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Lax Pairs, Painlevé Properties and Exact Solutions of the Calogero Korteweg-de Vries Equation and a New $(2 + 1)$-Dimensional Equation

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Abstract

We prove the existence of a Lax pair for the Calogero Korteweg-de Vries (CKdV) equation. Moreover, we modify the T operator in the the Lax pair of the CKdV equation, in the search of a $(2 + 1)$-dimensional case and thereby propose a new equation in (2+1) dimensions. We named this the (2+1)-dimensional CKdV equation. We show that the CKdV equation as well as the (2+1)-dimensional CKdV equation are integrable in the sense that they possess the Painlevé property. Some exact solutions are also constructed.

1 Introduction

In this paper we attempt to extend the Calogero Korteweg-de Vries (CKdV) equation to a $(2 + 1)$-dimensional equation. The CKdV equation is a $(1 + 1)$-dimensional nonlinear equation [1] of the form

$$w_t + \frac{1}{4}w_{xxx} + \frac{3w_x}{8w^2} + \frac{3w_x^3}{8w^2} - \frac{3w_x w_{xx}}{4w} = 0. \tag{1.1}$$

Pavlov constructed (1.1) using the new method for the description of an infinite set of differential substitutions and the KdV modifications [2]. We briefly describe how the CKdV equation was constructed by Pavlov. The Lax pair of the KdV equation

$$u_t + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x = 0 \tag{1.2}$$

has the form

$$L = \partial_x^2 + u, \tag{1.3}$$

$$T = \partial_x L + \frac{1}{2}u\partial_x - \frac{1}{4}u_x + \partial_t. \tag{1.4}$$

Pavlov obtained an infinite set of differential substitutions and the KdV modifications from the Taylor expansion of the linear system for (1.3) and (1.4) respectively (see [2]). The first order of an infinite set of differential substitutions is the Miura transformation

$$u = v^2 + \sigma v_x, \qquad (\sigma = \pm i). \tag{1.5}$$

After substitution of the Miura transformation (1.5) into the first order KdV modifications, we obtain the modified KdV (mKdV) equation

$$v_t + \frac{1}{4} v_{xxx} + \frac{3}{2} v^2 v_x = 0. \tag{1.6}$$

This equation admits the Lax representation

$$L = \partial_x^2 + 2\sigma v \partial_x, \tag{1.7}$$

$$T = \partial_x L + \sigma v \partial^2 - \left(\frac{3}{2} v^2 + \frac{1}{2} \sigma v_x \right) \partial_x + \partial_t. \tag{1.8}$$

The representation (1.7), (1.8) can be obtained from the Lax pair of the KdV equation (1.3), (1.4) by the gauge transformation [3]. In the second order, an infinite set of differential substitutions and the KdV modifications, lead to the Miura type transformation

$$v = -\frac{1}{2w}(1 + \sigma w_x) \tag{1.9}$$

and the CKdV equation (1.1). Hamiltonian structures for the CKdV equation are discussed in [2].

This paper is organized as follows. In Section 2, we construct a Lax pair of the CKdV equation (1.1) and propose a new equation in $(2+1)$ dimensions by the extension of the T operator for the CKdV equation. We named it the $(2+1)$-dimensional CKdV equation. Moreover, another dimensional extension is performed by changing the L operator [4, 5, 6] as follows:

$$L \mapsto L + \partial_y. \tag{1.10}$$

A $(2+1)$-dimensional equation obtained by the abovemethod is, however, reduced to the KP equation. In Section 3, the CKdV equation and the $(2+1)$-dimensional CKdV equation are proved to be integrable in the sense that they possess the Painlevé property. The solutions to these equations are constructed by the Miura transformation in Section 4. Section 5 is devoted to discussions.

2 The Lax pairs of the CKdV equation and the $(2+1)$-dimensional CKdV equation

We conjecture that a Lax pair of the CKdV equation (1.1) is of the form

$$L = \partial_x^2 + g[w]\partial_x + h[w], \tag{2.1}$$

$$T = \partial_x L + T' + \partial_t, \tag{2.2}$$

where $g[w]$, $h[w]$ are functions of w and its x-derivatives, and T' is an unknown operator. We can fix the form of $g[w]$, $h[w]$ and T' by the condition that the Lax equation

$$[L, T] = 0 \tag{2.3}$$

gives the CKdV equation. The result is

$$g[w] = \frac{\sigma}{w}, \tag{2.4}$$

$$h[w] = -\frac{1}{4w^2} - \frac{\sigma w_x}{2w^2}, \tag{2.5}$$

$$T' = \frac{\sigma}{2w}\partial_x^2 - \frac{1}{2w^2}\partial_x - \frac{3\sigma}{16w^3} + \frac{w_x}{4w^3} - \frac{\sigma w_x^2}{16w^3} + \frac{\sigma w_{xx}}{8w^2}. \tag{2.6}$$

