

# Poisson Homology of $r$ -Matrix Type Orbits I: Example of Computation

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## Abstract

In this paper we consider the Poisson algebraic structure associated with a classical  $r$ -matrix, i.e. with a solution of the modified classical Yang–Baxter equation. In Section 1 we recall the concept and basic facts of the  $r$ -matrix type Poisson orbits. Then we describe the  $r$ -matrix Poisson pencil (i.e the pair of compatible Poisson structures) of rank 1 or  $CP^n$ -type orbits of  $SL(n, C)$ . Here we calculate symplectic leaves and the integrable foliation associated with the pencil. We also describe the algebra of functions on  $CP^n$ -type orbits. In Section 2 we calculate the Poisson homology of Drinfeld–Sklyanin Poisson brackets which belong to the  $r$ -matrix Poisson family.

## Introduction

The canonical or Poisson homology of Poisson manifolds were introduced by Gelfand–Dorfman [11], Koszul [19] and Brylinsky [1]. Their algebraic analogue was considered by Huebschmann in [14].

Let  $(M, \pi)$  be a smooth Poisson manifold with a Poisson structure given by the anti-symmetric bivector field  $\pi$ , and  $f_i \in C^\infty(M)$ ,  $i = 0, \dots, k$  are smooth functions.

Recall that the formula for a canonical (Poisson) differential of the degree  $-1$  has the following simple form on the decomposable differential forms

$$\begin{aligned} \delta_\pi(f_0 df_1 \wedge \dots \wedge df_k) &= \sum_i (-1)^{i+1} \pi(df_0, df_i) df_1 \wedge \dots \wedge \hat{df}_i \wedge \dots \wedge df_k \\ &+ \sum_{i < j} (-1)^{i+j} f_0 df_1 \wedge \dots \wedge d\pi(df_i, df_j) \wedge \dots \wedge \hat{df}_j \wedge \dots \wedge df_k. \end{aligned}$$

The first definition of [11] was inspired by the integrable systems theory: the Poisson homologies are responsible for the (non)-existence of bihamiltonian structures involved in the so-called Magri–Lenard scheme. It is now clear that there are many other reasons to

study these homologies. We are unable to discuss all of them here and indicate only the most interesting points.

The Poisson appeared in [1] as an important tool in the computations of Hochschild and cyclic homologies in the frame of Connes-like double complex, where the Poisson differential  $\delta_\pi$  plays the role of the Hochschild boundary operator and the usual de Rham differential is very similar to the Connes cyclic cohomology operator.

Moreover this ideology was used later by Feng–Tsygan [9] for their description of the Hochschild complex for a “quantum” (deformed) algebra of smooth functions using the Poisson homology as the second term in the appropriate spectral sequence.

J.L. Brylinski, in the same paper [1], conjectured an interesting symplectic version of the Hodge theorem. The negative answer to this conjecture [24, 31] established a connection between the canonical cohomology complex of a symplectic manifold and the old topological problem, where the homotopy type of a space is a formal consequence of its real homology ring. The interesting development of the conjecture and attempts to generalize it to the case of the Jacobi manifold, are given in the paper [10].

O. Mathieu studied various families of Poisson complexes [25] and found that they are very different for different values of the parameter, even if the underlying Poisson structures are simply related (for example are parametrized by  $\mathbf{CP}_1$ ). We would like to stress this aspect because of some similarities to our study of the Poisson complex associated to a Poisson pencil of  $r$ -matrix Poisson structures.

O. Mathieu also linked the Poisson homology with the Gelfand-Fuks cohomology of the Lie algebra Hamiltonian vector fields, symplectic operates and, finally, his approach was used by G. Papadopoulos in his computations of cyclic (co)homology for Poisson and symplectic manifolds [26].

This list of interesting links would not be complete without mentioning the recent Weinstein definition of the modular class of Poisson manifold, which can be considered as a classical analogue of the modular form of a von Neumann algebra [29, 2]. If this class is equal to 0 the manifold is referred to as unimodular.

The Poisson homology of an unimodular Poisson structure are isomorphic to the Poisson cohomology [30] and in fact there is an interesting pairing (which is a degenerate in general) between the Poisson homology and the Poisson cohomology [7], whose algebraic roots are in the similar duality between the Lie-Reinhart Poisson algebra cohomology and homology [15].

We will finish our brief survey of the recent manifestations of the Poisson homology with the indication of the link between the very general construction of Quillen which was used by B. Fresse in his definition of Poisson homology, and their computations for some singular Poisson surfaces [8].

In this text we consider the Poisson algebraic structure associated with a classical  $r$ -matrix, i.e. with a solution of the modified classical Yang–Baxter equation [5, 27]. The classical  $r$ -matrix leads to Poisson orbits of two types. Below a brief description for both of them is given.

The first structure, called the Drinfeld–Sklyanin, arises on Poisson homogeneous spaces. It can be obtained as a result of Poisson reduction for Poisson–Lie groups.

The second one exists on special homogeneous spaces, which are known as  $r$ -matrix orbits and classified in [12]. Hereafter we will consider the  $r$ -matrix type orbits only.

It is clear that the Poisson brackets of  $r$ -matrix type arise both on the Poisson–Lie

groups and Poisson homogeneous spaces as the quasiclassical limit of the corresponding quantum objects. Here we mean only a deformation quantization developed in [3, 13]. Let us remark that, as was shown in [20], the geometric quantization does not exist on certain Poisson homogeneous spaces.

We now describe the structure of the paper.

In Section 1 we recall the concept and basic facts of the  $r$ -matrix type Poisson orbits. Then we describe the  $r$ -matrix Poisson pencil (i.e the pair of compatible Poisson structures) on the rank 1 or  $CP^n$ -type orbits of  $SL(n, C)$ . Here we calculate symplectic leaves and integrable foliation associated with the pencil. We also describe the algebra of functions on  $CP^n$ -type orbits.

In Section 2 we calculate the Poisson homology of Drinfeld–Sklyanin Poisson brackets which belong to the  $r$ -matrix Poisson family.

There are many interesting open questions the study of which we have postponed to the future. Among them the relations between our definition of “harmonic” forms and the theory of Poisson harmonic forms, Kostant harmonic forms and equivariant Poisson cohomology [6], the precise links between Hochschild and cyclic (co)homologies and the computations in this paper etc.

## 1 On the Poisson structure of $CP^n$ -type complex orbits

Let  $G$  be a semisimple Lie group and  $g$  be a Lie algebra of  $G$ . Assume that  $r$  is a standard Drinfeld–Jimbo  $r$ -matrix

$$r = \sum_{\alpha \in \Delta^+} e_\alpha \wedge e_{-\alpha},$$

where  $e_\alpha, e_{-\alpha}$  is the Cartan basis for  $g$ .

Let  $\mathcal{O}$  be a coadjoint orbit and let  $X_\alpha, X_{-\alpha}$  be generators of the action of  $G$  that corresponds to the basis  $\{e_\alpha, e_{-\alpha}\}$ . Then the orbit  $\mathcal{O}$  is called of  $r$ -matrix type if and only if the bivector field

$$\pi = \sum_{\alpha \in \Delta^+} X_\alpha \wedge X_{-\alpha},$$

corresponds to the  $r$ -matrix  $r$ , gives us the Poisson brackets.

