

Links Between Different Analytic Descriptions of Constant Mean Curvature Surfaces

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Abstract

Transformations between different analytic descriptions of constant mean curvature (CMC) surfaces are established. In particular, it is demonstrated that the system

$$\begin{aligned}\partial\psi_1 &= (|\psi_1|^2 + |\psi_2|^2)\psi_2 \\ \bar{\partial}\psi_2 &= -(|\psi_1|^2 + |\psi_2|^2)\psi_1\end{aligned}$$

descriptive of CMC surfaces within the framework of the generalized Weierstrass representation, decouples into a direct sum of the elliptic Sh-Gordon and Laplace equations. Connections of this system with the sigma model equations are established. It is pointed out, that the instanton solutions correspond to different Weierstrass parametrizations of the standard sphere $S^2 \subset E^3$.

1 Introduction

We investigate the system

$$\begin{aligned}\partial\psi_1 &= (|\psi_1|^2 + |\psi_2|^2)\psi_2 \\ \bar{\partial}\psi_2 &= -(|\psi_1|^2 + |\psi_2|^2)\psi_1\end{aligned}\tag{1.1}$$

which has been derived in [1] and governs constant mean curvature (CMC) surfaces in the conformal parametrization z, \bar{z} ($\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$). This system was subsequently discussed in [2,4,6]. In this paper, we demonstrate that system (1.1) can be decoupled into a direct

sum of the elliptic Sh-Gordon and Laplace equations. Firstly, we change from ψ_1, ψ_2 to the new dependent variables Q, R

$$Q = 2(\psi_2 \partial \bar{\psi}_1 - \bar{\psi}_1 \partial \psi_2), \quad R = 2(|\psi_1|^2 + |\psi_2|^2)^2 / |Q|.$$

Secondly, we introduce new independent variables $\eta, \bar{\eta}$ according to

$$d\eta = \sqrt{Q} dz, \quad d\bar{\eta} = \sqrt{\bar{Q}} d\bar{z}$$

(these formulae are correct since Q is holomorphic). In the new variables $Q, R, \eta, \bar{\eta}$, system (1.1) assumes the decoupled form

$$\begin{aligned} (\ln R)_{\eta\bar{\eta}} &= \frac{1}{R} - R, \\ Q_{\bar{\eta}} = \bar{Q}_{\eta} &= 0, \end{aligned}$$

which is a direct sum of the elliptic sh-Gordon and Laplace equations. This transformation is an immediate corollary of the known properties of CMC surfaces. Connection of system (1.1) with the sigma-model equations

$$\partial \bar{\partial} \rho - \frac{2\bar{\rho}}{1 + |\rho|^2} \partial \rho \bar{\partial} \rho = 0,$$

($\rho = i\bar{\psi}_1/\psi_2$) is also discussed. In terms of ρ , our transformation adopts the form:

$$Q = \frac{2\partial \rho \partial \bar{\rho}}{(1 + |\rho|^2)^2}, \quad R = \left| \frac{\partial \bar{\rho}}{\partial \rho} \right|.$$

2 Generalized Weierstrass representation of surfaces in \mathbb{R}^3

Following the results of [1], with any solution ψ_1, ψ_2 of the Dirac equations

$$\partial \psi_1 = p \psi_2, \quad \bar{\partial} \psi_2 = -p \psi_1, \tag{2.1}$$

($p(z, \bar{z})$ is a real potential), we associate a surface $M^2 \subset E^3$ with the radius-vector $\mathbf{r}(z, \bar{z})$ defined by the formulae

$$\begin{aligned} \partial \mathbf{r} &= (i(\psi_2^2 + \bar{\psi}_1^2), \quad \bar{\psi}_1^2 - \psi_2^2, \quad -2\psi_2 \bar{\psi}_1), \\ \bar{\partial} \mathbf{r} &= (-i(\bar{\psi}_2^2 + \psi_1^2), \quad \psi_1^2 - \bar{\psi}_2^2, \quad -2\psi_1 \bar{\psi}_2). \end{aligned}$$

The latter are compatible by virtue of (2.1). The unit normal \mathbf{n} of the surface M^2 can be calculated according to the formula $\mathbf{n} = \frac{1}{i} \partial \mathbf{r} \times \bar{\partial} \mathbf{r} / |\partial \mathbf{r} \times \bar{\partial} \mathbf{r}|$ and is represented as follows:

$$\begin{aligned} \mathbf{n} &= \frac{1}{q} (i(\bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2), \quad \bar{\psi}_1 \bar{\psi}_2 + \psi_1 \psi_2, \quad \psi_1 \bar{\psi}_1 - \psi_2 \bar{\psi}_2), \\ q &= |\psi_1|^2 + |\psi_2|^2. \end{aligned} \tag{2.2}$$