Hence the Lax pair of the CKdV equation is expressed as

$$L = \partial_x^2 + \frac{\sigma}{w}\partial_x - \frac{1}{4w^2} - \frac{\sigma w_x}{2w^2}, \tag{2.7}$$

$$T = \partial_x L + \frac{\sigma}{2w}\partial_x^2 - \frac{1}{2w^2}\partial_x - \frac{3\sigma}{16w^3} + \frac{w_x}{4w^3} - \frac{\sigma w_x^2}{16w^3} + \frac{\sigma w_{xx}}{8w^2} + \partial_t. \tag{2.8}$$

Next we construct a new equation in $(2+1)$ dimensions. For that, we modify the above T operator to include another spatial dimension as follows

$$T = \partial_z L + T'' + \partial_t, \tag{2.9}$$

where L is the same L operator (2.7) for the CKdV equation. The Lax equation (2.3) gives not only the form of T'' but also a new equation. They are

$$\begin{aligned} T'' = {} & \frac{1}{2}\sigma\partial_x^{-1}\left(\frac{1}{w}\right)_z \partial_x^2 - \frac{1}{2w}\partial_x^{-1}\left(\frac{1}{w}\right)_z \partial_x + \frac{\sigma w_{xz}}{8w^2} - \frac{\sigma w_x w_z}{8w^3} \\ & -\frac{\sigma}{8w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z + \frac{w_x}{4w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z - \frac{\sigma}{16w}\partial_x^{-1}\left(\frac{1}{w^2}\right)_z + \frac{\sigma}{16w}\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & w_t + \frac{1}{4}w_{xxz} + \frac{w_z}{4w^2} + \frac{1}{8}w_x\partial_x^{-1}\left(\frac{1}{w^2}\right)_z \\ & \quad + \frac{w_x^2 w_z}{2w^2} - \frac{1}{8}w_x\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z - \frac{w_x w_{xz}}{2w} - \frac{w_{xx}w_z}{4w} = 0, \end{aligned} \tag{2.11}$$

respectively. We name the above equation the $(2+1)$-dimensional CKdV equation. It follows from (2.9) and (2.10) that the Lax pair of $(2+1)$-dimensional CKdV equation is given by

$$L = \partial_x^2 + \frac{\sigma}{w}\partial_x - \frac{1}{4w^2} - \frac{\sigma w_x}{2w^2}, \tag{2.12}$$

$$T=\partial_z L+\frac{1}{2}\sigma\partial_x^{-1}\left(\frac{1}{w}\right)_z\partial_x^2-\frac{1}{2w}\partial_x^{-1}\left(\frac{1}{w}\right)_z\partial_x+\frac{\sigma w_{xz}}{8w^2}$$
$$-\frac{\sigma w_x w_z}{8w^3}-\frac{\sigma}{8w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z+\frac{w_x}{4w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z \tag{2.13}$$
$$-\frac{\sigma}{16w}\partial_x^{-1}\left(\frac{1}{w^2}\right)_z+\frac{\sigma}{16w}\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z+\partial_t.$$

Equation (2.11) and the Lax pair (2.12), (2.13) are reduced to the CKdV equation and the Lax pair of the CKdV equation in the case of $x=z$. In [7, 8, 9], we developed the construction method for higher-dimensional integrable equation. For example, we considered the Calogero–Bogoyavlenskij–Schiff (CBS) equation [10, 11, 12, 13, 14, 15, 16],

$$u_t+\frac{1}{4}u_{xxz}+uu_z+\frac{1}{2}u_x\partial_x^{-1}u_z=0, \tag{2.14}$$

and the modified Calogero–Bogoyavlenskij–Schiff (mCBS) [15],

$$v_t+\frac{1}{4}v_{xxz}+v^2v_z+\frac{1}{2}v_x\partial_x^{-1}\left(v^2\right)_z=0. \tag{2.15}$$

These equations admit the Lax representations, respectively [8, 9],

$$L=\partial_x^2+u, \tag{2.16}$$

$$T=\partial_z L+\frac{1}{2}\partial_x^{-1}u_z\partial_x-\frac{1}{4}u_z+\partial_t, \tag{2.17}$$

and

$$L=\partial_x^2+2\sigma v\partial_x, \tag{2.18}$$

$$T=\partial_z L+\sigma\partial_x^{-1}v_z\partial_x^2+\left(\frac{1}{2}\partial_x^{-1}\left(v^2\right)_z-2v\partial_x^{-1}v_z-\frac{1}{2}\sigma v_z\right)\partial_x$$
$$+\frac{\sigma}{4}v_{xz}+\frac{\sigma}{2}v\left(\partial_x^{-1}\left(v^2\right)_z\right)+\sigma\partial_x^{-1}v_t+\partial_t, \tag{2.19}$$

respectively. We obtained the mCBS equation from the CBS equation using the same Miura transformation (1.5) that connects the KdV equation with the mKdV equation [7, 8, 9]. We checked that the transformation (1.9) connects the mCBS equation (2.15) and the $(2+1)$-dimensional CKdV equation (2.11), i.e.,

$$v_t+\frac{1}{4}v_{xxz}+v^2v_z+\frac{1}{2}v_x\partial_x^{-1}\left(v^2\right)_z$$
$$=\left(\frac{1}{2w^2}(1+\sigma w_x)-\frac{\sigma}{2w}\partial_x\right)\left\{w_t+\frac{1}{4}w_{xxz}+\frac{w_z}{4w^2}+\frac{1}{8}w_x\partial_x^{-1}\left(\frac{1}{w^2}\right)_z\right. \tag{2.20}$$
$$\left.+\frac{w_x^2w_z}{4w^2}-\frac{1}{8}w_x\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z-\frac{w_xw_{xz}}{2w}-\frac{w_{xx}w_z}{4w}\right\}.$$

These results are depicted in Fig. 1.

