)^2 - \left(x^2\right)^2 \right), \quad F_2 = \frac{1}{2} \left(\left(x^1\right)^2 - \left(x^3\right)^2 \right). \tag{4.2}$$

So, because this system is NP and homogeneous, the previous proposition works. For a list of others homogeneous Hamiltonian vector fields see Appendix 1.

We return to the case $n = 2$ of Proposition 4.1 for local expressions. So, let $\{,\}$ be a Poisson bracket on $\mathbb{R}^2$ and $F \in C^\infty \left(\mathbb{R}^2\right)$. Then:

$$X_F = \frac{\partial F}{\partial x^2} \frac{\partial}{\partial x^1} - \frac{\partial F}{\partial x^1} \frac{\partial}{\partial x^2}$$

$$\omega_F = dF = \frac{\partial F}{\partial x^1} dx^1 + \frac{\partial F}{\partial x^2} dx^2$$

$$\mathcal{F}_1 = \omega_F (\Upsilon) = \frac{\partial F}{\partial x^1} x^1 + \frac{\partial F}{\partial x^2} x^2 = \mathcal{L}_\Upsilon F$$

and the p-homogeneity of X_F reads as follows:

$$\mathcal{L}_\Upsilon \frac{\partial F}{\partial x^i} = p \frac{\partial F}{\partial x^i}, \quad 1 \leq i \leq 2.$$

Therefore we get:

Proposition 4.3. *Let* $\{,\}$ *be a Poisson bracket on* $\mathbb{R}^2$ *and* $F \in C^\infty \left(\mathbb{R}^2\right)$ *such that the functions* $\frac{\partial F}{\partial x^i}, 1 \leq i \leq 2$ *are p-homogeneous. Then* $\mathcal{F}_1 = \mathcal{L}_\Upsilon F$ *is a first integral of the Hamiltonian vector field* X_F.

At the beginning of this section we use the P. Morando's result that every NP Hamiltonian vector field on $\mathbb{R}^n$ is solenoidal. In the following we give a local converse of this topic.

So, if $X \in \mathcal{X}(\mathbb{R}^n)$ is solenoidal, a generalized Euler theorem states that there are $F_1, \ldots, F_{n-1}$ smooth functions in a neighbourhood U of every regular point p of X(that is $X(p) \neq 0$) such that:

$$X = \nabla F_1 \times \ldots \times \nabla F_{n-1} \tag{4.3}$$

where ∇F means the gradient of $F \in C^\infty(U)$ and $\times$ is the vector product of $\mathbb{R}^n$. But (4.3) is exactly the expression Nambu-Poisson of X.

5 From adjoint pseudosymmetries to first integrals in Nambu-Poisson dynamics

In order to obtain a result similar to Proposition 2.4 in NP mechanics we need:

Definition 5.1. Let $X \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$. Then a given $\alpha \in \Omega^1(M)$ is called an *ω-adjoint pseudosymmetry* of X if

$$\mathcal{L}_X \alpha = \rho\omega$$

for some $\rho \in C^\infty(M)$.

A straightforward computation gives another main result of this paper:

Proposition 5.2. *Let $X_{F_1 \ldots F_{n-1}}$ be a Hamiltonian vector field and $\omega \in \Omega^1(M)$ be an adjoint symmetry of $X_{F_1 \ldots F_{n-1}}$. If $\omega_1, \ldots, \omega_{n-1} \in \Omega^1(M)$ are ω-adjoint pseudosymmetries of $X_{F_1 \ldots F_{n-1}}$ then:*

$$\mathcal{F} = \Lambda(\omega, \omega_1, \ldots, \omega_{n-1})$$

is a first integral of $X_{F_1 \ldots F_{n-1}}$. In particular, if $\omega, \omega_1, \ldots, \omega_{n-1}$ are adjoint symmetries then the above $\mathcal{F}$ is first integral of $X_{F_1 \ldots F_{n-1}}$.

