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On the Calculation of Finite-Gap Solutions of the KdV Equation

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Abstract

A simple and general approach for calculating the elliptic finite-gap solutions of the Korteweg-de Vries (KdV) equation is proposed. Our approach is based on the use of the finite-gap equations and the general representation of these solutions in the form of rational functions of the elliptic Weierstrass function. The calculation of initial elliptic finite-gap solutions is reduced to the solution of the finite-band equations with respect to the parameters of the representation. The time evolution of these solutions is described via the dynamic equations of their poles, integrated with the help of the finite-gap equations. The proposed approach is applied by calculating the elliptic 1-, 2- and 3-gap solutions of the KdV equations.

1 Introduction

In accordance with the finite-band theory, the integrable Korteweg-de Vries (KdV) equation can be considered as the compatibility condition of the two auxiliary linear differential matrix equations $\partial_{x_\pm}\mathbf{\Phi} = \mathbf{U}_\pm$ ($x_+ \equiv x$, $x_- \equiv t$, $\partial_x \equiv d/dx$) with matrix operators $\mathbf{U}_\pm$, coefficients of which, as is well known (see [1, 2, 3]), are expressed through solutions of the KdV equation. The finite-gap solutions of the KdV equation are solutions of the spectral problem for these equations with a finite-gap spectrum of eigenvalues. These are expressed through multi-dimensional Riemann theta functions with implicit parameters, the evaluation of which is the special algebraic geometrical problem (see [3, 4, 5, 6]). The class of elliptic finite-gap solutions leads to the problem of the reduction of n-dimensional theta-functions to the one-dimensional Jacobi theta functions (see [6]).

In the framework of the spectral problem the finite-gap solutions satisfy the matrix finite-gap equation of the form $\partial_\pm\mathbf{\Psi} = \mathbf{U}_\pm\mathbf{\Psi} - \mathbf{\Psi}\mathbf{U}_\pm$, where $\mathbf{\Psi}$ is the matrix components which are polynomial in the eigenvalue of the auxiliary equations (such as in the case of the “sine-Gordon” equation [7]). Usually, solving this equation is realized with help of the well known Abel transformation with a subsequent solving of the inverse Jacobi problem (see [4, 5]).

However, in the case under consideration the finite-gap solutions of the KdV equation are elliptic functions represented as rational functions of the elliptic Weierstrass function

($\wp$-function) [8]. Solving the finite-gap equations in terms of $\wp$-function can be reduced to solving simple algebraic equations. This yields the straightforward manner for calculating the elliptic finite-gap solutions in an explicit form.

We shown that in the initial time ($t = 0$) the finite-gap equations can be reduced to algebraic equations with respect to parameters of the above mentioned rational functions. The computation of these parameters gives all possible initial elliptic finite-gap solutions in the form of linear combinations of $\wp$-functions with shifted arguments.

In accordance with the KdV equation the time dependent elliptic solutions are built as linear combinations of $\wp$-functions with time dependent argument shifts (φ_i) (its poles) under condition of its compatibility with corresponding initial solutions. Their time evolution is determined by the poles which satisfy the system of linked dynamic equations which follow from the above mentioned auxiliary equations.

Using the finite-gap equations we shown that the dynamic system is transformed in the system of independent differential equations of the first order with separated variables of the form $\partial_t \varphi_i = X_i(\varphi_i)$. Here X_i-functions represent themselves roots of some polynomial equations which follow from the finite-gap equations the order of which equals the number of poles in the elliptic solutions.

This paper is organized as follows. In Section 2 the approach to the straightforward calculation of the elliptic finite-gap solutions of the KdV equation, based on the auxiliary system of finite-gap and dynamic equations, is formulated. In Section 3 this approach is applied to the calculation of the initial elliptic 1-, 2- and 3-gap solutions of the KdV equation. In Section 4 the time evolution of these elliptic solutions is investigated. We shown that the linked system of auxiliary dynamic equations, for poles of the corresponding time dependent elliptic solutions, can be integrated with the help of the finite-gap equations.

2 Finite-band equations and general elliptic solutions

In the class of elliptic functions (which we shall denote as $U(x,t)$) with the $\wp$-functional representation under consideration, the finite-gap solutions of the KdV equation can be considered as solutions of the finite-band equations. The latter gives the compatibility condition of the finite-gap and the general solutions of the auxiliary linear differential equations. These finite-band equations represent a system of equalities, obtained by equating coefficients of the power series in the eigenvalue E of the finite-gap and general solutions (see [2]).

