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Quest for Universal Integrable Models

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Abstract

In this paper we discuss a universal integrable model, given by a sum of two Wess-Zumino-Witten-Novikov (WZWN) actions, corresponding to two different orbits of the coadjoint action of a loop group on its dual, and the Polyakov-Weigmann cocycle describing their interactions. This is an effective action for free fermions on a torus with nontrivial boundary conditions. It is universal in the sense that all other known integrable models can be derived as reductions of this model. Hence our motivation is to present an unified description of different integrable models. We present a proof of this universal action from the action of the trivial dynamical system on the cotangent bundles of the loop group. We also present some examples of reductions.

1 Introduction

During the last two decades an essential progress has been achieved in the investigation of integrable models [3, 5, 7, 8, 19]. Recently one of us [16] proposed an universal action for integrable models. It turns out to be a sum of two Wess-Zumino-Witten-Novikov (WZWN) actions, corresponding to two different orbits of the coadjoint action of a loop group, and Polyakov-Weigmann cocycle [20] describing their interaction. The WZWN model is an universal object in the conformal field theory. It is conjectured that all conformal field theories are considered as some reductions of WZNW model in the spirit of Drinfeld-Sokolov [6], or of some appropriate coset construction, and all the symmetries of the conformal model are the symmetries of the WZNW model. In other words, all the algebraic structures (operator algebras) arising in different conformal field theories are considered as some reductions of the universal enveloping Kac-Moody algebras. Hence the theory of $2d$ conformal models is exhausted by the theory of the WZWN model.

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Let M be a closed two dimensional manifold and let B denote a three dimensional manifold with boundary M.

The WZWN action is given by

$$S_0(g) = -\frac{k}{4\pi}\int_M d^2z \, \mathrm{Tr}\left(g^{-1}\partial g\right)^2 + \frac{k}{12\pi}\int_B d^3y\epsilon^{ijk} \, \mathrm{Tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g\right),$$

where $g : M \longrightarrow \hat{G}$. The trace is the Killing form on the algebra $\hat{\mathcal{G}}$ of the loop group $\hat{G}$, and k is the level of the affine algebra.

There are two different ways to derive the WZWN actions. Firstly, it is an anamolous part of the effective action for fermions on a plane in a gauge field [20], and secondly it is obtained from the Kostant-Kirillov form on an orbit of the coadjoint action of a loop group [3]. In this paper we shall search for a similar type of construction for integrable models. The action considered here is universal in the sense that all known integrable models can be derived from it by reduction. This action can be interpreted as an effective action for free fermions on a torus with nontrivial boundary conditions, where the role of perturbating relevant operators is played by monodromies of the fermions.

This approach is useful from the point of view of string theory – the set of integrable models may play the role of configuration space in string dynamics [9, 13].

In the late seventies M. Adler, B. Kostant and W. Symes [1, 4, 12, 24] proposed a scheme to construct integrable Hamiltonian systems. The AKS scheme originated from their work and was subsequently developed by Reiman and Semenov-Tian-Shansky [21, 22]. The scenario of the classical R-matrix was unveiled by Semenov-Tian-Shansky [22]. Recently one of us has proposed a hierarchy of this formalism [10].

Our approach is complimentary to the AKS formulation. The Euler-Lagrange equation of motion of our proposed universal action, based on the Lie-Poisson structure, yields a zero curvature equation. This is of course necessary, but does not fulfill the sufficient condition of integrability. Only the choice of a special Hamiltonian, prescribed by the Adler-Kostant-Symes scheme, guarantees the integrability in the Liouville sense. Also, by choosing different coadjoint orbits and different matrix entries, one can obtain various sets of integrable systems. Thus one can associate different integrable systems to various symmetric spaces (see for example [11, 14, 17, 18]).

We organise this paper in the following way: In section 2 we discuss some background material like the Hamiltonian action, the moment map of the action of a loop group [23], etc. We describe how some canonical dynamical systems associate to a cotangent bundle of a Lie group [2]. A proof of our proposed universal action is presented in section three. We derive this action from the action of the trivial dynamical system on the cotangent bundles of the loop group. In the final section we give some explicit examples.