Let us consider a particular case when ω, ω_i are exact 1-forms: $\omega = df$, $\omega_i = df_i$ with $f, f_i \in C^\infty(M)$. From the definition of Λ we get that $\mathcal{F}$ given by Proposition 5.2 is $\mathcal{F} = \Lambda(df, df_1, \ldots, df_{n-1}) = \{f, f_1, \ldots, f_{n-1}\}$. Recall also that $d\mathcal{L}_X = \mathcal{L}_X d$ and then we have:

Proposition 5.3. *Let $X_{F_1 \ldots F_{n-1}}$ be a Hamiltonian vector field and $f, f_1, \ldots, f_{n-1} \in C^\infty(M)$ such that:*
(i) $\mathcal{L}_{X_{F_1 \ldots F_{n-1}}} f$ is a closed 1-form i.e $d\left(\mathcal{L}_{X_{F_1 \ldots F_{n-1}}} f\right) = \mathcal{L}_{X_{F_1 \ldots F_{n-1}}}(df) = 0$
(ii) $L_{X_{F_1 \ldots F_{n-1}}} df_i = \rho_i df$ for some $\rho_i \in C^\infty(M)$. Then:

$$\mathcal{F} = \{f, f_1, \ldots, f_{n-1}\}$$

is a first integral for $X_{F_1 \ldots F_{n-1}}$.

It is obvious that this result represents a generalization of Property 3.3 because if $f, f_1, \ldots, f_{n-1}$ are first integrals then (i) and (ii) hold with $\rho_i = 0$. The next scheme is significant:

property 3.3	$\{F_1, \ldots, F_{n-1}, f\} = 0$ $\{F_1, \ldots, F_{n-1}, f_i\} = 0$	$\rightarrow \{F_1, \ldots, F_{n-1}, \{f, f_1, \ldots, f_{n-1}\}\} = 0$

prop. 5.3	$d\{F_1, \ldots, F_{n-1}, f\} = 0$ $d\{F_1, \ldots, F_{n-1}, f_i\} = \rho_i df$	$\rightarrow \{F_1, \ldots, F_{n-1}, \{f, f_1, \ldots, f_{n-1}\}\} = 0.$

Example 2.6 revisited. For any $f\left(x^1, x^2, x^3\right)$ we have:

$$\{F_1, F_2, f\} = x^2 x^3 \frac{\partial f}{\partial x^1} + x^3 x^1 \frac{\partial f}{\partial x^2} + x^1 x^2 \frac{\partial f}{\partial x^3}$$

We seek a polynomial $f = A\left(x^1\right)^\alpha + B\left(x^2\right)^\beta + C\left(x^3\right)^\gamma, \alpha, \beta, \gamma \neq 0$ satisfies (i) of Proposition 5.3, that is $\{F_1, F_2, f\} = \text{constant}$. Then: $A\alpha\left(x^1\right)^{\alpha-1} x^2 x^3 + B\beta x^1\left(x^2\right)^{\beta-1} x^3 + C\gamma x^1 x^2\left(x^3\right)^{\gamma-1} = x^1 x^2 x^3\left(A\alpha\left(x^1\right)^{\alpha-2} + B\beta\left(x^2\right)^{\beta-2} + C\gamma\left(x^3\right)^{\gamma-2}\right) = \text{constant}$ which means: $A\alpha\left(x^1\right)^{\alpha-2} + B\beta\left(x^2\right)^{\beta-2} + C\gamma\left(x^3\right)^{\gamma-2} = 0$ that is: $\alpha = \beta = \gamma = 2$ and $A + B + C = 0$. Therefore: $f = (-B - C)\left(x^1\right)^2 + B\left(x^2\right)^2 + C\left(x^3\right)^2 = -2(BF_1 + CF_2)$. In conclusion f is a linear combination of F_1 and F_2.

Appendix 1: Remarkable homogeneous NP Hamiltonian vector fields

I) the Euler equations of the free rigid body in $I\!R^3$ [12], [8] :

$$\dot{x}^1 = \left(\frac{1}{I_2} - \frac{1}{I_3}\right) x^2 x^3$$

$$\dot{x}^2 = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) x^3 x^1$$

$$\dot{x}^3 = \left(\frac{1}{I_1} - \frac{1}{I_2}\right) x^1 x^2$$

with

$$F_1 = \frac{1}{2}\left[\frac{1}{I_1}\left(x^1\right)^2 + \frac{1}{I_2}\left(x^2\right)^2 + \frac{1}{I_3}\left(x^3\right)^2\right], F_2 = \frac{1}{2}\left[\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right]$$

is 2-homogeneous. Let us remark that F_1 is *the total angular momentum* and F_2 is *the kinetic energy* of the free rigid body.