The finite band equations (which are also known as "trace-formulae") can be written in the form [2, 9]

$$A_{n+1} = \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1}(x,t) \ (n = 0, 1, \dots),\ a_0 = 1 \tag{2.1}$$

$$A_n = \frac{1}{n!} \partial_z^n \left(\frac{\sqrt{\sum_{n=0}^{2g+1} a_n z^n}}{\sum_{n=0}^{g} b_n(x,t) z^n} \right) \Bigg|_{(z=0)},\ b_0 = 1 \tag{2.2}$$

(g is the number of a gaps in the spectrum of the eigenvalues E), where the χ_n-functions are determined by the recursion relation

$$\chi_{n+1} = \partial_x \chi_n + \sum_{k=1}^{n-1} \chi_k \chi_{n-k},\ \chi_1 = -U(x). \tag{2.3}$$

Here A_n and χ_n are coefficient functions of the power-series expansion in E of the general and the finite-gap solutions of the auxiliary equations, respectively. In view of (2.3) the χ-functions have the form of polynomials in the solutions $U(x,t)$ and its derivatives. As is well known, (2.1) is algebraicly solvable with respect to the coefficient functions $b_n(x,t)$. Therefore, the finite-gap equations (2.1) are reduced to the system containing expressions for the coefficient functions $b_n(x,t)$, $n = (1,\dots,g)$, which at $n \le g$ is a closed system of equations for the elliptic finite-gap U-solutions.

The elliptic finite-gap solutions as double periodic functions of the complex variable z which admit the rational functions representation namely the elliptic Weierstrass function $\wp$ [8]. In view of the equation (2.1), U is determined by the formula

$$U(z,t) = \alpha\wp(z) + \sum_i \sum_{n_i=1}^{2} \left\{ \frac{\alpha_{n_i}(t)}{(\wp(z) - h_i(t))^{n_i}} \right\} + \tilde{R}(z,t) \tag{2.4}$$

with poles of the secondary order in $\wp \equiv \wp(z|\omega,\omega')$ (ω and ω' are real and imaginary half-periods of the $\wp$-function), where $\tilde{R}(z,t)$ means an odd function of z.

At the initial time ($t = 0$) the expression (2.4) describes all possible so-called initial elliptic finite-gap solutions, which in the case of even functions ($\tilde{R} = 0$) under consideration, takes the form

$$U(z) = \alpha\wp(z) + \sum_i \sum_{n_i=1}^{2} \frac{\alpha_{n_i}}{(\wp(z) - h_i)^{n_i}}, \tag{2.5}$$

that reduces the finite-gap equations (2.1) to simple algebraic equations with respect to the α- and h_i-parameters.

The time evolution of the elliptic finite-gap solutions (2.4) are determined by the time dependence of their poles, which are described with the help of the known dynamic auxiliary equation

$$\partial_t \Psi(x,t,E) = \left(4\partial + 3(U\partial_x + \partial_x U)\right) \Psi(x,t,E), \tag{2.6}$$

$\Psi = \Psi(\{a_n)\},\{b_n\})$, $b_n = b_n(U(z,t),\, U^{(n)}(z,t))$ can be reduced to the system of dynamic equations with respect to the poles of the U-functions. This is a system of coupled differential equations of the first order which, as will be shown below, can be reduced to independent equations with separated variables.

3 A calculation of the initial elliptic solutions

The calculation of the initial finite-gap elliptic solutions of the KdV equation is based on the use of the finite-band equations (2.1) in the representation of the rational functions (2.5). On equating the corresponding coefficients of the Laurent expansion in $\wp$ of the left- and right-hand sides (2.1), yields simple algebraic equations which determine the parameters of the expression (2.5) for the initial elliptic finite-gap solutions of the KdV equation.

3.1 One-gap elliptic solutions

In the one-gap case the parameters of the elliptic solution $U(z)$ are determined by the system of three finite-gap equations of the form (2.1) at $n = \overline{0,2}$. A substitution in these equations, of the explicit expressions (2.2) for A_n and polynomials in U expressions, for the χ_n-functions which follow from (2.3), yields the system

$$\begin{aligned}
&\frac{1}{2}a_1 - b_1 = -\frac{1}{2}U;\\
&\frac{1}{2}\{(a_2 - \frac{1}{4}a_1^2) + 2(b_1^2 - b_2) - a_1 b_1\} = -\frac{1}{2^3}\{U^2 - U^{(2)}\};\\
&\frac{1}{3!}\{(\frac{3}{8}a_1^3 - \frac{3}{2}a_1 a_2 + \frac{3}{2}a_3) + 3(\frac{1}{4}a_1^2 - a_2)b_1 + 3a_1(b_1^2 - b_2)\\
&\qquad + (12b_1 b_2 - 4!b_3 - 6b_1^3)\} = \frac{1}{2^5}\{U^{(4)} - 5U^{(1)^2} + 6UU^{(2)} - 2U^3\}
\end{aligned} \tag{3.1}$$