Since there are no derivative terms appearing in the kinetic part of the action, it seems that we can not produce various nontrivial mechanical systems. This is similar to the Hamiltonian system on the cotangent bundle (T^*G, ω). The form ω is the symplectic form on the cotangent bundle. This does not have an immediate mechanical meaning (in the general sense). However, in both the cases they do enable us to produce an interesting family of Hamiltonian systems associated to a family of arbitrary Riemannian symmetric spaces. We explain this in one of our examples. We also add an appendix, where we present an explicit connection between the nonlinear Schrödinger equation and the Heisenberg ferromagnetic system, although this connection is known for some time (see for example [17, 18]).

2 Preliminaries

2.1 Hamiltonian action and moment map

Let us start this section with some standard definitions of Hamiltonian mechanics [15].

Let G be any compact semi-simple Lie group, $\mathcal{G}$ its Lie algebra, and $\mathcal{G}^*$ the dual space of $\mathcal{G}$. The left and right translation

$$L_g : h \longmapsto gh, \qquad R_g : h \longmapsto hg$$

induces a map

$$dL_g^*(\text{or } dR_g^*) : T_g^*G \longrightarrow T_e^*G \cong \mathcal{G}^*.$$

Thus if $(g, \kappa_g) \in T^*G$, where $\kappa_g \in T_g^*G$ is the coordinate of the fibre, then

$$(g, \kappa_g) \overset{L}{\longmapsto} (g, l_i), \qquad l_i = dL_g^* \kappa_g,$$
$$(g, \kappa_g) \overset{R}{\longmapsto} (g, r_i), \qquad r_i = -dR_g^* \kappa_g,$$

where r_i and l_i are related by

$$r_i = -\text{Ad}^* g(l_i).$$

Hence we can identify

$$T^*G \overset{L,R}{\longrightarrow} G \times \mathcal{G}^*.$$

Let $\hat{\mathcal{G}} = C^\infty(S^1, \mathcal{G})$ be the loop algebra and $\hat{G}$ the corresponding loop group. By left or right trivialization, induced from T^*G, we can identify

$$T^*\hat{G} \simeq \hat{G} \times \hat{\mathcal{G}}^*.$$

Let us consider a Hamiltonian action

$$\hat{G} \times T^*\hat{G} \longrightarrow T^*\hat{G},$$

such that

$$L_h(g, l_i) = (hg, l_i), \qquad R_h(g, l_i) = \left(gh, \text{Ad}^* h^{-1}(l_i)\right).$$

The Hamiltonian actions are given by

$$\mu_X^L(g, l_i) = -\langle X, \text{Ad}^* g(l_i)\rangle, \qquad \mu_X^R(g, l_i) = \langle X, l_i\rangle,$$

where $X \in \hat{\mathcal{G}}$, $g \in T^*\hat{G}$ and $l_i \in \hat{\mathcal{G}}^*$.

Then the corresponding moment maps associated to the Hamiltonian action are

$$\mu_L(g, l_i) = \text{Ad}^* g(l_i), \qquad \mu_R(g, l_i) = l_i.$$

Hence by symplectic reduction the reduced phase space is naturally identifiable with the coadjoint orbit.

Let $\alpha \in \hat{\mathcal{G}}^*$ be a constant element then there exist a canonical one form

$$\theta := \langle \alpha, g^{-1} dg \rangle$$

and the symplectic form

$$\Omega := d\langle \alpha, g^{-1} dg \rangle$$

on $\hat{\mathcal{G}}^*$.

We define a geometrical action $S = \int \theta$ on $T^*\hat{G}$ as a functional of trajectories on $T^*\hat{G}$. The symmetries of this geometrical actions are:

$$\alpha \to \alpha, \qquad g \to h_R g,$$
$$\alpha \to h_L^{-1} \alpha h_L, \qquad g \to g h_L,$$

where h_L and h_R are constant elements of $\hat{G}$.