II) the differential system of Jacobi elliptic functions [3, p. 137]:

$$\dot{x}^1 = x^2 x^3$$

$$\dot{x}^2 = -x^3 x^1$$

$$\dot{x}^3 = -k^2 x^1 x^2$$

with $0 < k^2 < 1$ and:

$$F_1 = \frac{1}{2}\left[\left(x^1\right)^2 + \left(x^2\right)^2\right], \quad F_2 = \frac{1}{2}\left[k^2\left(x^1\right)^2 + \left(x^3\right)^2\right]$$

is 2-homogeneous.

III) A more special NP system is given by the 2D isotropic harmonic oscillator [3, p. 2]:

$$\dot{x}^1 = y^1$$

$$\dot{x}^2 = y^2$$

$$\dot{y}^1 = -x^1$$

$$\dot{y}^2 = -x^2.$$

The following functions $F_1, \ldots, F_4$ are a basis for the vector space of all the quadratic integrals of above system [3, p. 11]:

$$F_1 = x^1 y^2 - x^2 y^1$$

$$F_2 = \frac{1}{2}\left(x^1 x^2 + y^1 y^2\right)$$

$$F_3 = \frac{1}{2}\left(\left(y^1\right)^2 + \left(x^1\right)^2 - \left(y^2\right)^2 - \left(x^2\right)^2\right)$$

$$F_4 = \frac{1}{4}\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(y^1\right)^2 + \left(y^2\right)^2\right).$$

If we define the NP bracket:

$$\{f_1, f_2, f_3, f_4\} = \frac{1}{F_4}\frac{\partial\left(f_1, f_2, f_3, f_4\right)}{\partial\left(x^1, x^2, y^1, y^2\right)}$$

then the differential system is given by the Hamiltonian vector field $X_{F_1 F_2 F_3}$ which is 1-homogeneous.

Appendix 2: The completeness of Hamiltonian vector fields in Nambu-Poisson dynamics

We recall that $X \in \mathcal{X}(M)$ is *complete* if for every $x_0 \in M$ the maximal interval of existence (t_-, t_+) for the solution of equation (2.1) with initial condition $x(0) = x_0$ is given by $t_\pm = \pm\infty$. A sufficient condition which assures this property is provided by:

Theorem A2.1 ([6]). *Let $X \in \mathcal{X}(M)$. If there exists $E, f \in C^\infty(M)$ with f proper, that is f^{-1} (compact) = compact, and $\alpha, \beta \in \mathbb{R}$ such that for each $x \in M$ we have:*

$$|X(E)(x)| \leq \alpha |E(x)|$$

$$|f(x)| \leq \beta |E(x)|$$

then X is complete.

Then we can prove:

Proposition A2.2. *Let* $X_{F_1 \ldots F_{n-1}}$ *be a NP Hamiltonian vector field. If there exists* $i \in \{1, \ldots, n-1\}$ *such that* F_i *is proper then* $X_{F_1 \ldots F_{n-1}}$ *is complete.*

Proof. Let us take in previous theorem $E = f = F_i$. Since $X_{F_1 \ldots F_{n-1}}(F_i) = 0$ it follows that all conditions of the theorem A2.1 are satisfied. ■

Remark A2.3. (i) The Poisson case of Proposition A2.2 is Theorem 3.1 from [13, p. 96]. In the paper cited it is added the assumption that F_1 is bounded below, say $F_1 \geq 0$. As Janusz Grabowski point out in the MR review of [13] this assumption is not necessary.
(ii) Also, using a remark of J. Grabowski in the review cited, we note that it is sufficient the bracket be just skew-symmetric and satisfying the Leibniz rule, i.e. be determined by a n-vector field.

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