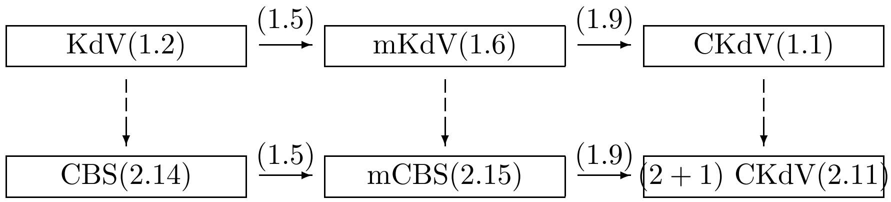


Figure 1: The dimensional extensions are indicated by broken arrows. These broken arrows indicate the modification of T operators for the search of the $(2+1)$-dimensional case. Full arrows mean the Miura transformations. The mKdV equation (6) and the mCBS equation (25) are induced by the Miura transformation (5) from the KdV equation (2) and CBS equation (24). We construct exact solutions of the CKdV equation (1) and the $(2+1)$-dimensional CKdV equation (21) from the solution of the mKdV equation (6) and the mCBS equation (25), by using using the Miura type transformation (9).

We also extend the CKdV equation via (1.10) [4, 5, 6]. Namely we consider

$$L = \partial_x^2 + \frac{\sigma}{w}\partial_x - \frac{1}{4w^2} - \frac{\sigma w_x}{2w^2} + \partial_y. \tag{2.21}$$

The T operator corresponding to (2.21) should be of the form

$$\begin{aligned} T = \partial_x^3 + \frac{3\sigma}{2w}\partial_x^2 + \left\{ -\frac{3}{4w^2} - \frac{3\sigma w_x}{2w^2} - \frac{3}{4}\sigma\partial_x^{-1}\left(\frac{1}{w}\right)_y \right\}\partial_x - \frac{\sigma}{8w^3} \\ + \frac{3w_x}{4w^3} + \frac{3\sigma w_x^2}{4w^3} - \frac{3\sigma w_{xx}}{8w^2} + \frac{3\sigma w_y}{8w^2} + \frac{3}{8}\partial_y^{-1}\left\{\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w}\right)_{yy}\right\} + \partial_t. \end{aligned} \tag{2.22}$$

We can construct the following equation from the Lax equation with (2.21) and (2.22),

$$\begin{aligned} w_t + \frac{1}{4}w_{xxx} + \frac{3w_x^3}{2w^2} - \frac{3w_x w_{xx}}{2w} - \frac{3\sigma w_y}{4w} - \frac{3}{4}w^2\partial_x^{-1}\left(\frac{1}{w}\right)_{yy} \\ -\frac{3}{4}\sigma w_x \partial_x^{-1}\left(\frac{1}{w}\right)_y - \frac{3}{4}\sigma w^2\left(\partial_y^{-1}\left\{\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w}\right)_{yy}\right\}\right)_x = 0. \end{aligned} \tag{2.23}$$

However, the above equation is reduced to the KP equation

$$\left(u_t + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x\right)_x + \frac{3}{4}u_{yy} = 0 \tag{2.24}$$

by the transformation

$$w = -\frac{\sigma}{2\partial_y^{-1}u_x}. \tag{2.25}$$

It follows that we cannot construct a new $(2+1)$-dimensional equation by this method.

In the previous papers [8, 9] we modified both L and T operators for the KdV equation in searching for a $(3+1)$-dimensional equation. However, the Lax equation was reduced to the $(2+1)$-dimensional equation. This equation separated the first and second order equations for the KP hierarchy [17]. Let us apply the same procedure to the CKdV equation and search for a (3 + 1)-dimensional Lax pair. That is, we consider the Lax pair (2.21) and

$$T = \partial_z L + T''' + \partial_t. \tag{2.26}$$

However, we cannot fix the form of T''' by the Lax equation and, therefore, cannot construct a new equation in $(3+1)$ dimensions from the Lax pair (2.21) and (2.26).