One can verify directly that the scalar products in E^3 are $(\mathbf{n}, \mathbf{n}) = 1$, $(\mathbf{n}, \partial\mathbf{r}) = (\mathbf{n}, \bar{\partial}\mathbf{r}) = 0$. Equations of motion of the complex frame $\partial\mathbf{r}$, $\bar{\partial}\mathbf{r}$, \mathbf{n} are of the form

$$\begin{aligned} \partial \begin{pmatrix} \partial\mathbf{r} \\ \bar{\partial}\mathbf{r} \\ \mathbf{n} \end{pmatrix} &= \begin{pmatrix} 2\frac{\partial q}{q} & 0 & Q \\ 0 & 0 & 2Hq^2 \\ -H & -\frac{Q}{2q^2} & 0 \end{pmatrix} \begin{pmatrix} \partial\mathbf{r} \\ \bar{\partial}\mathbf{r} \\ \mathbf{n} \end{pmatrix} \\ \bar{\partial} \begin{pmatrix} \partial\mathbf{r} \\ \bar{\partial}\mathbf{r} \\ \mathbf{n} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 2Hq^2 \\ 0 & 2\frac{\bar{\partial}q}{q} & \bar{Q} \\ -\frac{\bar{Q}}{2q^2} & -H & 0 \end{pmatrix} \begin{pmatrix} \partial\mathbf{r} \\ \bar{\partial}\mathbf{r} \\ \mathbf{n} \end{pmatrix} \end{aligned} \quad (2.3)$$

where $Q = 2(\psi_2\partial\bar{\psi}_1 - \bar{\psi}_1\partial\psi_2)$ and $H = p/q$ is the mean curvature. Formulae (2.3) are compatible with the scalar products

$$(\mathbf{n}, \mathbf{n}) = 1, \quad (\partial\mathbf{r}, \bar{\partial}\mathbf{r}) = 2q^2$$

(all other scalar products being equal to zero). Using (2.3), one can derive the following useful equation for the unit normal \mathbf{n} :

$$\partial\bar{\partial}\mathbf{n} + (\partial\mathbf{n}, \bar{\partial}\mathbf{n})\mathbf{n} + \bar{\partial}H\partial\mathbf{r} + \partial H\bar{\partial}\mathbf{r} = 0. \quad (2.4)$$

The first fundamental form $I = (d\mathbf{r}, d\mathbf{r})$ and the second fundamental form $II = (d^2\mathbf{r}, \mathbf{n})$ of the surface M^2 are given by

$$\begin{aligned} I &= 4q^2 dz d\bar{z}, \\ II &= Q dz^2 + 4Hq^2 dz d\bar{z} + \bar{Q} d\bar{z}^2. \end{aligned} \quad (2.5)$$

The quantity $Q dz^2$ is called the Hopf differential. The real potential p and the spinors ψ_1, ψ_2 satisfying (1.1) can be viewed as the "generalized Weierstrass data" of the surface M^2 . The corresponding Gauss-Codazzi equations which are the compatibility conditions for (2.3), are of the form

$$\begin{aligned} \partial\bar{\partial}(\ln q^2) &= \frac{1}{2}\frac{Q\bar{Q}}{q^2} - 2H^2q^2, \\ \bar{\partial}Q &= 2q^2\partial H, \\ \partial\bar{Q} &= 2q^2\bar{\partial}H. \end{aligned} \quad (2.6)$$

In fact, equations (2.6) are a direct differential consequence of (2.1), as can be checked by a straightforward calculation. Gauss-Codazzi equations of surfaces in conformal parametrization z, \bar{z} have been discussed in [5]. We recall also that the Gaussian curvature K of the surface M^2 can be calculated as follows:

$$K = -\frac{1}{q^2}\partial\bar{\partial}(\ln q). \quad (2.7)$$

The unit normal $\mathbf{n} = (n_1, n_2, n_3)$, given by (2.2), maps a surface M^2 onto the unit sphere S^2 . Combining this map with the stereographic projection, we obtain a map ρ of the surface M^2 onto the complex plane, called the complex Gauss map. In our notation, ρ assumes the form

$$\rho = \frac{n_1 + in_2}{1 - n_3} = i \frac{\bar{\psi}_1}{\psi_2}. \quad (2.8)$$