If  $G = SL(n)$  and  $\mathcal{O}$  is an orbit of rank 1 matrices then the Poisson structure is called  $CP^n$ -type.

Now let us give an explicit description of such orbits and  $r$ -matrix type brackets on them.

Let us consider a standard action of  $SL(n)$  on  $C^n$  and its cotangent lift to  $T^*C^n$ . The generators of the action corresponding to the root basis  $e_{ij}$  of  $gl(n)$  are in dual coordinates  $(z_i, \xi_i)$

$$X_{ij} = z_j \partial_{z_i} - \xi_j \partial_{\xi_i}$$

The Drinfeld–Jimbo  $r$ -matrix  $r = \sum_{i < j} e_{ij} \wedge e_{ji}$  maps to the bivector field

$$\pi = \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} - \sum_{i < j} \xi_i \xi_j \partial_{\xi_i} \wedge \partial_{\xi_j} + \sum_{i < j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j} - \sum_{j < i} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j}$$

The structure is compatible with the natural symplectic one on  $T^*C^n$  given by the form

$$\omega = \sum_i dz_i \wedge d\xi_i.$$

The momentum map  $\mu : T^*C^n \rightarrow sl^*(n)$  is

$$\mu(z, \xi) = \text{tr}(z^t \xi, *), \quad (z^t \xi)_{ij} = z_i \xi_j.$$

Thus the orbits of cotangent action of  $SL(n)$  cover the orbits of rank 1 on  $sl^*(n)$ . If  $\text{tr } z^t \xi = (z, \xi) = \sum_i z_i \xi_i \neq 0$  then the corresponding orbit is semisimple and symmetric. If  $\text{tr } z^t \xi = \sum_i z_i \xi_i = 0$  and  $\mu(z, \xi) \neq 0$  then this is a nilpotent orbit of height 2.

The symplectic restriction to the level  $(z, \xi) = \text{const}$  is a pull-back of the Kirillov form. The momentum map is  $SL(n)$ -equivariant so the  $r$ -matrix structure is compatible with the momentum map. Moreover the momentum map gives us an isomorphism between rank 1 orbits and Poisson reduction under the Hamiltonian action of  $H = \sum_i z_i \xi_i$ .

Now we obtain the eigenvalues of the bivector  $\pi_r$  with respect to the symplectic form. We introduce an operator field  $A$ ,  $A(\phi) = V_\Omega(i_\phi \pi)$  for all 1-forms  $\phi$ . Here  $V_\Omega$  is a Hamiltonian operator of the symplectic structure  $\Omega$  acting as

$$V_\Omega : \partial_{\xi_i} \rightarrow dz_i, \quad \partial_{z_i} \rightarrow -d\xi_i$$

First of all we check its eigenvalues and find its eigen-vectors. Then we “forget” about the tangent direction to the fibres of  $\mu$ , i.e. we separate only the eigen-vectors that are the pull-back from the coadjoint orbits.

Let

$$\phi_i = \xi_i dz_i + z_i d\xi_i, \quad \psi_i = \xi_i dz_i - z_i d\xi_i.$$

(i) We calculate how  $A$  acts on the basis  $\{\phi_i, \psi_i\}$  of 1-forms:

$$\begin{aligned} i_{\phi_i} \pi &= \sum_{i < j} (\xi_i z_i) z_j \partial_{z_j} - \sum_{i < j} (\xi_i z_i) \xi_j \partial_{\xi_j} - \sum_{j < i} (\xi_i z_i) z_j \partial_{z_j} + \sum_{j < i} (\xi_i z_i) \xi_j \partial_{\xi_j} \\ &+ \sum_{j < i} (\xi_j z_j) \xi_i \partial_{\xi_i} - \sum_{j < i} (\xi_j z_j) z_i \partial_{z_i} - \sum_{i < j} (\xi_j z_j) \xi_i \partial_{\xi_i} + \sum_{i < j} (\xi_j z_j) z_i \partial_{z_i}, \\ i_{\psi_i} \pi &= \sum_{i < j} (\xi_i z_i) z_j \partial_{z_j} + \sum_{i < j} (\xi_i z_i) \xi_j \partial_{\xi_j} - \sum_{j < i} (\xi_i z_i) z_j \partial_{z_j} - \sum_{j < i} (\xi_i z_i) \xi_j \partial_{\xi_j} \\ &+ \sum_{j < i} (\xi_j z_j) \xi_i \partial_{\xi_i} + \sum_{j < i} (\xi_j z_j) z_i \partial_{z_i} - \sum_{i < j} (\xi_j z_j) \xi_i \partial_{\xi_i} - \sum_{i < j} (\xi_j z_j) z_i \partial_{z_i}. \end{aligned}$$

Hence we see that

$$\begin{aligned} A(\phi_i) &= - \sum_{i < j} (\xi_i z_i) z_j d\xi_j - \sum_{i < j} (\xi_i z_i) \xi_j dz_j + \sum_{j < i} (\xi_i z_i) z_j d\xi_j + \sum_{j < i} (\xi_i z_i) \xi_j dz_j \\ &+ \sum_{j < i} (\xi_j z_j) \xi_i dz_i + \sum_{j < i} (\xi_j z_j) z_i d\xi_i - \sum_{i < j} (\xi_j z_j) \xi_i dz_i - \sum_{i < j} (\xi_j z_j) z_i d\xi_i. \end{aligned}$$

If we define  $a_i = \xi_i z_i$  then

$$A(\phi_i) = \left( \sum_{j < i} a_j - \sum_{i < j} a_j \right) \phi_i + a_i \left( \sum_{j < i} \phi_j - \sum_{i < j} \phi_j \right).$$

Similarly

$$A(\psi_i) = \left( \sum_{j < i} a_j - \sum_{i < j} a_j \right) \psi_i - a_i \left( \sum_{j < i} \psi_j - \sum_{i < j} \psi_j \right).$$

(ii) Note that if  $\phi_0 = \sum_i \phi_i$ , then

$$A(\phi_0) = \sum_i \left( \sum_{j < i} a_j - \sum_{i < j} a_j \right) \phi_i + \sum_i a_i \left( \sum_{j < i} \phi_j - \sum_{i < j} \phi_j \right) = 0.$$

**Explanation 1** This form vanishes on the level  $\sum_i z_i \xi_i = \text{const}$ . Therefore, if we make a Poisson reduction via the Hamiltonian field  $H = \sum_i z_i \xi_i$ , then one needs to consider  $A \bmod (\phi_0)$ .