Principle Bundle Construction: If $\hat{\mathcal{O}}_\nu$ is the coadjoint orbit in $\hat{\mathcal{G}}^*$ through the point $\nu \in \hat{\mathcal{G}}^*$, then there is a natural canonical imbedding

$$i_\nu : \hat{\mathcal{O}}_\nu \longrightarrow \hat{\mathcal{G}}^*.$$

The left translation is already indentified by $T^*\hat{G} \equiv \hat{G} \times \hat{\mathcal{G}}^*$. Then the pullback map $i_\nu^*(T^*\hat{G})|_{\hat{\mathcal{O}}_\nu}$, with restriction of $T^*\hat{G}$ on $\hat{\mathcal{O}}_\nu$, is the principle bundle over $\hat{\mathcal{O}}_\nu$. Let $\hat{G} \times \hat{\mathcal{O}}_\nu \longrightarrow \hat{\mathcal{O}}_\nu$ is the trivial bundle over $\hat{\mathcal{O}}_\nu$. Let $f : \hat{G} \times \hat{\mathcal{O}}_\nu \longrightarrow \hat{\mathcal{G}}$ be an equivariant function. Let $\alpha_g \in i_\nu^*(T^*\hat{G})$ and $dR_g^*(\alpha_g) \in T_e^*\hat{G} \equiv \hat{\mathcal{G}}^*$ then it is easy to prove

Proposition 2.1. *$f(\alpha_g) = dR_g^*(\alpha_g)$ is the moment map associated to the action of $\hat{G}$ on $T^*\hat{G}$.*

In the next section we describe a canonical integrable system associated with T^*G, which can easily be lifted to $T^*\hat{G}$.

2.2 A Universal Integrable System on T^*G

In this section we present a brief description on the construction of a universal integrable system on the cotangent bundle of the Lie group [2].

The moment map $\mu : T^*G \simeq G \times \mathcal{G}^* \longrightarrow \mathcal{G}$, associated with the Hamiltonian action $G \times T^*G \to T^*G$, is a Poisson map whenever $\mathcal{G}^*$ is endowed with a natural Poisson structure.

Definition 2.2. *A bivector $\Lambda \in \wedge^2 \mathcal{G}$ is called a Poisson bivector if it commutes with itself*

$$[\Lambda, \Lambda] = 0.$$

Two Poisson bivectors Λ_1, Λ_2 are called compatible if they commute with one another

$$[\Lambda_1, \Lambda_2] = 0.$$

This is equivalent to the fact that any linear combination $l\Lambda_1 + m\Lambda_2$ is a Poisson bivector. This is called pencil of Poisson bivectors.

A well known example is the rigid body system. In this case the moment map is a Poisson map

$$\mu : T^*SO(3) \longrightarrow so(3)^*,$$

with the linear Poisson structure

$$\Lambda_{so(3)^*} = \epsilon_{ijk} p_i \partial_j \otimes \partial_k.$$

We consider differential 1-form η on $\mathcal{G}^*$, which is annihilated by the natural Poisson structure $\Lambda_{\mathcal{G}^*}$ on $\mathcal{G}^*$ associated with the Lie bracket. Such a form is called a Casimir form.

Definition 2.3. *We define the vector field by $\Gamma_\eta = \Lambda(\mu^*(\eta))$, and the dynamical system by*

$$g^{-1}\dot{g} = \eta(g, \alpha) = \eta(\alpha),$$
$$\dot{\alpha} = 0,$$

where $\Omega = d\langle \alpha, g^{-1} dg \rangle$.

This system can be integrated by quadratures on each level set, obtained by fixing α's in $\mathcal{G}^*$, so that this particular dynamical system coincides with a one parameter group of the action of G on that particular level set.