According to the results of [3], Gauss map ρ satisfies the nonlinear equation

$$(\partial\bar{\partial}\rho - \frac{2\bar{\rho}}{1 + |\rho|^2}\partial\rho\bar{\partial}\rho)H = \partial H\bar{\partial}\rho, \quad (2.9)$$

which formally can be viewed as a differential consequence of (2.1). In terms of ρ and H , the initial data ψ_1 , ψ_2 and p assume the form

$$\psi_1 = \frac{\bar{\rho}}{\sqrt{H}} \frac{\sqrt{i\partial\bar{\rho}}}{1 + |\rho|^2}, \quad \psi_2 = \frac{1}{\sqrt{H}} \frac{\sqrt{i\partial\bar{\rho}}}{1 + |\rho|^2}, \quad p = \frac{|\partial\bar{\rho}|}{1 + |\rho|^2}$$

while the expressions for q and Q take the form

$$q = \frac{1}{H} \frac{\bar{\partial}\rho\partial\bar{\rho}}{1 + |\rho|^2}, \quad Q = \frac{2}{H} \frac{\partial\rho\partial\bar{\rho}}{(1 + |\rho|^2)^2}.$$

Remark 1. Linear problem (2.3) can be rewritten in terms of ψ_1 , ψ_2 as follows. First of all, we point out that

$$\partial q = \psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2.$$

Combining this equation with the definition of Q :

$$\frac{1}{2}Q = \psi_2\partial\bar{\psi}_1 - \bar{\psi}_1\partial\psi_2,$$

and solving these two equations for $\partial\bar{\psi}_1$, $\partial\psi_2$, we can “close” system (2.1) as follows:

$$\begin{aligned} \partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} 0 & qH \\ -\frac{Q}{2q} & \frac{\partial q}{q} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} \frac{\bar{\partial} q}{q} & \frac{\bar{Q}}{2q} \\ -qH & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned} \quad (2.10)$$

The compatibility conditions for system (2.10) coincide with (2.6). We point out that the 2×2 matrix approach to surfaces in E^3 has been extensively developed in [5]. From the point of view of the theory of integrable systems, linear system (2.3) can be regarded as the squared eigenfunction equations corresponding to (2.10) (indeed, $\partial\mathbf{r}$, $\bar{\partial}\mathbf{r}$ and \mathbf{n} are quadratic expressions in ψ_1 and ψ_2).

3 CMC-1 surfaces

The class of CMC-1 surfaces is characterized by the constraint $H = 1$ or, equivalently, $p = q$. Introducing this ansatz in (2.1), we arrive at the nonlinear system (1.1)

$$\begin{aligned}\partial\psi_1 &= (|\psi_1|^2 + |\psi_2|^2)\psi_2, \\ \bar{\partial}\psi_2 &= -(|\psi_1|^2 + |\psi_2|^2)\psi_1,\end{aligned}$$

which is the main subject of our study. According to the previous section, system (1.1) is equivalent to

$$\begin{aligned}\partial\bar{\partial}\ln q^2 &= \frac{1}{2}\frac{Q\bar{Q}}{q^2} - 2q^2, \\ \bar{\partial}Q &= \partial\bar{Q} = 0,\end{aligned}\tag{3.1}$$

where $q = |\psi_1|^2 + |\psi_2|^2$, $Q = 2(\psi_2\partial\bar{\psi}_1 - \bar{\psi}_1\partial\psi_2)$. Thus, for CMC surfaces the Hopf differential $Q dz^2$ is holomorphic. Applying to system (3.1) the reciprocal transformation

$$d\eta = \sqrt{Q} dz, \quad d\bar{\eta} = \sqrt{\bar{Q}} d\bar{z}$$

(that is, changing from z, \bar{z} to the new independent variables $\eta, \bar{\eta}$ which are correctly defined in view of the holomorphicity of Q) and introducing

$$R = \frac{2q^2}{|Q|},$$

we transform system (3.1) into the decoupled form

$$\begin{aligned}(\ln R)_{\eta\bar{\eta}} &= \frac{1}{R} - R, \\ Q_{\bar{\eta}} &= \bar{Q}_{\eta} = 0.\end{aligned}\tag{3.2}$$

This result provides the rationale for the change of variables which we introduce in Section 1.