Thus

$$\begin{aligned} A(\phi_i) &= \left( \sum_{j < i} a_j - \sum_{i < j} a_j \right) \phi_i + a_i \left( \sum_{j < i} \phi_j - \sum_{i < j} \phi_j \right) \\ &= \left( \sum_{j \leq i} a_j - \sum_{i < j} a_j \right) \phi_i + a_i \left( -\phi_0 + 2 \sum_{j < i} \phi_j \right). \end{aligned}$$

We obtain the “triangle” basis  $\{\phi_0, \phi_1, \dots, \phi_{n-1}, \psi_0, \psi_1, \dots, \psi_{n-1}\}$  for  $A$ :

$$\begin{aligned} &A(\phi_0) = 0; \\ &\dots\dots\dots \\ &A(\phi_i) = \left( \sum_{j \leq i} a_j - \sum_{i < j} a_j \right) \phi_i + a_i \left( -\phi_0 + 2 \sum_{j < i} \phi_j \right); \\ &\dots\dots\dots \\ &A(\phi_{n-1}) = \left( \sum_{j < n} a_j - a_n \right) \phi_{n-1} + a_{n-1} \left( -\phi_0 + 2 \sum_{j < n-1} \phi_j \right). \end{aligned}$$

We see that the eigenvalues of  $A$  are equal to  $\sum_{j \leq i} a_j - \sum_{i < j} a_j$ ,  $i = 1, \dots, n-1$  or  $\lambda_i = \sum_{j \leq i} z_j \xi_j - \sum_{i < j} z_j \xi_j$ ,  $i = 1, \dots, n-1$ . Moreover,

$$\begin{aligned} A \left( \sum_{i \leq k} \phi_i \right), \text{mod}(\phi_0) &= \sum_{i \leq k} \sum_{j \leq i} a_j \phi_i - \sum_{i \leq k} \sum_{i < j} a_j \phi_i + 2 \sum_{i \leq k} \sum_{j < i} a_i \phi_j \\ &= \sum_{i \leq k} \sum_{j \leq i} a_j \phi_i - \sum_{i < k} \sum_{j \leq k} a_j \phi_i - \sum_{i \leq k} \sum_{j > k} a_j \phi_i + 2 \sum_{j \leq k} \sum_{i < j} a_j \phi_i = \\ &= \sum_{i \leq k} \sum_{j \leq i} a_j \phi_i + \sum_{j \leq k} \sum_{i < j} a_j \phi_i - \sum_{i \leq k} \sum_{j > k} a_j \phi_i = \sum_{i \leq k} \sum_{j \leq k} a_j \phi_i - \sum_{i \leq k} \sum_{j > k} a_j \phi_i = \\ &= \left( \sum_{j \leq k} a_j - \sum_{j > k} a_j \right) \sum_{i \leq k} \phi_i. \end{aligned}$$

Hence the diagonal  $\text{mod}(\phi_0)$ -basis is

$$\bar{\phi}_k = \sum_{i \leq k} \phi_i, \quad A(\bar{\phi}_k) = \lambda_k \bar{\phi}_k, \quad k = 1, \dots, n-1.$$

Note that  $\bar{\phi}_k = d \left( \sum_{i \leq k} z_i \xi_i \right)$ .

It is not difficult to see also that  $A(\bar{\psi}_k) = \lambda_k \bar{\psi}_k$ ,  $k = 1, \dots, n-1$ , where

$$\bar{\psi}_k = \frac{\psi_{k+1}}{a_{k+1}} - \frac{\psi_k}{a_k}.$$

Note that

$$\bar{\psi}_k = d(\ln(z_{k+1}) - \ln(\xi_{k+1}) - \ln(z_k) + \ln(\xi_k)) = d \left( \ln \left( \frac{\xi_k z_{k+1}}{z_k \xi_{k+1}} \right) \right).$$

**Explanation 2** All forms  $\bar{\phi}_k$ ,  $\bar{\psi}_k$  vanish on the orbits of the  $H$ -Hamiltonian action and are  $H$ -invariant, so one has to consider them as a pull-back of the almost everywhere independent basis of differential 1-forms from coadjoint orbits.

Hence we get an integrable foliation on the orbits associated with a Poisson pencil  $\pi_r + \lambda \pi_{kir}$ , where  $\pi_{kir}$  is the Kirillov Poisson structure that is also called “Lie–Poisson”. It is well-known that  $A$ , as defined above, is “integrable” since  $\pi_r$  and  $\pi_{kir}$  are compatible structures. This means that its eigen- (or adjacent) spaces give us an integrable (singular) foliation. As proved above, the foliation is given by the equations  $\bar{\phi}_k = 0$ ,  $\bar{\psi}_k = 0$ ,  $k \in I \subset (1, \dots, n-1)$  or

$$\sum_{i \leq k} z_i \xi_i = c_k, \quad \frac{\xi_k z_{k+1}}{z_k \xi_{k+1}} = c'_k, \quad k \in I.$$

We know at least two natural Poisson algebras associated with an  $r$ -matrix on the rank 1 orbits.

The first one is an algebra  $A$  generated by the Hopf bundle on  $CP^n$ . It might be identified with the algebra of polynomials  $C[z_1, \dots, z_n]$ , where  $SL_n$  acts as follows:

$$e_i = z_i \partial_{z_{i+1}}, \quad f_i = z_{i+1} \partial_{z_i}, \quad h_i = z_i \partial_{z_i} - z_{i+1} \partial_{z_{i+1}}.$$

Here  $e_i, f_i, h_i$  are Cartan generators of the Lie algebra.

Recall that

$$C^k[z_1, \dots, z_n] \cong V_{\omega_1},$$

where  $C^k[z_1, \dots, z_n]$  is the space of polynomials of degree  $k$ ,  $V_{\omega_1}$  is an  $sl_n$ -module with highest weight  $\omega_1$ .

The second one is the algebra  $B$  of algebraic (holomorphic) functions on  $\mathcal{O}$ . The algebra of functions on  $CP^n$ -type orbits is generated by polynomials  $z_i \xi_j$  with the relation  $\sum_i z_i \xi_i = \text{const} \neq 0$ .

The main facts concerning the structure of the algebra of functions on  $\mathcal{O}$  as  $sl_n$ -module are as follows:

1.  $\text{Fun}(\mathcal{O}) \cong \bigoplus_{k \geq 0} V_{k(\omega_1 + \omega_{n-1})},$
2.  $C^k[z] \otimes C^k[\xi] \stackrel{\text{def}}{=} C^{k,k}[z, \xi] \cong \bigoplus_{l \leq k} V_{l(\omega_1 + \omega_{n-1})}.$

**Proposition 1** *The subcomplex of algebraic differential forms  $\Omega^k(\mathcal{O})$  on  $\mathcal{O}$ , generated by subalgebra  $B^k(\mathcal{O}) \cong \bigoplus_{l \leq k} V_{l(\omega_1 + \omega_{n-1})}$  of  $\text{Fun}(\mathcal{O})$ , is isomorphic to the subcomplex  $A^k$  of forms on  $T^*C^n$ , generated by  $z_i \xi_j$ .*

**Proof.** As we know from the Hochschild–Konstant–Rosenberg theorem [16]

$$\Omega^*(B) \cong HH^*(B),$$

where  $HH^*(B) = H^*(C^*(B), b)$  are Hochschild homologies of  $B$ . They are calculated as the homology of complex of  $B$ -chains  $C^*(B) = \bigoplus_m B \otimes B^{\otimes m}$ , with differential  $b$  acting as follows

$$\begin{aligned} b(a_0 \otimes a_1 \dots \otimes a_m) \\ = \sum_{j \leq m-1} (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \dots \otimes a_m + (-1)^m a_m a_0 \otimes a_1 \dots \otimes a_{m-1}. \end{aligned}$$

Poisson and Hochschild differentials on  $B$ -chains are compatible with the introduced filtration, so we can deduce that

$$A^k \cong HH_*^{k,k}(C[z, \xi]) \cong HH_*^k(B) \cong \Omega^k(\mathcal{O}).$$

Let  $\mathcal{O}$  be a  $CP^n$ -type orbit and  $P = \bigoplus_k P^k$  – subalgebra of polynomials on  $z_i, \xi_j$ , generated by  $z_i \xi_j$ . Let  $B$  be an algebra of functions on  $\mathcal{O}$ . Then, without losing generality, we establish that

$$B = P/J, \quad J = \{p(\xi, z) = (H(\xi, z) - 1)f(\xi, z) | f \in P\},$$

where  $H(\xi, z) = \sum_i z_i \xi_i$ .