in which $b_n|_{n\geq 2} = 0$ (in view of the relation $b_n|_{n\geq g+1} = 0$), where g is the number of gaps in the spectrum of the auxiliary linear differential equation. Excluding b_n from the system (3.1) we can obtain the equations

$$\begin{aligned}
b_2 = 0 = &\frac{1}{8}(3U^2 - U^{(2)}) + \frac{1}{4}a_1 U + \frac{1}{2}a_2 - \frac{1}{8}a_1^2;\\
b_3 = 0 = &-\frac{1}{32}(U^{(4)} + 10U^3 - 5U^{(1)^2} - 10UU^{(2)}) - \frac{1}{16}a_1(3U^2 - U^{(2)})\\
&+ \frac{1}{16}U(a_1^2 - 4a_2) + \frac{1}{2}a_3 + \frac{1}{4}a_1 a_2 - \frac{1}{16}a_1^3,
\end{aligned} \tag{3.2}$$

which is known [2] as the one-gap "trace formulae". These equations form a closed system which determines the initial elliptic solutions (2.5). Inserting the rational expression for $U(z)$ (2.5) into the system (3.2) and equating coefficients of the Laurent expansion in $\wp$ in the right-hand sides to zero, we obtain algebraic relations which lead to the equalities

$$\begin{aligned}
&1)\quad \alpha = 2\,\alpha_{1_i} = \alpha_{2_i} = 0,\ a_1 = 0,\ a_2 = -\frac{1}{4}g_2 E,\ a_3 = -\frac{1}{4}g_3;\\
&2)\quad \alpha = 2,\ \alpha_{(1,2)_i} = \frac{1}{4}\beta_{(1,2)},\ a_1 = 0,\ a_2 = \frac{1}{4}(11g_2 - 120\wp^2(\varphi_1)),\\
&\qquad a_3 = -\frac{1}{4}g_3 + 4\wp(\varphi_1)(12\wp^2(\varphi_1) - g_2) - 4\wp'^2(\varphi_1).
\end{aligned} \tag{3.3}$$

Here and below $\beta_1 = 12h^2 - g_2$, $\beta_2 = 2h'^2$, φ_1 is the argument of the function $h = \wp(\varphi_1)$ which satisfies the equation $(12h^2 - g_2)^2 = 48hh'^2$, from which $\varphi_1 = (2/3)\omega_i|_{i=(1,2,4)}$, where $\omega_4 = (\omega - \omega')$ (see also [10]).

The two systems (3.3) determine two types of initial elliptic solutions:

$$\begin{aligned}
&1)\quad U(z) = 2\wp(z);\\
&2)\quad U(z) = 2\wp(z) + 2[\wp(z - \varphi_1) + \wp(z + \varphi_1)] - 4\wp(\varphi_1),
\end{aligned} \tag{3.4}$$

where the first type is the known [8] Lamé potential and the second type is a new potential obtained in [10].

3.2 2-gap elliptic initial solutions

Coefficients of the initial elliptic 2-gap solutions are determined by the system of the four finite-gap equations of the form (2.1) at $n = \overline{0,3}$. In analogy to the one-gap case, the explicit form can be obtained by substituting the expressions (2.2) for A_n and the expressions for χ_n (from (2.3)) into (2.1). In doing so, the first two equations, which coincide with the first two equations of the system (3.2), are solvable with respect to b_1 and b_2. Excluding the latter from the fourth and fifth equation and taking into account the equality $b_n|_{n\geq 3} = 0$ we obtain the finite-gap system

$$\begin{aligned} b_3 = 0 = {} & \frac{1}{2^5}(16a_3 + 8a_2U + 10U^3 - 5U'^2 - 2a_1U'' \\ & - 10UU'' + U^{(4)}); \\ b_4 = 0 = {} & \frac{1}{2^7}(-16a_2^2 + 64 * a_4 + 32a_3U + 24a_2U^2 + 35U^4 \\ & - 70UU'^2 - 8a_2U'' - 70U^2U'' + 21U''^2 \\ & + 28U'U^{(3)} + 14UU^{(4)} - U^{(6)}). \end{aligned} \tag{3.5}$$