Remark 2. System (1.1) is invariant under the $SU(2)$ -symmetry

$$\zeta_1 = \alpha\psi_1 + \beta\bar{\psi}_2, \quad \zeta_2 = -\beta\bar{\psi}_1 + \alpha\psi_2,\tag{3.3}$$

where α, β are complex constants subject to the constraint $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. One can check directly, that the quantities Q and R are invariant under transformations (3.3), so that the surfaces, corresponding to (ψ_1, ψ_2) and (ζ_1, ζ_2) , have coincident fundamental forms. Thus, they are identical up to a rigid motion in E^3 . The passage from ψ_1, ψ_2 to Q, R can thus be viewed as a passage to the differential invariants of the point symmetry group (3.3).

Remark 3. For CMC-1 surfaces, system (2.10) allows the introduction of a spectral parameter

$$\begin{aligned}\partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} 0 & q \\ -\lambda\frac{Q}{2q} & \frac{\partial q}{q} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ \bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} \frac{\bar{\partial} q}{q} & \frac{1}{\lambda}\frac{\bar{Q}}{2q} \\ \frac{q}{-q} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\end{aligned}\tag{3.4}$$

where λ is a unitary constant, $|\lambda| = 1$. The gauge transformation

$$\tilde{\psi}_1 = \psi_1, \quad \tilde{\psi}_2 = q^{-1} \psi_2$$

reduces linear spectral problem (3.4) to the $SL(2)$ form

$$\begin{aligned} \partial \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} &= \begin{pmatrix} 0 & q^2 \\ -\lambda \frac{Q}{2q^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} \\ \bar{\partial} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} &= \begin{pmatrix} \frac{\bar{\partial}q}{q} & \frac{\bar{Q}}{2\lambda} \\ -1 & -\frac{\bar{\partial}q}{q} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}. \end{aligned} \tag{3.5}$$

The compatibility conditions for both systems (3.4) and (3.5) coincide with (3.1). From the linear system (3.5) the radius-vector \mathbf{r} can be recovered via the so-called Sym formula: we refer to [5] and [9] for the further discussion of the Sym approach.

Linear system (3.5) can be readily rewritten in terms of ψ_1, ψ_2 . Indeed, observing that (3.5) implies

$$\partial \ln q = \partial \ln(|\psi_1|^2 + |\psi_2|^2), \quad \bar{\partial} \ln q = \bar{\partial} \ln(|\psi_1|^2 + |\psi_2|^2),$$

we can take $q = c(|\psi_1|^2 + |\psi_2|^2)$, $c \in \mathbb{C}$, which, upon substitution in (3.5), produces system (1.1). Thus transformation from (3.1) to (1.1) consists of rewriting (3.5) in terms of the ψ . Representations in terms of ψ are called eigenfunction equations, and are fundamental in soliton theory – see eg [7]. The Lax pair for system (1.1) is of the form [6]

$$\begin{aligned} \partial \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \frac{2}{\mu+1} \begin{pmatrix} -\bar{\psi}_1 \psi_2 + \frac{Q}{2q^2} \psi_1 \bar{\psi}_2 & -\bar{\psi}_1^2 - \frac{Q}{2q^2} \bar{\psi}_2^2 \\ \psi_2^2 + \frac{Q}{2q^2} \psi_1^2 & \bar{\psi}_1 \psi_2 - \frac{Q}{2q^2} \psi_1 \bar{\psi}_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ \bar{\partial} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \frac{2}{\mu-1} \begin{pmatrix} -\psi_1 \bar{\psi}_2 + \frac{\bar{Q}}{2q^2} \bar{\psi}_1 \psi_2 & \bar{\psi}_2^2 + \frac{\bar{Q}}{2q^2} \bar{\psi}_1^2 \\ -\psi_1^2 - \frac{\bar{Q}}{2q^2} \psi_2^2 & \psi_1 \bar{\psi}_2 - \frac{\bar{Q}}{2q^2} \bar{\psi}_1 \psi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \end{aligned}$$

It is interesting to note that the compatibility conditions for this linear system, which is of the first order in the derivatives of ψ , give us exactly system (1.1). The latter is also of the first order in ψ .