The algebra  $B$  has a natural filtration  $B^{(k)}$ , arising from the filtration on  $P$ ,  $P^{(k)} = \bigoplus_{j \leq k} P^j$ .

$$B^{(k)} = P^{(k)} / (H - 1)P^{(k-1)}.$$

**Proposition 2** *We have the following isomorphism:*

$$B^{(k)} \cong P^k.$$

**Proof.** Let  $f \in B^{(k)}$  and  $f = \sum_{j=0}^k f_j$  be a decomposition on homogeneous components.

Consider the mapping

$$\mathcal{P} : f \mapsto \hat{f} = \sum f_j H^{k-j}, \quad B^{(k)} \rightarrow P^k.$$

Then  $\text{Ker } \mathcal{P} = J^{(k)} = (H - 1)B^{(k-1)}$ . Indeed

$$\hat{f} = \sum f_j H^{k-j} = f + \sum_{j < k} (H^{k-j} - 1)f_j = 0 \implies f \in J^{(k)}$$

so  $\text{Ker } \mathcal{P} \subset J^{(k)}$ . On the other hand, if  $f \in J^{(k)}$  then

$$f_k = Hg_k, \quad f_j = Hg_{j-1} - g_j.$$

Therefore it is easy to see that  $\text{Ker } \mathcal{P} = J^{(k)}$ .

Thus  $\mathcal{P}$  gives rise to the isomorphism of filtered spaces

$$\mathcal{P} : B^{(k)} \rightarrow P^k.$$

**Proposition 3** *The map  $\mathcal{P}$  satisfies the following conditions:*

1.  $\mathcal{P}$  is a homomorphism of algebras, i.e.

$$\hat{f}_1 \hat{f}_2 = \widehat{f_1 f_2};$$

2.  $\mathcal{P}$  is a homomorphism of  $SL_n$ -modules, i.e. the following diagram is commutative

$$\begin{array}{ccc} B^{(k)} & \rightarrow & P^k \\ g \in SL_n, \quad \downarrow g & & \downarrow g \\ B^{(k+1)} & \rightarrow & P^{k+1}. \end{array}$$

**Proof.** 1. Let  $f_1 = \sum_{j=0}^k f_{1,j}$ ,  $f_2 = \sum_{j=0}^l f_{2,j}$ . Then

$$\hat{f}_1 = \sum_{j=0}^k f_{1,j} H^{k-j}, \quad \hat{f}_2 = \sum_{j=0}^l f_{2,j} H^{l-j}.$$

Hence

$$\hat{f}_1 \hat{f}_2 = \sum_{i+j \leq k+l} f_{1,i} f_{2,j} H^{k+l-i-j} = \widehat{f_1 f_2}.$$



2. It follows from the fact that  $H$  is invariant under the  $SL_n$ -action.

It is very important that two algebras of complex smooth functions on  $CP^n$  and of algebraic functions on  $\mathcal{O}$  are isomorphic as  $sl_n$ -modules. So the following diagrams are commutative

$$\begin{array}{ccc} \Omega^*(CP^{n-1}) & \rightarrow & \Omega^*(\mathcal{O}) \\ \downarrow d & & \downarrow d \\ \Omega^{*+1}(CP^{n-1}) & \rightarrow & \Omega^{*+1}(\mathcal{O}) \\ \Omega^*(CP^{n-1}) & \rightarrow & \Omega^*(\mathcal{O}) \\ \downarrow \delta_\omega, \delta_\pi & & \downarrow \delta_\omega, \delta_\pi \\ \Omega^{*-1}(CP^{n-1}) & \rightarrow & \Omega^{*-1}(\mathcal{O}), \end{array}$$

where  $d$  is the de Rham differential and  $\delta_\pi, \delta_\omega$  are Poisson differentials, corresponding to the  $r$ -matrix structure  $\pi$  and to the symplectic form  $\omega$ , respectively.

As a direct consequence we get the identity

$$H_{*, \delta'_\pi}(CP^n) \cong H_{*, \delta'_\pi}(\mathcal{O})$$

for all Poisson differentials  $\delta'_\pi = a\delta_\omega + b\delta_\pi$  from the Poisson pencil  $\pi, \pi_\omega$ .

We remark that the structure of the  $sl_n$ -module on the algebra of complex functions arises as a simple complexification of the  $su_n$ -module structure on the smooth function algebra.

Let

$$\{f_1, f_2\} \stackrel{\text{def}}{=} \langle \pi, df_1 \wedge df_2 \rangle, \quad \{f_1, f_2\}_\omega \stackrel{\text{def}}{=} H \langle \pi_\omega, df_1 \wedge df_2 \rangle$$

and  $\{f_1, f_2\}_{a,b} = a\{f_1, f_2\} + b\{f_1, f_2\}_\omega$ . Then we can prove the following proposition.

**Proposition 4** *The following identity holds:*

$$\{\hat{f}_1, \hat{f}_2\}_{a,b} = \{\hat{f}_1, \hat{f}_2\}_{a,b}.$$

**Proof.** Straightforward calculations show that the mapping  $\mathcal{P}$  satisfies this formula because  $H(z, \xi)$  is Casimir with respect to the whole pencil  $\pi_{a,b}$ . The structure  $\pi_\omega$  is symplectic and nongenerate on  $T^*C^n = \{(z, \xi)\}$ . We easily obtain  $\{H, z_i \xi_j\}_\omega = 0$ .

## 2 On the Poisson homology of $CP^n$ -type orbits

The Poisson homology was introduced as the second term in the spectral sequence associated with the Hochschild complex for a deformed algebra of smooth functions.

We give a short description of the Poisson homology.

Let  $X$  be a smooth manifold and  $A_0 = C^\infty(X)$  be an algebra of smooth functions on  $X$ . An associative algebra  $A$  over the ring of formal series  $C[[h]]$  is called a deformation of  $A_0$  if  $A$  is isomorphic to  $A_0 \otimes C[[h]]$  as a  $C[[h]]$ -module and a multiplication on  $A$  coincide with the multiplication on  $A_0 \otimes C[[h]] \bmod O(h)$  [22].

Every such deformation can be obtained by deformation quantization [22] of Poisson brackets on  $X$ , which means that the commutator on the deformed algebra give us a Lie algebra structure on the functions on  $X$

$$(a * b - b * a) \bmod o(h) = \{a, b\}.$$

Recently it was shown by M. Kontsevich [18] that every Poisson structure on a flat space is quantizable.

It is well-known fact that there exists a natural chain complex associated with any associative algebra. Here we recall the definition and some basic facts about this complex.