3 Painlevé analysis for the CKdV equation and the (2 + 1)-dimensional CKdV equation

To prove the Painlevé property [18, 19] of the CKdV equation (1.1) and the (2 + 1)-dimensional CKdV equation (2.11), we rewrite these equations by the change of of variable

$$W = \frac{1}{w}, \tag{3.1}$$

so that

$$W^2 W_t + \frac{1}{4} W^2 W_{xxx} + \frac{3}{8} W_x^3 + \frac{3}{8} W^4 W_x - \frac{3}{4} W W_x W_{xx} = 0, \tag{3.2}$$

$$\begin{aligned} &W^3 W_x W_{xt} - W^3 W_{xx} W_t + \frac{1}{4} W^3 W_x W_{xxxz} - \frac{1}{4} W^3 W_{xx} W_{xxz} + \frac{3}{4} W W_x^2 W_{xx} W_z \\ &\quad + \frac{3}{4} W W_x^3 W_{xz} - \frac{3}{4} W_x^4 W_z + \frac{3}{4} W^4 W_x^2 W_z + \frac{1}{4} W^5 W_x W_{xz} - \frac{1}{4} W^5 W_{xx} W_z \\ &\quad - \frac{1}{2} W^2 W_x^2 W_{xxz} - \frac{1}{4} W^2 W_x W_{xxx} W_z - \frac{1}{4} W^2 W_x W_{xx} W_{xz} + \frac{1}{4} W^2 W_{xx}^2 W_z = 0. \end{aligned} \tag{3.3}$$

The solutions to (3.2) and (3.3) have the form

$$W \sim W_0 \gamma^\alpha. \tag{3.4}$$

Here γ is single valued about an arbitrary movable singular manifold and α is a negative integer (leading order). By using leading order analysis, we obtain

$$\alpha = -1, \qquad W_0^2 + \gamma_x^2 = 0. \tag{3.5}$$

Substituting

$$W = \sum_{j=0} W_j \gamma^{j-1} \tag{3.6}$$

into (3.2) and (3.3), leads to the resonances of (3.2), namely

$$j = -1, 1, 3, \tag{3.7}$$

and the resonances of (3.3), namely

$$j = -1, 1, 2, 3. \tag{3.8}$$

The resonance $j = -1$ corresponds to the arbitrary singularity manifold γ. We used *MATHEMATICA* [20] to handle the calculation for the existence of arbitrary functions at the above resonances (except for $j = -1$). We find that W_1, W_3 are arbitrary for equation (3.2), and W_1, W_2, W_3 are arbitrary for equation (3.3). Thus the general solution W to (3.2) and (3.3) admits a sufficient number of arbitrary functions, thus satisfying the Painlevé property. Therefore the CKdV equation and the $(2 + 1)$-dimensional CKdV equation are integrable.

4 Exact solutions to the CKdV equation and the $(2 + 1)$-dimensional CKdV equation

In the previous section, the integrability of the CKdV equation and the $(2+1)$-dimensional CKdV equation was shown by the use of the Painlevé test. In this section we shall construct exact solutions of the CKdV equation and the $(2 + 1)$-dimensional CKdV equation.

The mKdV equation (1.6) has the solutions [21]

$$v_N = \sigma \left\{ \log\left(\frac{f_N}{g_N}\right) \right\}_x, \tag{4.1}$$

where f_N and g_N can be expressed as

$$f_N = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} \eta_{i_1 \cdots i_n} \exp(\lambda_{i_1} + \cdots + \lambda_{i_n}), \tag{4.2}$$

$$g_N = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} (-1)^n \eta_{i_1 \cdots i_n} \exp(\lambda_{i_1} + \cdots + \lambda_{i_n}), \tag{4.3}$$

$$\lambda_j = p_j x + r_j t + s_j, \qquad r_j = -\frac{1}{4} p_j^3, \tag{4.4}$$

$$\eta_{jk} = \frac{(p_j - p_k)^2}{(p_j + p_k)^2}, \tag{4.5}$$

$$\eta_{i_1 i_2 \cdots i_{n-1} i_n} = \eta_{i_1,i_2} \cdots \eta_{i_1,i_n} \cdots \eta_{i_{n-1},i_n}. \tag{4.6}$$

Here ${}_N C_n$ indicates summation over all possible combinations of n elements taken from N, and symbols s_j always denote arbitrary constants. We can solve the Miura type transformation (1.9) for w using solutions to the mKdV equation (4.1). The solutions to the CKdV equation (1.1) are

$$w_N = \sigma \left(\frac{f_N}{g_N}\right)^2 \int \left(\frac{f_N}{g_N}\right)^{-2} dx + c \left(\frac{f_N}{g_N}\right)^2, \tag{4.7}$$