Consider, for example, the rigid rotator $\eta = f dH$, where $H = \sum_i p_i^2/2$ is the Hamiltonian and $f = f(p)$ is an arbitrary function. If $\{X_i\}$ is the basis of $so(3)$. Then it is not difficult to see that

$$\Gamma_\eta = f(p) p_i \hat{X}_i,$$

where $\hat{X}_i$ are left invariant vector fields on $SO(3)$. Hence, the dynamical system is given by

$$g^{-1}\dot{g} = f(p) p_i X_i,$$
$$\dot{p}_i = 0.$$

In particular, if we restrict ourselves to the abelian Lie group R^n, then $\mu : T^*R^n \longrightarrow (R^n)^*$, induced by the natural action of R^n on itself by a translation, is a Poisson map. Let $\eta = \nu_k dI^k$ be a one form on $(R^n)^*$ in terms of action-angle variables. After pulling it back to T^*R^n we obtain a vector field $\Gamma_\eta = \Lambda(\mu^*(\eta))$, where Λ is the canonical Poisson structure in the cotangent bundle. Then the associated equations of motion on T^*R^n or $T^*\mathbf{T}^n$ is

$$\dot{I}^k = 0, \qquad \dot{\phi}_k = \nu_k.$$

We can recover this from another point of view. Let the Ad^*-invariance function $H : T^*G \longrightarrow R$ satisfies

$$H = \frac{1}{2} ||g^{-1}\dot{g}||_G^2.$$

This is a free particle Hamiltonian. Now it is easy to see that if $(g, g^{-1}\dot{g}) \in T^*G$, the equation assumes the form

$$\dot{(g^{-1}\dot{g})} = 0.$$

If we assume

$$H(g, g^{-1}\dot{g}) = \frac{1}{2}||g^{-1}\dot{g}||^2_G - \langle \mathrm{Ad}^*_g \alpha, \beta \rangle$$

for $\alpha \in \mathcal{G}^*$, the equation becomes

$$(\dot{\overline{g^{-1}\dot{g}}}) = [\mathrm{Ad}^*_{g^{-1}}(\beta), \alpha].$$

This equation is the nontrivial part of the canonical system of equations for the free particle of the Hamiltonian system (T^*G, Ω, H).

3 Universal Integrable Model

Recently [16] Olshanetsky proposed an action based on WZWN theory which has the following form

$$S = S^u(A) - H(A),$$

where

$$S^u(A) = 2 \int \mathrm{tr}(u\bar{\partial}gg^{-1})d^2z + kS_{WZWN}.$$

Here $H(A)$ is a Hamiltonian and A is a current, given by

$$A = g^{-1}ug + g^{-1}\partial g.$$

It defines a point on the coadjoint orbit through a point (u, k) in $\hat{\mathcal{G}}^*$.

Let us assume, for simplicity, that $k = 1$. The equation of motion, based on the Lie Poisson structure

$$(\bar{\partial} - \mathrm{ad}^*_{\bar{A}})A = 0,$$

is given by

$$\bar{\partial}A = [\bar{A}, A] + \partial\bar{A},$$

where $\bar{A} = \mathrm{grad}\ H$, the gradient of the Hamiltonian. This is a zero curvature equation, which is a necessary but not sufficient condition of integrablity. Only the special Hamiltonians guarantee this distinguish property. We must emphasize here that $\partial/\partial\bar{z}$ arises along with the "time" derivative $\frac{\partial}{\partial z}$, when the central extension of the classical algebra is considered.

Conformal models are distinguished by their holomorphicity property: these are theories of massless scalars with the equation of motion

$$\bar{\partial}A = 0, \qquad \text{and} \qquad A = \partial\phi.$$

We have already seen that in the WZWN model the role of A is played by the Kac-Moody currents.

The Adler-Kostant-Symes scheme can be used to choose a particular subset of the zero curvature equation which are integrable in the Liouville sense.

Apparently there is a drawback in this model, there appear no z derivative in the kinetic part of the action. This is the reflection of the lack of central charge in the R-algebra. This is exactly the necessary condition for the description of ordinary integrable systems. But we shall show how to overcome this difficulty by constructing the mechanical systems associated to the Riemannian symmetric spaces via Fordy-Kulish decomposition.