Let  $A$  be an associative algebra with unity over the field  $C$ .

Let us denote  $C_n(A) = A \otimes A^{\otimes n}$  as the space of  $A$ -value  $n$ -chains. We can consider it as  $A$ -bimodule with the usual left and right actions

$$\begin{aligned} (a, a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n) &\rightarrow aa_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n \\ (a, a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n) &\rightarrow a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_na. \end{aligned}$$

Recall that there is an exact sequence of right  $A$ -modules

$$C_{n+1}(A) \xrightarrow{b'} C_n(A) \xrightarrow{b'} \dots \xrightarrow{b'} C_1(A) \xrightarrow{b'} A,$$

where  $b' = \sum_{0 \leq i \leq n-1} b_i$  and

$$b_i(a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n) = (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n.$$

We introduce an operator of homotopy  $s : C_* \rightarrow C_{*+1}$ ,

$$s(a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n,$$

such that the identity  $b's + sb' = 1$  holds. So the complex  $(C(A), b')$  is acyclic.

We say that an associative algebra is  $H$ -unital if the complex  $(C(A), b')$  is homotopically trivial.

On can consider another operator  $b' = \sum_{0 \leq i \leq n} b_i$ , where  $b_n(a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n) = (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$ .

We see again that  $b^2 = 0$ , so there is a new complex named the Hochschild ones

$$C_{n+1}(A) \xrightarrow{b} C_n(A) \xrightarrow{b} \dots \xrightarrow{b} C_1(A) \xrightarrow{b} A.$$

The homologies of this complex are the Hochschild homologies.

One can also define it as  $\text{Tor}_{A \otimes A^o}(A, A)$  in the category of  $A$ -bimodules or  $A \otimes A^o$ -modules, where  $A^o$  is an “opposite” algebra. We used the free resolution defined above to establish an explicit formula for the Hochschild complex. The result of course does not depend on the choice of the projective resolution.

The dual construction  $HH^n(A, A) = \text{Ext}_{A \otimes A^o}^n(A, A)$  is called a Hochschild cohomology.

**Example 1** (Hochschild–Kostant–Rosenberg [16]) Let  $A = C^\infty(X)$ , where  $X$  is a smooth manifold. Then  $HH_n(A, A) \simeq \Omega_n(X)$ . Two maps

$$\chi : a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes a_n \rightarrow \frac{1}{n!} a_o da_1 \wedge \dots \wedge da_n$$

and

$$\chi^{-1} : a_o da_1 \wedge \dots \wedge da_n \rightarrow a_0 \otimes \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma 1} \otimes \dots \otimes a_{\sigma n}$$

give us a quasi-isomorphism of complexes.

Now let  $A$  be the deformed algebra of smooth functions on  $X$  introduced above. Let us consider a filtration for the Hochschild complex of  $A$  defined as follows:

$$F^p C_*(A) = h^p C_*(A)$$

Note that the filtration is served by the Hochschild differential and there is an isomorphism  $F^p C_*(A)/F^{p+1} C_*(A) \simeq A_0$  as vector spaces.

Apparently the zero-order term in the spectral sequence associated with the filtration coincides with the Hochschild complex for  $A_0$  as a free  $C[[h]]$ -module. That is why we get (from the Hochschild–Kostant–Rosenberg theorem)

$$E_0^p(C_*(A)) \simeq \Omega^*(X) \otimes C[[h]].$$

As shown in [1], the first-order term in the corresponding sequence coincides with the Poisson homology complex  $(\Omega^*(X), \delta_\pi) \otimes C[[h]]$ , where the differential  $\delta_\pi$  is defined as follows:

$$\begin{aligned} \delta_\pi(f_0 df_1 \wedge \cdots \wedge df_k) &= \sum_i (-1)^{i+1} \{f_0, f_i\} df_1 \wedge \cdots \wedge \hat{df}_i \wedge \cdots \wedge df_k \\ &+ \sum_{i < j} (-1)^{i+j} f_0 df_1 \wedge \cdots \wedge \hat{df}_i \wedge \cdots \wedge \hat{df}_j \wedge \cdots \wedge df_k \end{aligned}$$

or, equivalently

$$\delta_\pi(\tau) = di_\pi(\tau) - i_\pi d(\tau), \quad \tau \in \Omega^*.$$

Now the task is to study the Poisson homology of  $r$ -matrix type coadjoint orbits of rank 1. The programme of investigation for the Hochschild and cyclic homology of the  $CP^n$ -type orbits is supposed.

Recall that if  $\pi = \sum_i X_i \wedge Y_i$  then  $\delta_\pi = \sum_i (L_{X_i} i_{Y_i} - i_{X_i} L_{Y_i})$ , where  $L_X$  and  $i_X$  are the Lie derivatives along  $X$  and the interior multiplication by  $X$ , respectively.

**Example 2** The  $r$ -matrix structure introduced above might be written on the  $CP^n$ -type orbits as a restriction of the bivector field

$$\pi = - \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} + \sum_{i < j} \xi_i \xi_j \partial_{\xi_i} \wedge \partial_{\xi_j} - \sum_{i < j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j} + \sum_{j < i} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j}$$

to the space of forms on  $(z, \xi)$  of equal degree with the only relation  $\langle z, \xi \rangle = \sum_i z_i \xi_i = \text{const.}$

We assume that  $\langle z, \xi \rangle = \sum_i z_i \xi_i = 1$ . Hence the Kirillov–Kostant–Souriau structure comes from the restriction of the bivector field

$$\pi_\omega = \langle z, \xi \rangle \sum_i \partial_{z_i} \wedge \partial_{\xi_i}.$$

That is why

$$\begin{aligned} \pi_{a,b} &= a\pi_\omega + b\pi = a \sum_i z_i \xi_i \partial_{z_i} \wedge \partial_{\xi_i} - b \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} + b \sum_{i < j} \xi_i \xi_j \partial_{\xi_i} \wedge \partial_{\xi_j} \\ &+ (a+b) \sum_{i > j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j} + (a-b) \sum_{i < j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j}. \end{aligned}$$

Now we compute a Poisson homology for

$$\begin{aligned}\pi_{DS} &= \pi_\omega + \pi = \sum_i z_i \xi_i \partial_{z_i} \wedge \partial_{\xi_i} \\ &\quad - \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} + \sum_{i < j} \xi_i \xi_j \partial_{\xi_i} \wedge \partial_{\xi_j} + 2 \sum_{i > j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j}.\end{aligned}$$

One can introduce a grading on the algebras of algebraic forms and polyvectors with formal coefficients such that all natural operations (i.e exterior differential and multiplication, interior multiplication and Schouten–Nijenhuis brackets) preserve the grading. Here we attach the grading degree  $i$  to  $z_i$ ,  $\xi_i$ ,  $dz_i$ ,  $d\xi_i$  and  $-i$  to  $\partial_{z_i}$ ,  $\partial_{\xi_i}$ .