where c is an integration constant. The above integral factor is rewritten as

$$\int \left(\frac{f_N}{g_N}\right)^{-2} dx = x + \frac{H_N}{f_N}, \tag{4.8}$$

where

$$H_N = 4\sum_{n=1}^{N}\frac{f_{N-1}(\hat{n})}{p_n} \tag{4.9}$$

and

$$\begin{aligned} f_{N-1}(\hat{j}) = 1 &+ e^{\lambda_1} + \cdots + e^{\lambda_{j-1}} + e^{\lambda_{j+1}} + \cdots + e^{\lambda_n} + \eta_{12}e^{\lambda_1+\lambda_2} + \cdots \\ &+\eta_{1j-1}e^{\lambda_1+\lambda_{j-1}} + \eta_{1j+1}e^{\lambda_1+\lambda_{j+1}} + \cdots + \eta_{1N}e^{\lambda_1+\lambda_N} + \cdots \\ &+\eta_{j-1j+1}e^{\lambda_{j-1}+\lambda_{j+1}} + \cdots + \eta_{j-1N}e^{\lambda_{j-1}+\lambda_N} + \cdots + \eta_{j+1j+2}e^{\lambda_{j+1}+\lambda_{j+2}} + \cdots \\ &+\eta_{j+1N}e^{\lambda_{j+1}+\lambda_N} + \cdots + \eta_{N-1N}e^{\lambda_{N-1}+\lambda_N} + \cdots \\ &+\eta_{12\cdots j-1j+1\cdots N-1N}e^{\lambda_1+\cdots+\lambda_{j-1}+\lambda_{j+1}+\cdots+\lambda_N}, \end{aligned} \tag{4.10}$$

that is, f_{N-1} is of the same structure as f_N, except for the j index. Equation (4.8) is differentiated with respect to x, i.e.,

$$H_{N,x}f_N - H_N f_{N,x} + f_N^2 - g_N^2 = 0. \tag{4.11}$$

We checked equation (4.11) up to $N = 6$ by the use of *MATHEMATICA*. Fig. 2 shows the solution (4.7) with $N = 1$, $p_1 = 1$, $s_1 = 2$ and $c = 0$. In Fig. 3, we depict the case of $N = 2$, $p_1 = 1$, $s_1 = 2$, $p_2 = 0.5$, $s_2 = 5$ and $c = 0$.

We obtain solutions of the (2+1)-dimensional CKdV equation (2.11) using the identical procedure as with the construction of solutions (4.7). Therefore, the form of the solutions are the same as (4.7). The difference between solutions of the $(2+1)$-dimensional CKdV equation and the CKdV equation, is the dimensional extension of (4.4):

$$\lambda_j = p_j x + q_j z + r_j t + s_j, \qquad r_j = -\frac{1}{4}p_j^2 q_j. \tag{4.12}$$

The propagation of the solution to the $(2+1)$-dimensional CKdV equation with $N = 1$, $p_1 = 1$, $q_1 = 3$, $s_1 = 2$ and $c = 0$ is shown in Fig. 4. Fig. 5 shows the solution with $N = 2$, $p_1 = 1$, $q_1 = 3$, $s_1 = 0$, $p_2 = 0.5$, $q_2 = -3$, $s_2 = 0$ and $c = 0$.

5 Conclusions

In this paper, we obtained the Lax pair of the CKdV equation and searched for the Lax pair of the higher dimensional CKdV equation using three methods. The first method is to modify the T operator for the Lax pair of the CKdV equation. We then have obtained the $(2+1)$-dimensional CKdV equation (2.11) and the Lax pair (2.12) and (2.13). The second method is to modify the L operator. We constructed the Lax pair (2.21), (2.22) and the equation (2.23). Equation (2.23) is, however, reduced to the KP equation by the transformation (2.25). In the last method, we unified the first and second methods. Using this method we can expect a new $(3+1)$-dimensional equation. It, however, gives no consistent Lax equation, unlike the first and second methods. We also discussed the Painlevé property and exact solutions of equation (1.1) and equation (2.11), which proves that the equations are integrable.