Inserting the rational expression (2.5) for $U(z)$ into the system (3.5) and equating coefficients of the Laurent expansion in $\wp$, on the right- and left-hand sides, we obtain algebraic relations which lead to the equalities

$$\begin{aligned} 1) \quad & \alpha = 6,\ \alpha_{1_i} = \alpha_{2_i} = 0,\ a_1 = 0,\ a_2 = -\tfrac{21}{4}g_2, \\ & a_3 = -\tfrac{27}{4}g_3,\ a_4 = \tfrac{27}{4}g_2^2,\ a_4 = -\tfrac{81}{4}g_2g_3; \\ 2) \quad & \alpha = 6,\ \alpha_{1_i} = \delta_{i,j}(12e_j^2 - \tfrac{1}{2}g_2),\ a_2 = -7(9e_j^2 + 2\Lambda_j), \\ & a_3 = 18(3e_j^3 - 5e_j\Lambda_j),\ a_4 = 27(36e_j^4 + 16e_j^2\Lambda_j + 3\Lambda_j^2), \\ & a_5 = -54(36e_j^5 - 52e_j^3\Lambda_j - 9e_j\Lambda_j^2), \\ & \Lambda_j = 3e_j^2 - \tfrac{1}{4}g_2; \\ 3) \quad & \alpha = 6,\ \alpha_{1_i} = (\delta_{i,j} + \delta_{i,k})(12e_i^2 - g_2),\ a_2 = 161\Lambda_j - 378e_j^2, \\ & a_3 = 531e_j\Lambda_j + 108e_j^3,\ a_4 = 27(240\Lambda_j^2 + 1280e_j^2 - 1159e_j^2\Lambda_j), \\ & a_5 = 27(1594e_j\Lambda_j^2 + 8100e_j^5 + 120e_j^3\Lambda_j^2); \\ 4) \quad & \alpha = 6,\ \alpha_{1_i} = \tfrac{1}{4}\beta_1\delta_{i,1},\ \alpha_{2_i} = \tfrac{1}{4}\beta_2\delta_{i,1}, \\ & a_2 = \tfrac{7}{2}(18h^2 + \tfrac{7}{2}\alpha_1),\ a_3 = \tfrac{9}{4}(24h^3 - 25h'^2) - \tfrac{34h}{2}\alpha_1), \\ & a_4 = \tfrac{27}{4}(144h^4 + 44hh'^2 + 56h^2\alpha_1 + \tfrac{23}{4}\alpha_1^2), \\ & a_5 = -\tfrac{27}{2}(144h^5 + 6h^2h'^2 - 74h^3\alpha_1 - \tfrac{41}{2}h'^2\alpha_1 - \tfrac{75h}{2}\alpha_1^2), \end{aligned} \tag{3.6}$$

where φ_2 is determined by the equality $h = \wp(\varphi_2)$ which satisfies the equation $64h'^4 + 48hh'^2\alpha_1 - \alpha_1^3 = 0$. These four types of equalities (3.6) lead to the following four expressions

$$\begin{aligned} 1) \quad & U(z) = 6\wp(z); \\ 2) \quad & U(z) = 6\wp(z) + 2\wp(z + \omega_i) - 2e_i; \\ 3) \quad & U(z) = 6\wp(z) + 2\wp(z + \omega_i) + 2\wp(z + \omega_k) - 2(\wp(\omega_i) + \wp(\omega_j)); \\ 4) \quad & U(z) = 6\wp(z) + 2\wp(z - \varphi_2) + 2\wp(z + \varphi_2)) - 4\wp(\varphi_2). \end{aligned} \tag{3.7}$$

which describe all possible initial two-gap solutions of the KdV equations (see [11, 10]). The first type coincides with the known two-gap Lamé potential, while the second and third type corresponds to the Treibich-Verdier potential [8, 12]. The fourth solution is the new two-gap potential obtained in [10].

3.3 Initial elliptic three-gap solutions

The parameters of the initial elliptic three-gap solutions (2.5) are determined by the system of the finite-band equations (2.1) at $b_{n\leq 3} \neq 0$ and $b_{n>3} = 0$. Using the expressions (2.2) and (2.3) this system can be expressed through U-functions. In doing so, the first three equations coinciding formally with (3.1) and are solvable with respect to $\overline{b_1, b_3}$. Therefore, taking into account the equality $b_n|_{n\geq 4} = 0$, we can obtain the equation

$$\begin{aligned} b_4 = 0 = \; & -16a_2^2 + 64a_4 + 32a_3U + 24a_2U^2 + 35 * U^4 \\ & -70 * UU'^2 - 8 * a_2U'' - 70 * U^2U'' \\ & +21U''^2 + 28U'U''' + 14UU^{(4)} - U^{(6)}, \end{aligned} \tag{3.8}$$

which is solvable with respect to parameters of the rational expression $U(z)$ (2.5). Substituting (2.5) into the equation (3.8) and equating coefficients of the Laurent expansion in $\wp$ with its right-hand side to zero we can obtain closed algebraic relations for α-parameters of the U-solution (2.5).