The algebra of formal differential forms  $\Omega_{\text{form}}^*(z_i, \xi_i)$  is graded (as well as the algebra  $A^k$ ). The corresponding increasing filtration is

$$F^p A = \bigoplus_{k \geq p} A_p, \quad A = F^0 A \supset F^1 A \supset \dots \supset F^p A \supset \dots$$

The Poisson (Brylinsky) differential  $\delta_\pi$  agrees with this filtration. In order to see it we decompose  $\pi_{DS}$  in two components,  $\pi_{DS} = \pi_0 + \pi_1$ , where

$$\begin{aligned}\pi_0 &= - \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} + \sum_{i < j} \xi_i \xi_j \partial_{\xi_i} \wedge \partial_{\xi_j} + \sum_i z_i \xi_i \partial_{z_i} \wedge \partial_{\xi_i} \\ \pi_1 &= 2 \sum_{i > j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j}.\end{aligned}$$

The first component of  $\pi_{DS}$  is of degree 0, the second is the sum of bivector fields of positive degrees. So  $\delta_\pi(F^p A) \subset F^p A$ . Moreover in the corresponding spectral sequence  $E_r^*$

$$E_0^* = \bigoplus_p F^p A / F^{p+1} A \simeq \bigoplus_p A_p = A, \quad \delta_o = \delta_{\pi_0}.$$

One has to compute the Poisson homology for  $\pi_0$ . The Poisson differential  $\delta_{\pi_0}$  is

$$\begin{aligned}\delta_{\pi_0} &= \sum_{i < j} (-L_{z_i \partial_{z_i}} i_{z_j \partial_{z_j}} + i_{z_i \partial_{z_i}} L_{z_j \partial_{z_j}} + L_{\xi_i \partial_{\xi_i}} i_{\xi_j \partial_{\xi_j}} - i_{\xi_i \partial_{\xi_i}} L_{\xi_j \partial_{\xi_j}}) \\ &\quad + \sum_i (L_{z_i \partial_{z_i}} i_{\xi_i \partial_{\xi_i}} - L_{\xi_i \partial_{\xi_i}} i_{z_i \partial_{z_i}}) = \sum_j \left( - \sum_{i < j} L_{z_i \partial_{z_i}} + \sum_{j < i} L_{z_i \partial_{z_i}} - L_{\xi_i \partial_{\xi_i}} \right) i_{z_j \partial_{z_j}} \\ &\quad - \sum_j \left( - \sum_{i < j} L_{\xi_i \partial_{\xi_i}} + \sum_{j < i} L_{\xi_i \partial_{\xi_i}} - L_{z_j \partial_{z_j}} \right) i_{\xi_j \partial_{\xi_j}}.\end{aligned}$$

Now our task is to compute the Poisson homology for the subcomplex of forms of equal total degree on  $z$  and  $\xi$  that in addition are in the kernel of  $i_H \stackrel{\text{def}}{=} i_{X_H}$ , where  $X_H = \sum_i z_i \partial_{z_i} - \sum_i \xi_i \partial_{\xi_i}$ .

Let us denote  $A^k(m, l) = \{\omega \in A^k \mid \deg_{z_i} \omega = m_i, \deg_{\xi_i} \omega = l_i\}$ ,  $m, l \in \mathbb{Z}_+^n$ .

It is clear that  $A^k(m, l)$  is a subcomplex of  $A^k$  under  $\delta_{\pi_0}$ . On the subcomplex this differential can be written as

$$\delta_{\pi_0} = \sum_j \left( -\sum_{i < j} m_i + \sum_{j < i} m_i - l_j \right) i_{z_j} \partial_{z_j} - \sum_j \left( -\sum_{i < j} l_i + \sum_{j < i} l_i - m_j \right) i_{\xi_j} \partial_{\xi_j},$$

or

$$\begin{aligned} \delta_{\pi_0} &= \sum_j a_j i_{z_j} \partial_{z_j} - \sum_j b_j i_{\xi_j} \partial_{\xi_j}, \\ a_j &= -\sum_{i < j} m_i + \sum_{j < i} m_i - l_j, \quad b_j = -\sum_{i < j} l_i + \sum_{j < i} l_i - m_j. \end{aligned}$$

Now we find the homotopy operators in the form

$$s = \sum_j x_j z_j^* dz_j + \sum_j p_j \xi_j^* d\xi_j = \sum_j x_j i_{z_j}^* \partial_{z_j} + \sum_j p_j i_{\xi_j}^* \partial_{\xi_j}$$

such that

$$si_H + i_H s = 0.$$

In this case  $s : \text{Ker } i_H \rightarrow \text{Ker } i_H$ .

From the well known commutation relations

$$\begin{aligned} \{dz_i, dz_j^*\} &= dz_i dz_j^* + dz_j^* dz_i = \delta_{i,j}, & [z_i^*, z_j] &= z_i^* z_j - z_j z_i^* = \delta_{i,j}, \\ \{d\xi_i, d\xi_j^*\} &= d\xi_i d\xi_j^* + d\xi_j^* d\xi_i = \delta_{i,j}, & [\xi_i^*, \xi_j] &= \xi_i^* \xi_j - \xi_j \xi_i^* = \delta_{i,j} \end{aligned}$$

we obtain that

$$\{i_{z_i}^* \partial_{z_i}, i_{z_j} \partial_{z_j}\} = L_{z_i \partial_{z_i}} \delta_{i,j} = m_i \delta_{i,j}, \quad \{i_{\xi_i}^* \partial_{\xi_i}, i_{\xi_j} \partial_{\xi_j}\} = L_{\xi_i \partial_{\xi_i}} \delta_{i,j} = l_i \delta_{i,j}$$

and

$$\{s, i_H\} = si_H + i_H s = 0 \iff \sum_i x_i m_i - \sum_i p_i l_i = 0.$$

From the identity

$$\delta_{\pi_0} s + s \delta_{\pi_0} = \sum_i a_i x_i m_i - \sum_i b_i p_i l_i$$

we see that the subcomplex  $A^k(m, l)$  is acyclic if there exist such  $(x_1, \dots, x_n, p_1, \dots, p_n)$  that

$$\sum_i a_i x_i m_i - \sum_i b_i p_i l_i = 0, \quad \sum_i x_i m_i - \sum_i p_i l_i = 0.$$

This is false if and only if for some  $\lambda$

$$a_j m_j = \lambda m_j, \quad b_j l_j = \lambda l_j.$$

Let

$$I = \{i | m_i \neq 0\}, \quad J = \{j | l_j \neq 0\}$$

Thus we have the equation

$$\begin{cases} m_i = l_j = 0, & i \in I, j \in J \\ a_i = b_j, & i \in \{0, \dots, n\} - I, j \in \{0, \dots, n\} - J \end{cases}$$

or

$$\begin{cases} m_i = l_j = 0, & i \in I, j \in J \\ \sum_{i < j} m_i - \sum_{j < i} m_i + l_j = \sum_{i < k} l_i - \sum_{k < i} l_i + m_k, \\ i \in \{0, \dots, n\} - I, & k \in \{0, \dots, n\} - J. \end{cases}$$

Now we can solve it. Let  $m_j, m_k \neq 0, j < k$ . Then  $a_j = a_k$  and

$$\begin{aligned} \sum_{i < j} m_i - \sum_{j < i} m_i + l_j &= \sum_{i < k} m_i - \sum_{k < i} m_i + l_k \implies \\ \implies 2 \sum_{j < i < k} m_i + m_j + m_k - l_j + l_k &= 0. \end{aligned}$$

Thus  $l_j > l_k \geq 0$ .