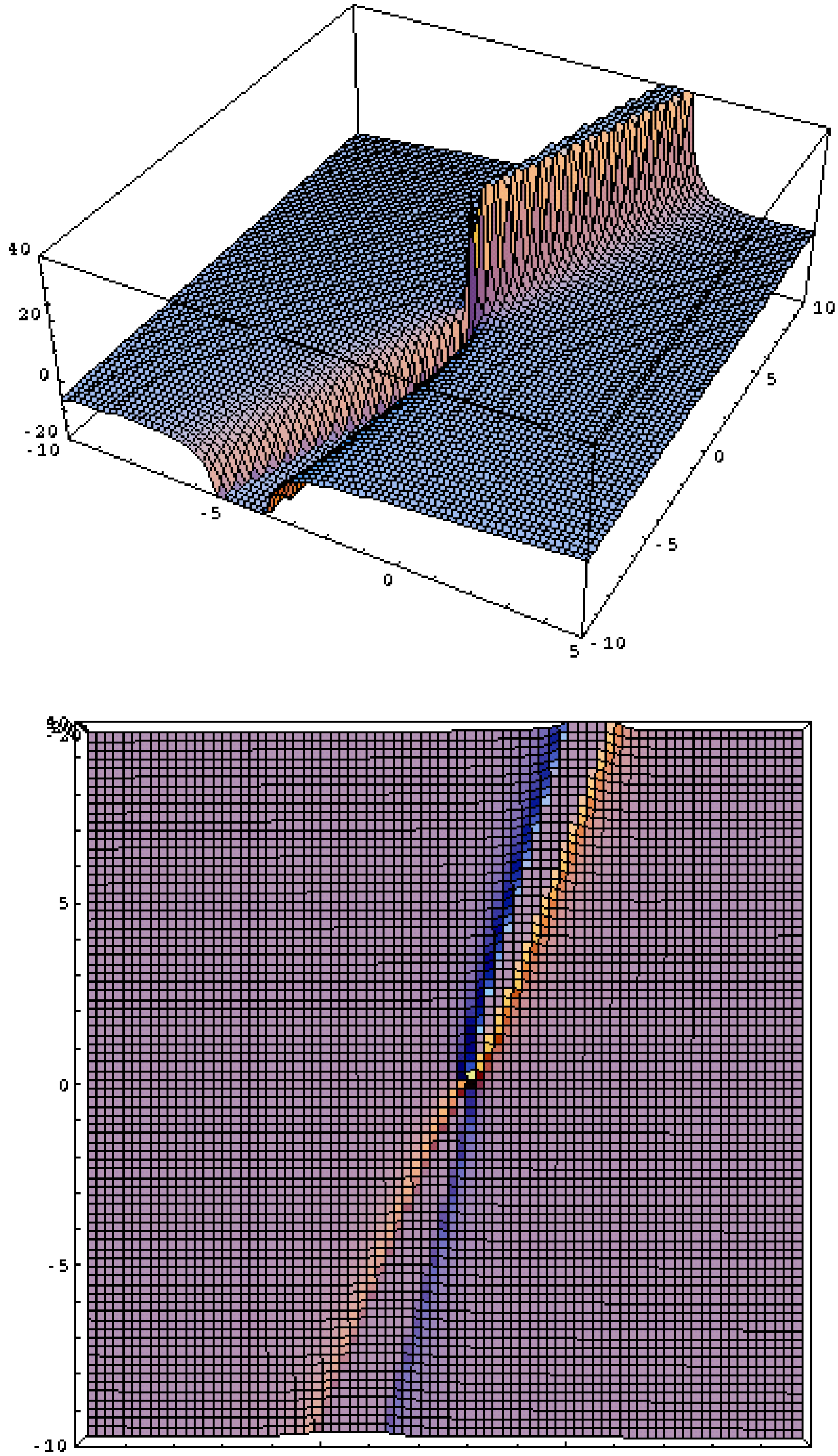


Figure 2: $\frac{w_1(x,t)}{\sigma}$ with $N = 1$, $p_1 = 1$, $s_1 = 2$ and $c = 0$.