It can be shown that there are four possible types of solutions for the α-parameters. The corresponding four types of initial elliptic solutions can be written in the form

$$\begin{aligned} &1) \quad U(z) = 12\wp(z); \\ &2) \quad U(z) = 12\wp(z) + 2\wp(z+\omega_i) - 2e_i, \; e_i = \wp(\omega_i); \\ &3) \quad U(z) = 12\wp(z) + 2(\wp(z+\omega_i) + \wp(z+\omega_j)) - 2(e_i + e_j); \\ &4) \quad U(z) = 12\wp(z) + 2(\wp(z+\varphi_3) + \wp(z-\varphi_3)) - 4\wp(\varphi_3). \end{aligned} \tag{3.9}$$

Here the argument φ_3 is determined by the equation

$$h^6 + \frac{101}{196}g_2h^4 + \frac{29}{49}g_3h^3 - \frac{43}{784}g_2^2h^2 - \frac{23}{196}g_2g_3h - \left(\frac{1}{3136}g_2^3 + \frac{5}{98}g_3^2\right) = 0$$

where $h = \wp(\varphi_3)$. The initial solutions 1 and 2, 3 are the 3-gap Lamé ([8]) and generated on the 3-gap case Treibich-Verdier [12] potentials, respectively. The initial elliptic solution 4 is the 3-gap generalization of the solution obtained in [10].

Note that the above described algorithm is general and applicable to computing arbitrary initial elliptic n-gap solutions of the KdV equation.

4 A dynamics of the elliptic finite-gap solutions

The time dependent elliptic finite-gap solutions of the KdV equation have the general form of the rational functions of the $\wp$-function (2.4), parameters of which are functions of time t. These parameters are described by the system of the auxiliary dynamic equation (2.6) and the finite-gap equation (2.1).

By substituting the expression (2.4) for the elliptic finite-gap solutions $U(z,t)$ into the KdV equation $\partial_t U(z,t) = 6U(z,t)U'(z,t) - U'''(z,t)$ and equating the coefficients of the Laurent expansion in $\wp$, in its left- and right-hand sides, lead to the known general formula

$$U(z,t) = 2\sum_{i=1}^{N} \wp(z - \varphi_i(t)) + C, \tag{4.1}$$

in which the integer number N and the constant C are determined by the condition of the reduction of $U(z,t)$ to the corresponding initial elliptic finite-gap solution at $t \to 0$. In the cases of the elliptic 1-, 2- and 3-gap solutions the numbers N must provide the reduction of the time dependent solutions (4.1) to the corresponding initial elliptic solutions of the systems (3.4), (3.7) and (3.9), respectively.

The substitution of the expression (4.1) into the KdV equation reduces the problem of the time evolution of $U(z,t)$ to the time evolution its poles. The latter is described by the system

$$\begin{aligned} &\partial_t \varphi_i(t) = -12X_i(t) + C,\ \ X_i(t) = \sum_{j=1, j\neq i}^{N-1} \wp(\varphi_i(t) - \varphi_j(t))\ (g \geq 2), \\ &\partial_t \sum_{n=1}^{N} \wp(z - \varphi_i(t)) = 0\ \ (g = 1), \end{aligned} \tag{4.2}$$

which is a system of coupled equations. However, in view of the symmetry properties of the finite-gap equations, (2.1) determine $X_i(t)$ as function $X_i(\varphi_i(t))$. Then the system (4.2) can be transformed to

$$\int_{\varphi_{0i}}^{\varphi_i} \frac{d\varphi_i}{X_i(h_i(\varphi_i)) + C} = -12t, \tag{4.3}$$

which describe the time dynamics of the poles in the expression (4.1). Here, initial values $\varphi_{0i} \equiv \varphi_i(0)$ are determined from the expressions for initial elliptic finite-gap solutions.

The functions $X_i(\varphi_i)$ are determined by the finite-gap equations (2.1) with U-functions in the form (4.1) as roots of Nth order polynomials in X_i. These polynomials are followed from the algebraic equations which can be obtained by equating coefficients of the Laurent expansion in $\wp$ of the left- and right-hand sides of the indicated equations (2.1).