Let  $l_k > 0$ . Then  $a_j = a_k = b_j = b_k$  and

$$\begin{cases} 2 \sum_{j < i < k} m_i + m_j + m_k + l_j - l_k = 0 \\ 2 \sum_{j < i < k} l_i - m_j + m_k + l_j + l_k = 0 \end{cases} \implies 2 \sum_{j < i \leq k} (m_i + l_i) = 0 \implies l_k = m_k = 0.$$

Therefore,  $l_k = 0$ . That's why  $I \cap J = \{j\}$ ,  $\#(I \cap J) = 1$ .

Let  $i < j < k, m_i, m_j, m_k \neq 0$ . Then  $l_i > l_j > l_k \geq 0$  and  $\#(I \cap J) \geq 2$ . So there exists no more than 2 different indexes  $i, j$  such that  $m_i, m_j > 0$ . Hence  $\#I, \#J \leq 2$ .

Let  $m_i, m_j, l_j \neq 0$ . Then we obtain  $a_i = a_j = b_j$

$$\text{if } i < j \quad \begin{cases} a_i = m_j \\ a_j = -m_i - l_j \end{cases} \quad a_i = a_j = b_j \quad \text{then} \quad m_i + m_j + l_j = 0$$

and

$$\text{if } i > j, \quad \begin{cases} a_i = -m_j \\ a_j = m_i - l_j \end{cases} \quad a_i = a_j \quad \text{then} \quad m_i + m_j = l_j.$$

Since  $\sum m_i = \sum l_i$  we get  $l_i = 0, i \neq j$ , and the solution is  $I = \{j, i | j < i\}, J = \{j\}$  and  $J = \{j, i | j < i\}, I = \{j\}$ .

In this case

$$\delta_{\pi_0} = i_H = \sum_i i_{z_i \partial_{z_i}} - \sum_i i_{\xi \partial_{\xi_i}} = 0$$

on the subcomplex  $A^k(m, l)$ .

Now one can declare that

$$H_{\delta\pi_0}^* \cong \lim_{k \rightarrow \infty} \left\{ \omega \in A^k \mid \omega = \sum_{i \geq j} \sum_{0 \leq \mu \leq 1, 0 \leq \nu \leq 1} \sum_{p+q+\mu+\nu} \alpha_{i,j,\mu,\nu} (z_i \xi_j)^p d(z_j \xi_j)^\mu \right. \\ \left. + \beta_{i,j,\mu,\nu} (z_j \xi_i)^q d(z_j \xi_j)^\nu \right\}.$$

Hence one can calculate the first term in the spectral sequence. Now let us prove that the spectral sequence converges in the first term.

Since  $(z_i \xi_j)^p d(z_i \xi_j)^\mu (z_j \xi_j)^q d(z_j \xi_j)^\nu$ ,  $i > j$  is of weight  $i(p + \mu) + j(p + \mu + 2q + 2\nu)$ , it is sufficient to find such cocycle  $\sigma$  that

$$(z_i \xi_j)^p d(z_i \xi_j)^\mu (z_j \xi_j)^q d(z_j \xi_j)^\nu - \sigma = 0 \pmod{F^{i(p+\mu)+j(p+\mu+2q+2\nu)+1}}.$$

Let

$$\sigma = (z_i \xi_j)^p d(z_i \xi_j)^\mu \left( \sum_{k \geq j} z_k \xi_k \right)^q d \left( \sum_{k \geq j} z_k \xi_k \right)^\nu.$$

Then

1.  $\delta_\pi \sigma = 0$ ;
2.  $\sigma \in F^{i(p+\mu)+j(p+\mu+2q+2\nu)}$ ;
3.  $(z_i \xi_j)^p d(z_i \xi_j)^\mu (z_j \xi_j)^q d(z_j \xi_j)^\nu - \sigma = 0 \pmod{F^{i(p+\mu)+j(p+\mu+2q+2\nu)+1}}$ .

Statement 1 follows from  $\left\langle \pi, d(z_i \xi_j) d \left( \sum_{k \geq j} z_k \xi_k \right) \right\rangle = 0$ ,  $i \geq j$ .

Indeed, since

$$\pi_{DS} = \sum_i z_i \xi_i \partial_{z_i} \wedge \partial_{\xi_i} - \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} + \sum_{i < j} \xi_i \xi_j \partial_{\xi_i} \wedge \partial_{\xi_j} + 2 \sum_{i > j} z_i \xi_i \partial_{z_j} \wedge \partial_{\xi_j},$$

we obtain

$$\left\langle \pi, d(z_i \xi_j) d \left( \sum_{k \geq j} z_k \xi_k \right) \right\rangle = \left\langle \pi, \sum_{k > i} z_i z_k d\xi_j d\xi_k + \sum_{k > i} \xi_j \xi_k dz_i dz_k \right. \\ \left. - z_i \xi_j dz_j d\xi_j + z_i \xi_j dz_i d\xi_i + \sum_{k > i} (z_i \xi_k dz_j d\xi_k + \xi_j z_k dz_i d\xi_k) \right\rangle \\ = \sum_{k > j} z_i z_k \xi_j \xi_k + \sum_{i > k \geq j} z_i z_k \xi_j \xi_k - \sum_{k > i} z_i z_k \xi_j \xi_k - z_i \xi_j dz_j d\xi_j \\ - 2 \sum_{k > j} z_i z_k \xi_j \xi_k + z_i \xi_j dz_i d\xi_i + 2 \sum_{k > i} z_i z_k \xi_j \xi_k = 0.$$

Hence the Poisson homologies of Drinfeld–Sklyanin brackets are represented by cocycles

$$\sigma = (z_i \xi_j)^p d(z_i \xi_j)^\mu \left( \sum_{k \geq j} z_k \xi_k \right)^q d \left( \sum_{k \geq j} z_k \xi_k \right)^\nu, \quad i \geq j.$$

Let us consider another example of computation for some standard Poisson structures.

**Example 3.** Let

$$\pi = \sum_{i < j} z_i z_j \partial_{z_i} \partial_{z_j}.$$

It is well-known that this Poisson structure arises from the skew-polynomial deformation

$$z_i z_j = q z_j z_i, \quad i < j.$$

It is quite easy to show that

$$\delta_\pi = \sum_{i < j} (L_{z_i \partial_{z_i}} i_{z_j \partial_{z_j}} - i_{z_i \partial_{z_i}} L_{z_j \partial_{z_j}}) = \sum_j \left( \sum_{i < j} L_{z_i \partial_{z_i}} - \sum_{j < i} L_{z_i \partial_{z_i}} \right) i_{z_j \partial_{z_j}}$$

because  $[z_i \partial_{z_i}, z_j \partial_{z_j}] = 0$ .

Now one can introduce the adjoint operator  $\delta_\pi^*$  and Laplacian  $\Delta_\pi$  as follows:

$$\delta_\pi^* = \sum_j \left( \sum_{i < j} L_{z_i \partial_{z_i}} - \sum_{j < i} L_{z_i \partial_{z_i}} \right) i_{z_j \partial_{z_j}}^*,$$

where  $i_{z_i \partial_{z_i}}^* = dz_i \partial_{z_i}$  and  $i_{z_i \partial_{z_i}} i_{z_j \partial_{z_j}}^* + i_{z_j \partial_{z_j}}^* i_{z_i \partial_{z_i}} = \delta_{i,j} L_{z_i \partial_{z_i}}$ .