3.1 AKS Scheme and Zero Curvature Equations

Let $\hat{\mathcal{G}} = gl(n, \mathbf{C}) \times \mathbf{C}[\lambda, \lambda^{-1}]$ be the loop algebra of a semi-infinite formal Laurent series in λ with coefficients in $gl(n, \mathbf{C})$. For example, an element $X(\lambda) \in \hat{\mathcal{G}}$ can be expressed as a formal series in the form

$$X(\lambda) = \sum_{i=-\infty}^{m} x_i \lambda^i \qquad \forall\, x_i \in gl(n, \mathbf{C}).$$

The Lie bracket, with $Y(\lambda) = \sum_{j=-\infty}^{l} y_j \lambda^j$, is given by

$$[X(\lambda), Y(\lambda)] = \sum_{k=-\infty}^{m+l} \sum_{i+j=k} [x_i, y_j] \lambda^k.$$

We define a nondegenerate bilinear two form on $\hat{\mathcal{G}}$

$$\langle A(\lambda), B(\lambda) \rangle := \mathrm{Res}_{\lambda=0}(\lambda^{-1} A(\lambda) B(\lambda)) = \mathrm{tr}(A(\lambda)B(\lambda))_0.$$

There is a natural splitting in the loop algebra $\hat{\mathcal{G}} = \hat{\mathcal{G}}^+ \oplus \hat{\mathcal{G}}^-$, where $\hat{\mathcal{G}}^+$ denotes the subalgebra of $\hat{\mathcal{G}}$, given by the polynomial in λ, and $\hat{\mathcal{G}}^-$ is the subalgebra of strictly negative series.

The above decomposition of $\hat{\mathcal{G}}$ do not correspond to a global decomposition of the loop group $\hat{G}$, but we have a dense open subset

$$\hat{G}^- \hat{G}^+ \subset \hat{G},$$

consisting of all loops ϕ that can be factorized in the form

$$\phi = \phi^- \phi^+$$

with $\phi^- \in \hat{\mathcal{G}}^-$, $\phi^+ \in \hat{\mathcal{G}}^+$. We refer to this subset of $\hat{G}$ as the *big cell*.

Let us consider the Grassmannian like homogeneous space $\hat{\mathcal{G}}/\hat{\mathcal{G}}^+$. The image in $\hat{\mathcal{G}}/\hat{\mathcal{G}}^+$ of the complement of the big cell in $\hat{\mathcal{G}}$ is a divisor in $\hat{\mathcal{G}}/\hat{\mathcal{G}}^+$. It therefore corresponds to a holomorphic line bundle

$$\mathcal{L} \longrightarrow \hat{\mathcal{G}}/\hat{\mathcal{G}}^+.$$

We denote by LG the automorphism group of $\mathcal{L}$. The pullback of $\mathcal{L}$ to LG is canonically trivial. Hence LG turns out to be the central extension of $\hat{G}$ by $C^\times$:

$$1 \longrightarrow \mathbf{C}^\times \longrightarrow LG \longrightarrow \hat{G} \longrightarrow 1.$$

The loop algebra

$$L\mathcal{G} = \hat{\mathcal{G}} \oplus \mathbf{C}$$

satisfies the following commutation relation

$$[(A(\lambda), a), (B(\lambda), b)] := ([A, B](\lambda), \omega(A, B)),$$

where $\omega(A,B)$ is the Maurer-Cartan $\mathbf{C}$-valued two cocycle

$$\omega(A,B) = \int_0^{2\pi} (A, \partial_{\bar{z}} B),$$

which satisfies

$$\omega(A,[B,C]) + \omega(B,[C,A]) + \omega(C,[A,B]) = 0.$$

$L\mathcal{G}$ is called the central extension of $\hat{\mathcal{G}}$, obtained through ω. In this particular case $L\mathcal{G}$ is also called a Kac-Moody algebra on S^1.

In general, the map

$$\kappa : \hat{\mathcal{G}} \to L\mathcal{G}$$

is not a Lie algebra homomorphism, but only its restriction to $\hat{\mathcal{G}}^+$ is a Lie algebra homomorphism, since the central extension term vanishes identically. The corresponding induced map

$$\kappa : \hat{G}^+ \longrightarrow LG$$

yields a canonical holomorphic trivialization of the part of the fibration lying over $\hat{\mathcal{G}}$.