Thus, the problem of time evolution of the elliptic finite-gap solutions of the KdV equations is reduced to the solution of the finite-band equations, with respect to the X_i-functions from relations (4.3). The proposed approach will now be applied in calculating the time evolution of elliptic 1-, 2- and 3-gap solutions.

4.1 Elliptic 1-gap solutions

The types of time dependent elliptic 1-gap solutions of the KdV equation are determined by the values of the number N in the expression (4.1). The condition of a coincidence in the general expression (4.1) with the initial elliptic 1-gap solutions (3.4) at $t \to 0$, yields the values, namely $N = 1$ and $N = 3$. The corresponding elliptic 1-gap solutions have the form

$$
\begin{aligned}
&1)\quad U(z,t) = 2\wp(z - \varphi_1(t)),\\
&2)\quad U(z,t) = 2\sum_{1}^{3}\wp(z - \varphi_i(t)) - 4\wp(\varphi_1).
\end{aligned}
\tag{4.4}
$$

The time dependence of poles of the elliptic solutions 1 and 2 is determined by the second equation of the system (4.2) at $N = 1$ and $N = 2$, respectively. In accordance with the initial conditions defined by the system (3.4), the solutions of the dynamic equation have the form $\varphi_1 = c_1 t$ at $N = 1$ and

$$
\varphi_1(t) = c_1 t,\ \varphi_2(t) = c_2 t + \varphi_i^{(1)},\ \varphi_3(t) = c_3 t - \varphi_i^{(1)}
$$

at $N = 3$. The substitution of these expressions into (4.4) yields the following two expressions

$$
\begin{aligned}
&1)\quad U(z,t) = 2\wp(z - c_1 t);\\
&2)\quad U(z,t) = 2\{\wp(z - c_1 t) + \wp(z - c_2 t + \varphi_i^{(1)}) + \wp(z - c_3 t - \varphi_i^{(1)}) - 4\wp(\varphi_1),
\end{aligned}
$$

which describes two possible types of the elliptic 1-gap solutions in the form of superpositions of one and three independent traveling waves, respectively.

4.2 Elliptic 2-gap solutions

The possible types of the time dependent elliptic 2-gap solutions of the KdV equation (4.1) are determined by the values N which can be obtained from the compatibility condition between (4.1) and (3.7) as $t \to 0$. Under this condition the number N equal 3, 4 and 5 in the formula (4.1).

1. The time dependent 2-gap elliptic solution corresponding to the initial condition 1 in the system (3.7), which is described by the expression (4.1) at $N = 3$, has the form

$$
U(z,t) = 2\sum_{i=1}^{3}\wp(z - \varphi_i(t)). \tag{4.5}
$$

The time evolution of poles $\varphi_i(t)$ is described by the equation (4.3) in which $X_i = \sum_{j\neq i,j=1}^{2}\wp(\varphi_i - \varphi_j)$. Under initial conditions, the lower limits of the integration in (4.3) are $\varphi_{0i} = 0$.

To comput the X_i-function we substitute the expression (4.5) into the finite-gap equation (3.2). Then, equating coefficients of the Laurent expansion in $\wp$ of the left- and right-hand sides, we obtain algebraic equations which are reduced to the polynomial equation

$$
X^3 + c_2 X^2 + c_1 X + c_0 = 0, \tag{4.6}
$$

where $c_n = c_n(\varphi)$, $n = \overline{1,3}$ (here and below the subscript i of φ is omitted). Three solutions of (4.6) describe three unknown functions $X_i(\varphi_i)$, $i = \overline{1,3}$. The dependence of the coefficient functions c_n on φ is described by the expressions

$$
\begin{aligned}
c_0 &= -\frac{36}{125}g_3 - \frac{2}{125}a_2 - \frac{1}{250}(149g_2 + 76a_2)h + \frac{1}{400}(g_2 + 4a_2)\Big(\frac{\beta_1}{h'}\Big)^2;\\
c_1 &= \frac{29}{250}g_2 + \frac{8}{125}a_2;\\
c_2 &= -\frac{42}{125}h + \frac{3}{800}\Big(\frac{\beta_1}{h'}\Big)^2,
\end{aligned}
$$

where $\beta_1 = 12h^2 - g_2$ and $h = \wp(\varphi)$ so that X_i depend on φ through the h-function.