So

$$\Delta_\pi = \delta_\pi \delta_\pi^* + \delta_\pi^* \delta_\pi = \sum_j \left( \sum_{i < j} L_{z_i \partial_{z_i}} - \sum_{j < i} L_{z_i \partial_{z_i}} \right)^2 L_{z_j \partial_{z_j}}.$$

Recall that  $L_{z_i \partial_{z_i}}$  is grading operator, i.e.  $L_{z_i \partial_{z_i}}(z_j^k dz_j^l) = (k+l)\delta_{i,j} z_i^k dz_i^l$ . Then, if  $\omega$  is a “harmonic” homogeneous form and  $m_i = \deg_{z_i} \omega$ , we get the equation

$$\sum_j \left( \sum_{i < j} m_i - \sum_{j < i} m_i \right)^2 m_j = 0, \quad m_j \geq 0.$$

It means that  $\left( \sum_{i < j} m_i - \sum_{j < i} m_i \right) m_j = 0$ .

Let  $j_0 = \max\{j, m_j \neq 0\}$ . Then  $\left( \sum_{i < j_0} m_i - \sum_{j_0 < i} m_i \right) m_{j_0} = 0$  implies  $\sum_{i < j_0} m_i = 0$  or  $m_i = 0, i \neq j_0$ .

Therefore, the solution is

$$\bigcup_j \{(0, \dots, m_j, 0, \dots, 0), m_j \in \mathbb{Z}\}.$$

Hence the space of “ $\Delta_\pi$ -harmonic” forms, that naturally set the space of homologies, is

$$\mathcal{H}_\pi = \bigoplus_j C(z_j, dz_j) = \bigoplus_j C[z_j] \otimes C(dz_j).$$



Now we consider the formal Poisson homology complex associated with the Drinfeld–Sklyanin  $r$ -matrix structure.

**Example 4** In the affine coordinates on large Schubert cell of  $CP^n$ , the Drinfeld–Sklyanin structure is as follows:

$$\begin{aligned} \pi = & \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} - \sum_{i < j} \bar{z}_i \bar{z}_j \partial_{\bar{z}_i} \wedge \partial_{\bar{z}_j} + \sum_{i,j} (1 + \delta_{i,j}) z_i \bar{z}_j \partial_{z_i} \wedge \partial_{\bar{z}_j} \\ & + 2 \sum_{i < j} z_i \bar{z}_i \partial_{z_j} \wedge \partial_{\bar{z}_j} + 2 \left( \sum_k z_j \bar{z}_j \right) \sum_{i,j} z_i \bar{z}_j \partial_{z_i} \wedge \partial_{\bar{z}_j}. \end{aligned}$$

One can introduce a grading on the algebra of algebraic forms and polyvectors with formal coefficients such that all natural operations (i.e exterior differential and multiplication, interior multiplication and Schouten–Nijenhuis brackets) preserve the grading. Here we attach the grading degree  $i$  to  $z_i$ ,  $\bar{z}_i$ ,  $dz_i$ ,  $d\bar{z}_i$  and  $-i$  to  $\partial_{z_i}$ ,  $\partial_{\bar{z}_i}$ . So the algebra of formal differential forms  $A = \Omega_{\text{form}}^*(z_i, \bar{z}_i)$  is graded. The corresponding increasing filtration is

$$F^p A = \bigoplus_{k \geq p} A_k, \quad A = F^0 A \supset F^1 A \supset \dots \supset F^p A \supset \dots$$

The Poisson (Brylinsky) differential  $\delta_\pi$  agrees with this filtration. In order to see it we decompose  $\pi$  in two components,  $\pi = \pi_0 + \pi_1$ , where

$$\begin{aligned} \pi_0 = & \sum_{i < j} z_i z_j \partial_{z_i} \wedge \partial_{z_j} - \sum_{i < j} \bar{z}_i \bar{z}_j \partial_{\bar{z}_i} \wedge \partial_{\bar{z}_j} + \sum_{i,j} (1 + \delta_{i,j}) z_i \bar{z}_j \partial_{z_i} \wedge \partial_{\bar{z}_j}, \\ \pi_1 = & 2 \sum_{i < j} z_i \bar{z}_i \partial_{z_j} \wedge \partial_{\bar{z}_j} + 2 \left( \sum_k z_j \bar{z}_j \right) \sum_{i,j} z_i \bar{z}_j \partial_{z_i} \wedge \partial_{\bar{z}_j}. \end{aligned}$$

The first component is of degree 0, and the second one is the sum of bivector fields of positive weights. Thus,  $\delta_\pi(F^p A) \subset F^p A$ . Moreover, in the corresponding spectral sequence  $E_r^*$

$$E_0^* = \bigoplus_p F^p A / F^{p+1} A \simeq \bigoplus_p A_p = A, \quad \delta_o = \delta_{\pi_0}.$$

Thus, one has to compute the Poisson homology for  $\pi_0$ . The Poisson differential  $\delta_{\pi_0}$  is

$$\begin{aligned} \delta_{\pi_0} = & \sum_{i < j} (L_{z_i \partial_{z_i}} i_{z_j \partial_{z_j}} - i_{z_i \partial_{z_i}} L_{z_j \partial_{z_j}} - L_{\bar{z}_i \partial_{\bar{z}_i}} i_{\bar{z}_j \partial_{\bar{z}_j}} + i_{\bar{z}_i \partial_{\bar{z}_i}} L_{\bar{z}_j \partial_{\bar{z}_j}}) \\ & + \sum_{i,j} (1 + \delta_{i,j}) (L_{z_i \partial_{z_i}} i_{\bar{z}_j \partial_{\bar{z}_j}} - i_{\bar{z}_i \partial_{\bar{z}_i}} L_{z_j \partial_{z_j}}) \\ = & \sum_j \left( \sum_{i < j} L_{z_i \partial_{z_i}} - \sum_{j < i} L_{z_i \partial_{z_i}} + \sum_i (1 + \delta_{i,j}) L_{\bar{z}_i \partial_{\bar{z}_i}} \right) i_{z_j \partial_{z_j}} \\ & - \sum_j \left( \sum_{i < j} L_{\bar{z}_i \partial_{\bar{z}_i}} - \sum_{j < i} L_{\bar{z}_i \partial_{\bar{z}_i}} + \sum_i (1 + \delta_{i,j}) L_{z_i \partial_{z_i}} \right) i_{\bar{z}_j \partial_{\bar{z}_j}}. \end{aligned}$$

The homologies of complex are calculated as in the previous example:

$$H_{\pi_0}(A) \simeq \mathcal{H}_{\pi_0} = \bigoplus_j C(z_j, dz_j) \oplus \bigoplus_j C(\bar{z}_j, d\bar{z}_j).$$

Since all “harmonic” forms are  $\delta_\pi$ -closed, we see that the spectral sequence converges in the first term and

$$H_\pi(A) \simeq \bigoplus_j C(z_j, dz_j) \oplus \bigoplus_j C(\bar{z}_j, d\bar{z}_j).$$

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