4 CMC surfaces and sigma model equations

For CMC surfaces, equations (2.9) imply the nonlinear sigma model

$$\partial \bar{\partial} \rho - \frac{2\bar{\rho}}{1+|\rho|^2} \partial \rho \bar{\partial} \rho = 0, \tag{4.1}$$

descriptions of the stationary two-dimensional $SU(2)$ magnet. The transformation of the system (4.1) into the decoupled form (3.2) now assumes the form

$$\begin{aligned} Q &= 2 \frac{\partial \rho \partial \bar{\rho}}{(1 + |\rho|^2)^2}, & R &= \left| \frac{\partial \bar{\rho}}{\partial \rho} \right|, \\ d\eta &= \sqrt{Q} dz, & d\bar{\eta} &= \sqrt{\bar{Q}} d\bar{z}. \end{aligned} \quad (4.2)$$

In terms of the unit normal vector \mathbf{n} , equation (2.4) adopts the form of the $SO(3)$ sigma model

$$\bar{\partial} \partial \mathbf{n} + (\partial \mathbf{n}, \bar{\partial} \mathbf{n}) \mathbf{n} = 0, \quad (\mathbf{n}, \mathbf{n}) = 1. \quad (4.3)$$

Formula (2.8) establishes a link between sigma models (4.1) and (4.3). The topological charge

$$\frac{1}{4\pi} \int \int (\mathbf{n}, [\partial \mathbf{n} \times \bar{\partial} \mathbf{n}]) dz \wedge d\bar{z}$$

can be written as

$$\frac{1}{2\pi i} \int \int \partial \bar{\partial} \ln q dz \wedge d\bar{z}$$

which, in view of (2.7), is the topologically invariant integral curvature of the surface M^2 .

Instanton solutions of system (4.3) are specified by the ansatz

$$\partial \mathbf{n} = \pm i \mathbf{n} \times \partial \mathbf{n}, \quad \bar{\partial} \mathbf{n} = \mp i \mathbf{n} \times \bar{\partial} \mathbf{n},$$

which, after a simple calculation, implies $Q = 0$. Solutions of system (1.1) specified by a constraint $Q = 0$, can be represented in the form

$$\psi_1 = \frac{\rho \sqrt{\partial \bar{\rho}}}{1 + |\rho|^2}, \quad \psi_2 = \frac{\sqrt{\partial \bar{\rho}}}{1 + |\rho|^2}, \quad p = \frac{|\partial \rho|}{1 + |\rho|^2}, \quad (4.4)$$

where $\rho(z)$ is an arbitrary holomorphic function. In the case when the energy

$$E = \int \int \frac{\partial \rho \bar{\partial} \rho}{1 + |\rho|^2} dz \wedge d\bar{z}$$

is finite, the function $\rho(z)$ is rational in z [8].

Geometrically, instanton solutions (4.4) parametrize the standardly embedded sphere $S^2 \subset E^3$. This can be readily seen from formulae (2.5), which, in case $Q = 0$, imply the proportionality of fundamental forms I and II. This example shows that different Weierstrass data (ψ_1, ψ_2, p) can correspond to different parametrizations of one and the same surface $M^2 \subset E^3$.

Introducing the two-component complex vector

$$\mathbf{N} = \left(\frac{\psi_1}{\sqrt{q}}, \frac{\bar{\psi}_2}{\sqrt{q}} \right), \quad q = |\psi_1|^2 + |\psi_2|^2,$$

one can check that \mathbf{N} satisfies the equations of the $\mathbb{C}P^1$ sigma model

$$\begin{aligned} (\mathbf{N}, \bar{\mathbf{N}}) &= 1, \\ \partial \bar{\partial} \mathbf{N} &= (\bar{\mathbf{N}}, \bar{\partial} \mathbf{N}) \partial \mathbf{N} + (\bar{\mathbf{N}}, \partial \mathbf{N}) \bar{\partial} \mathbf{N} - k \mathbf{N}, \end{aligned} \tag{4.5}$$

where

$$k = -2(\bar{\mathbf{N}}, \partial \mathbf{N})(\mathbf{N}, \bar{\partial} \bar{\mathbf{N}}) + \frac{1}{2}(\partial \mathbf{N}, \bar{\partial} \bar{\mathbf{N}}) + \frac{1}{2}(\bar{\partial} \mathbf{N}, \partial \bar{\mathbf{N}}).$$

Equations (4.5) are associated with the Lagrangian

$$L = \int \int \{(\bar{\partial} \mathbf{N}, \partial \bar{\mathbf{N}}) + (\partial \mathbf{N}, \bar{\partial} \bar{\mathbf{N}}) + 2(\bar{\mathbf{N}}, \partial \mathbf{N})(\bar{\mathbf{N}}, \bar{\partial} \bar{\mathbf{N}}) - 2k[(\mathbf{N}, \bar{\mathbf{N}}) - 1]\} dz d\bar{z},$$

where k is the Lagrange multiplier.

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