2. The time dependent elliptic 2-gap solution corresponding to the condition 2 in the system (3.7) is described by the expression (4.1) at $N = 4$ and has the form

$$
U(z,t) = 2\sum_{i=1}^{4}\wp(z - \varphi_i(t)) - 2\wp(\varphi_{04}). \tag{4.7}
$$

The time evolution of poles $\varphi_i(t)$ are described via the function $X_i = \sum_{j\neq i,j=1}^{3}\wp(\varphi_i - \varphi_j)$. In view of the initial conditions, the lower limits of an integration in (4.3) are $\varphi_{0i}|_{i\leq 3} = 0$ and $\varphi_{04} = \omega_j$.

For computing the X_i-functions we substitute the expression (4.7) into the system (3.5). Then, equating coefficients of the Laurent expansion of the left- and right-hand sides, we obtain the equation

$$
X^4 + c_3X^3 + c_2X^3 + c_1X + c_0 = 0, \tag{4.8}
$$

solutions of which describe the functions $X_n(\varphi)$, $n = \overline{1,4}$. The dependence of the coefficient functions c_n on φ are described by the expressions

$$
\begin{aligned}
c_0 =& -\tilde{m}_1^0 g_2^2 + \tilde{m}_2^0 g_2 a_2 + \tilde{m}_3^0 a_2^2 - \tilde{m}_4^0 a_4;\\
c_1 =& \tilde{m}_1^1 g_3 + \tilde{m}_2^1 a_3 + \tilde{m}_3^1 g_2 h + \tilde{m}_4^1 a_2 h - \tilde{m}_5^1 g_2\Big(\frac{\beta_1}{h'}\Big)^2 - \tilde{m}_6^1 a_2\Big(\frac{\beta_1}{h'}\Big)^2;.\\
c_2 =& -\tilde{m}_1^2 g_2 - \tilde{m}_2^2 a_2;\; c_3 = -\tilde{m}_1^3 h + \tilde{m}_2^3\Big(\frac{\beta_1}{h'}\Big)^2.
\end{aligned} \tag{4.9}
$$

Here $\tilde{m}_j^i$ denotes numerical parameters which have the form of some rational fractions.

3. The time dependent elliptic 2-gap solutions, corresponding to the initial conditions 3 and 4 in the system (3.7) which are described by the expression (4.1) at $N = 5$, have the form

$$
U(z) = 2\sum_{i=1}^{5}\wp(z - \varphi_i(t)) - 2(\wp(\varphi_{04}) + \wp(\varphi_{05})). \tag{4.10}
$$

In view of the initial conditions, the lower limits of the integration in (4.3) have the common values $\varphi_{0i}|_{i\leq 3} = 0$ and $\varphi_{04} = \omega_i$, $\varphi_{05} = \omega_j$ at the condition 3 and $\varphi_{04} = -\varphi_{05} = \varphi_2$ at the condition 4.

For computing the X_i-functions we substitute the expression (4.10) into the system (3.5). Then, equating coefficients of the Laurent expansion in $\wp$ of the left- and right-hand sides, we obtain the equation

$$
X^5 + c_3X^3 + c_2X^2 + c_1X + c_0 = 0,
$$

solutions of which describe functions $X_n(\varphi)$, $n = \overline{1,5}$. Coefficient functions c_n are described by the expressions

$$c_0 = \sum_{i=1}^{5}(m_i^0 g_3 + n_i^0 a_3)F_i(\varphi) + m_6^0 g_2 g_3 + m_7^0 g_3 a_2 - m_8^0 g_2 a_3 + m_9^0 a_2 a_3 - m_{10}^0 a_5;$$

$$c_1 = \sum_{i=1}^{5}(m_i^1 g_2 + n_i^1 a_2)F_i(\varphi) - m_6^1 g_2^2 - m_7^1 g_2 a_2 + m_8^1 a_2^2 + m_9^1 a_4;$$

$$c_2 = m_1^2 g_3 + m_2^2 a_3; \quad c_3 = \sum_{i=1}^{5} m_i^3 F_i(\varphi) - m_2^3 g_2 + m_3^3 a_2,$$

where

$$F_i(\varphi) \equiv \left(\delta_{i,1}\beta_1 + \delta_{i,2}h^2 + \delta_{i,3}\Big(\frac{\beta_1}{h'}\Big)^4 + \delta_{i,4}h\Big(\frac{\beta_1}{h'}\Big)^2 + \delta_{i,5}\Big(\frac{\beta_1 h^{(4)}}{h'^2}\Big)\right).$$

Here m_i^j, n_i^j denotes some numerical rational fractions, $\beta_1 = 12\wp^2(\varphi) - g_2$, $h = \wp(\varphi)$, $a_i = a_i(\wp(\varphi))$ denote some complicate functions, the explicit form of which we do not present here.