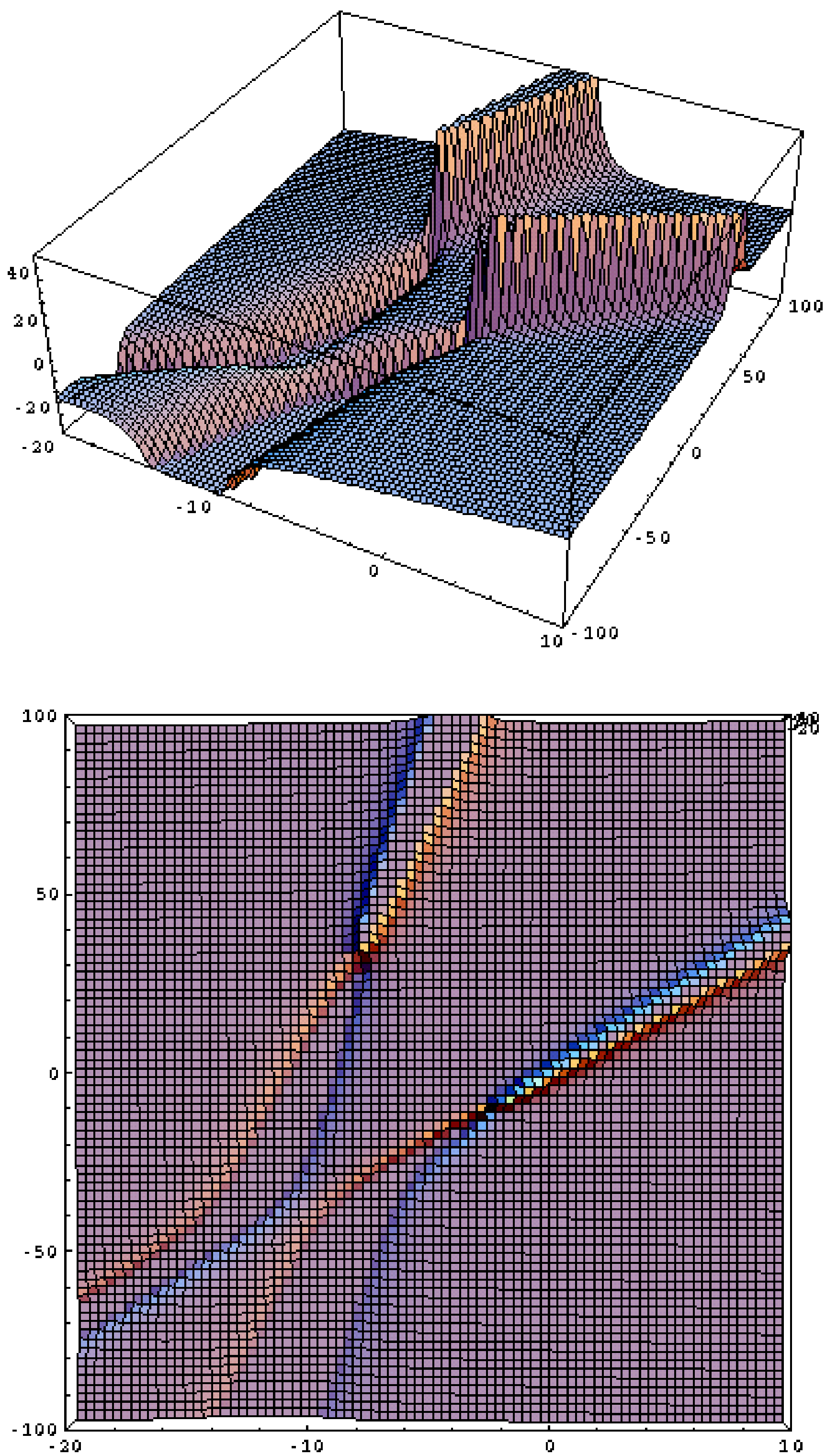
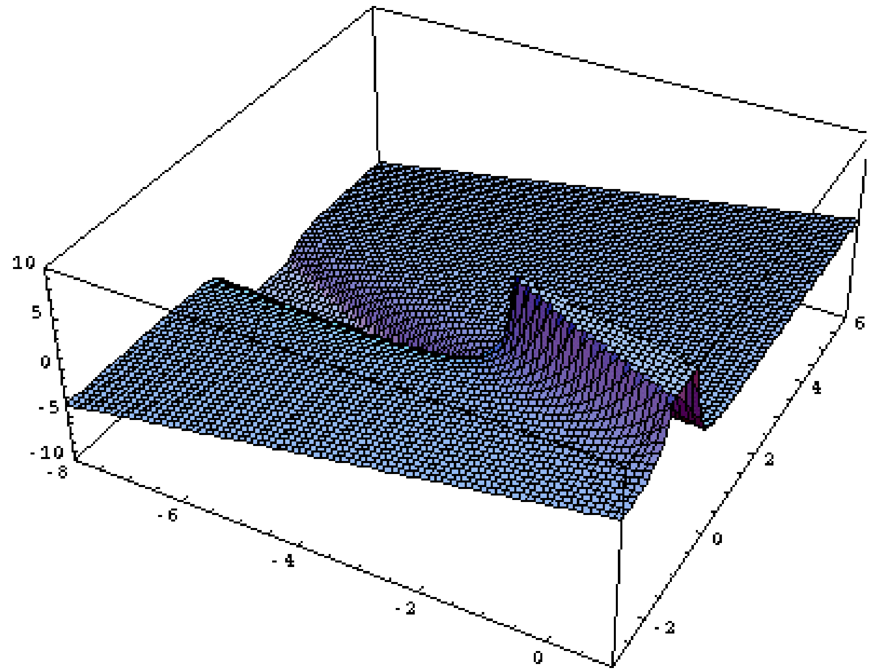


Figure 3: $\dfrac{w_2(x,t)}{\sigma}$ with $N = 2$, $p_1 = 1$, $s_1 = 2$, $p_2 = 0.5$, $s_2 = 5$ and $c = 0$.