Let $\phi = \phi^- \phi^+$ be an element of the big cell. Then $\kappa\phi$ satisfies

$$\kappa(\phi) = \kappa(\phi^-)\kappa(\phi^+),$$

where $\kappa(\phi)$ is the dense open subset of $L\mathcal{G}$ that lies over the big cell of $\hat{\mathcal{G}}$.

We also define the bilinear form on $L\mathcal{G}$

$$\langle (A,a),(B,b)\rangle = ab + \int_{s^1} \mathrm{tr}(AB).$$

Let $R \in \mathrm{End}\,\mathcal{G}$ be the linear operator on $\mathcal{G}$. The Kostant-Kirillov-Souriau R-bracket is given by

$$[X,Y]_R = \frac{1}{2}([RX,Y] + [X,RY]) \qquad \forall\, X,Y \in \mathcal{G}.$$

This satisfies the Jacobi identity if R-satisfies the modified Yang-Baxter equation.

Definition 3.1. *Let $(\hat{\mathcal{G}}, R)$ be a double loop algebra on which we define two algebraic structures. Suppose also that ω is a 2-cocycle on $\hat{\mathcal{G}}$. Then*

$$\omega_R(X,Y) = \omega(RX,Y) + \omega(X,RY)$$

is a 2-cocycle on $\hat{\mathcal{G}}_R$.

The gradient $\nabla F : \mathcal{G}^* \to \mathcal{G}$ is defined by

$$\frac{d}{dt} F(U+tV)|_{t=0} = \langle V, \nabla F(U)\rangle.$$

Lemma 3.2. *Let H be an ad-invariant function on $\hat{\mathcal{G}}^*$. Then the gradient of H satisfies*

$$\mathrm{ad}^*(R\nabla H(\alpha), a)(\alpha, 1) = (\mathrm{ad}^*(R\nabla H(\alpha))(\alpha) + R\nabla H', 0).$$

Sketch of the **Proof**: By using the identity

$$\langle \mathrm{ad}^*_R(X,a)(\beta,c),(Y,b)\rangle + \langle(\beta,c),\mathrm{ad}_R(X,a)(Y,b)\rangle = 0,$$

we obtain our result. ■

Definition 3.3. *There exists a natural Poisson structure on the space $C^\infty(\hat{\mathcal{G}}^*,\mathbf{C})$ of smooth real valued functions on $\hat{\mathcal{G}}^*$*

$$\{\xi,\chi\}(U,c) = \int_{S^1} \mathrm{Tr}\left(c\frac{d(\nabla\xi)}{d\bar{z}} + [\nabla\xi,\nabla\chi],U\right)d\bar{z} \qquad \forall\, \xi\chi \in C^\infty(\hat{\mathcal{G}}^*).$$

We observe that the central parameter c is fixed under the coadjoint action of the group. So $L\mathcal{G}^*$ stratifies into Poisson submanifolds, corresponding to different values of the parameter.

The differential equation appears from the ad-invariant condition, which should be satisfied by the gradients of the local Hamiltonians

$$(\partial_{\bar{z}} - \mathrm{ad}^*\,\alpha)\nabla H = 0.$$

It is known that the good substitutes for local Hamiltonians are Casimir functions. Hence we choose Hamiltonian $H = \frac{1}{2}\mathrm{tr}(\alpha^2)$.

Theorem 3.4. *Let α be the orbit. The Hamiltonian equations of motion on the $\hat{\mathcal{G}}^*$, generated by the gradient of the Hamiltonian H (the ad-invariant function), have the form*

$$\frac{d\alpha}{dz} = \frac{Rd(\nabla H)}{d\bar{z}} + [R(\nabla H),\alpha].$$

3.2 Derivation of the Action

Let us derive the generic form of the action for IM from the action of the trivial dynamical system on the cotangent bundle $T^*\hat{G}$

Let $(g,m;u,n) \in T^*LG$. Then by the left action of $\hat{G}$ follows

$$g \xrightarrow{L} gh, \qquad m \to m, \qquad n \to n, \qquad \forall\, h \in \hat{G}.$$