4.3 Elliptic 3-gap solutions

The possible types of the time dependent elliptic 3-gap solutions of the KdV equation (4.1) are determined by the values N which are obtained from the compatibility condition between (4.1) and (3.9) as $t \to 0$. Under this condition, the number N takes values $\overline{6,8}$ in the formula (4.1).

1. The values $N = 6$ and $N = 7$ determine two elliptic 3-gap solutions of the KdV equations with initial conditions 1 and 2 of the system (3.9), which have the form

$$U(z,t) = 2\sum_{i=1}^{6}\wp(z - \varphi_i(t)), \quad \text{and} \quad U(z,t) = 2\sum_{i=1}^{7}\wp(z - \varphi_i(t)) - 2\wp(\varphi_{07}), \tag{4.11}$$

respectively. The time evolution of the poles $\varphi_i(t)$ are described by relations (4.3) with the lower integration limits $\varphi_{0,i}|_{i=\overline{1,6}} = 0$ at the initial condition 1 and $\varphi_{0,i}|_{i=\overline{1,6}} = 0$, $\varphi_{0,7} = \omega_i$ at the initial condition 2.

By substituting the expressions (4.11) into the finite-band equation (3.8) and equating coefficients of the Laurent expansion in $\wp$ of the right-hand sides to zero, lead to the two equations

$$\sum_{i=0}^{6} c_{6,i}X^i = 0, \quad \text{and} \quad \sum_{i=0}^{7} c_{7,i}X^i = 0, \tag{4.12}$$

corresponding to the two 3-gap solutions with the values $N = 6$ and $N = 7$, respectively. Coefficient functions $c_{6,i}$ and $c_{7,i}$ in (4.12) are rational functions on $\wp(\varphi_i)$ and $\wp'(\varphi_i)$.

Therefore X_i as roots of (4.12) are functions of φ_i, i.e. $X_i = X_i(\varphi_i)$ where $i = \overline{1,6}$ and $i = \overline{1,7}$ at initial conditions 1 and 2, respectively.

2. The value $N = 8$ determines two similar elliptic 3-gap solutions of the KdV equations with the initial conditions 3 and 4 of the system (3.9), which have the form

$$U(z,t) = 2\sum_{i=1}^{8} \wp(z - \varphi_i(t)) - 2(\wp(\varphi_{07}) + \wp(\varphi_{08})). \tag{4.13}$$

The poles $\varphi_i(t)$ of (4.13) are described by relations (4.3), with the lower integration limits $\varphi_{0i}|_{i=\overline{1,6}} = 0$, $\varphi_{07} = \omega_i$, $\varphi_{08} = \omega_j$ and $\varphi_{0i}|_{i=\overline{1,6}} = 0$, $\varphi_{07} = -\varphi_{08} = -\varphi_3$ at initial conditions 3 an 4, respectively.

Substituting the expressions (4.13) into the finite-gap equation (3.8) and equating the coefficients of the Laurent expansion in $\wp$ of the right-hand sides to zero, we obtain two equations of the form

$$\sum_{i=0}^{8} c_{6,i} X^i = 0, \tag{4.14}$$

where $c_i = c_i(\varphi)$. Eight solutions of (4.14) coincide with eight functions $X_i(\varphi_i)$, $i = \overline{1,8}$.

The proposed approach is applicable for computing arbitrary elliptic n-gap solutions of the KdV equation. It can also be applied for computing finite-gap elliptic solutions for other integrable equations.

5 Conclusion

The solution of the KdV equation in the class of elliptic finite-gap functions is reduced to the solution of the system of finite-gap equations and auxiliary dynamic equations. In terms of the elliptic $\wp$-function this system reduces to simple algebraic relations which determin the parameters of the unknown solutions which are represented as rational functions of the $\wp$-function. This approach gives a simple algorithm for calculating arbitrary elliptic finite-gap solutions of the KdV equations at an initial time, which was demonstrated by the example of 1-, 2- and 3-gap solutions. The time evolutions of the unknown solutions with a known $\wp$-functional representation, are determined by the dynamics of their poles, which is described by coupled systems of dynamic equations. The latter always can be integrated with the help of the finite-gap equations. This was demonstrated by the example of elliptic 1-, 2- and 3-gap solutions.

The above approach will also be applied to other integrable nonlinear equations in a future paper.

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