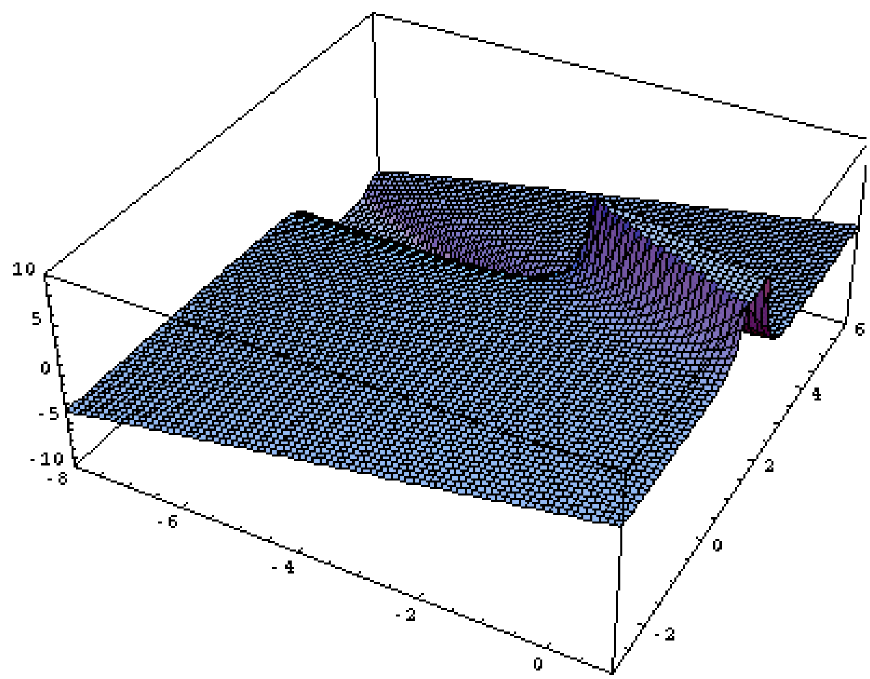


Figure 4: Time evolution of $\dfrac{w_1(x,z,t)}{\sigma}$ with $N=1$, $p_1=1$, $q_1=3$, $s_1=2$ and $c=0$.

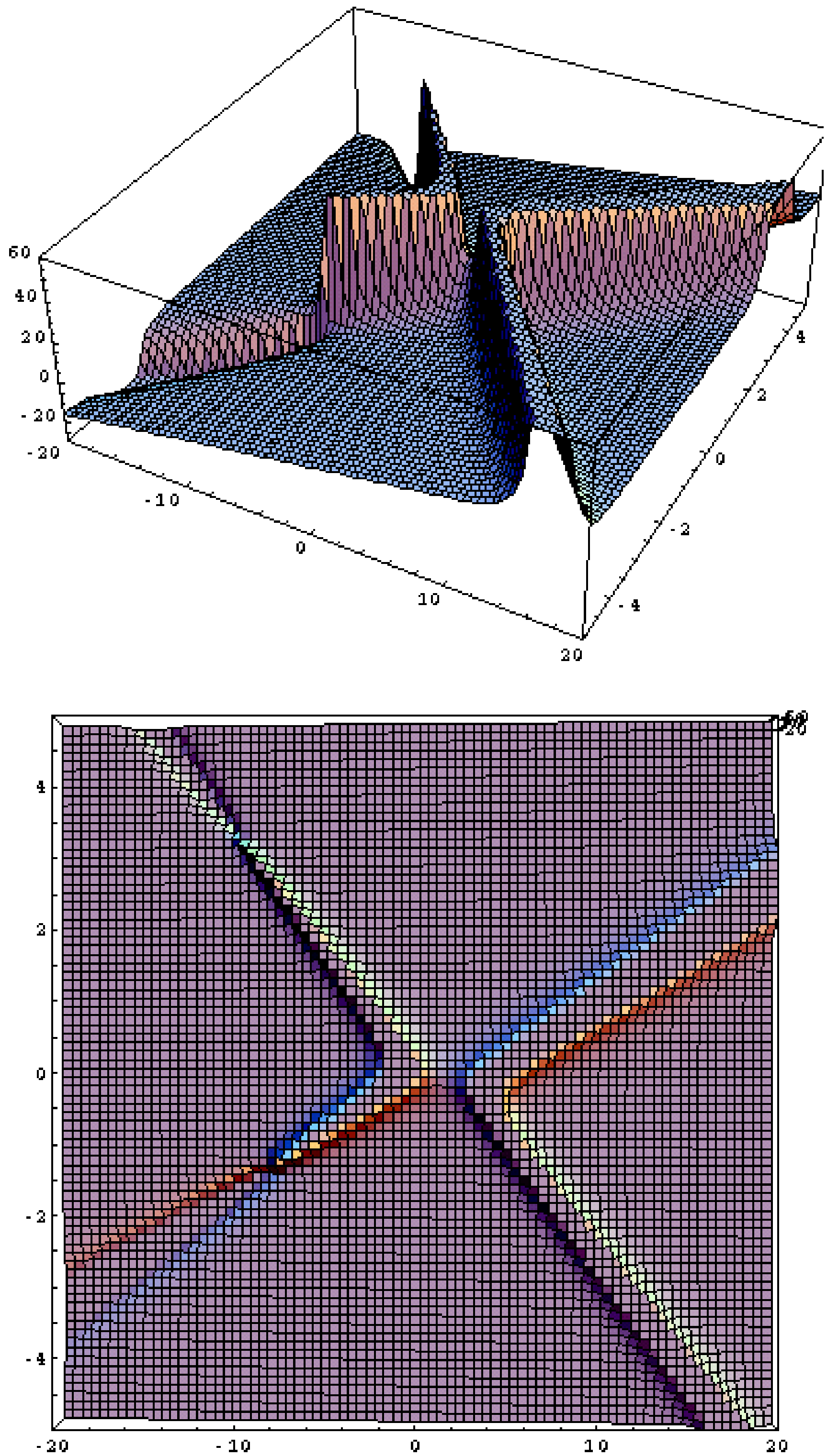


Figure 5: $\dfrac{w_2(x,z,t)}{\sigma}$ with $N=2$, $p_1=1$, $q_1=3$, $s_1=0$, $p_2=0.5$, $q_2=-3$, $s_2=0$, $c=0$ and $t=0$.

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