Since m, n are invariant under the action of group $h \in \hat{G}$, the action foliates T^*LG into hyperplanes. Let us confine to a particular hyperplane $m = 0$, $n = 1$. We again consider the two form $\omega(g)$ on $\hat{G}$

$$\omega(g) = \int dz\ \mathrm{Tr}\langle dgg^{-1},\partial(dgg^{-1})\rangle.$$

Let us replace u by a new field

$$h = P\exp\int dz'u(z').$$

Definition 3.5. *A symplectic form on $T^*\hat{G}$ is given by*

$$\Omega = \omega(g) + \omega(h) + 2\int dz\langle u,(dgg^{-1})^2\rangle.$$

The corresponding one form β satisfies $d\beta = \Omega$. We define a Hamiltonian

$$H(v) = 2\int d^2z\langle v, (hg)^{-1}\partial(hg)\rangle.$$

Hence the action is given by

$$S = \int dz'(\beta - H)$$

$$\simeq S_{WZWN}(h) + S_{WZWN}(g) + 2\int d^2z\langle u, dgg^{-1}\rangle - H.$$

After the gauge fixing condition we arrive at

$$S = S^u(A) - H(v, A),$$

where $H(v, A) = \int d^2z\langle v, A\rangle$.

It was demonstrated by Polyakov and Wiegmann [20], that an effective action from the fermionic Lagrangian on a plane

$$\mathcal{L} = \bar{\psi}^L(\partial - A^u)\psi^L + \bar{\psi}^R(\bar{\partial} - A^{\bar{v}})\psi^R + \frac{1}{2\alpha_0}\langle A^u, A^{\bar{v}}\rangle$$

, gives rise to a sum of the WZWN action in a gauge invariant form. In this case we have

$$\log\frac{\det(\partial - A^u)\det(\bar{\partial} - A^{\bar{v}})}{\det(\partial - u)\det(\bar{\partial} - v)} = S - 2\int d^2z\langle u, v\rangle.$$

Proposition 3.6. *The equation of motion, corresponding to* $S = S^u(A) - H(v, A)$ *for* $H(v, A) = \int d^2z\langle v, A\rangle$, *is*

$$\bar{\partial}A = [v, A] + \partial v.$$

Suppose g is any arbitrary element of the loop group. Then there exist a gauge transformation

$$u \longrightarrow g^{-1}ug + g^{-1}\partial g.$$

Hence the matrices u, v satisfy the zero curvature equation

$$\bar{\partial}u = [v, u] + \partial v.$$

We assume that u and v depend on a spectral parameter λ which lives on a rational curve $\mathbf{CP}^1$

$$u = u_0 + \sum_{j=1}^{m_1}\frac{u_j}{\lambda - a_j}, \qquad v = v_0 + \sum_{k=1}^{m_2}\frac{v_k}{\lambda - b_k}$$

, such that u and v satisfy the zero curvature condition.

Since the zero curvature condition is preserved under the gauge transformation $u \longmapsto u^g = g^{-1}ug + g^{-1}\partial g = A$, the corresponding linear equations

$$\partial\psi = A^u\psi, \qquad \bar{\partial}\psi = A^{\bar{v}}\psi$$

of the zero curvature equation satisfy

Proposition 3.7. *If $\partial\psi = -u\psi$, then A also satisfies the same equation for $g^{-1}\psi$.*

Sketch of the **Proof:**

$$\partial(g^{-1}\psi) = -g^{-1}\partial g g^{-1}\psi + g^{-1}\partial\psi = -A(g^{-1}\psi). \qquad \blacksquare$$

If we consider fermions with monodromies, then due to the zero curvature condition the left and right function can be identified by $\psi^L = \psi^R = \psi$.

We may regard ψ as a function on $\mathbf{R}$ with values in $\hat{G}$. Its value at $x = 2\pi$ is called the monodromy matrix T_A. The coadjoint orbits are described by Floquet's theorem.

Theorem 3.8. (Floquet) *Two periodic potentials A and A' are gauge equivalent if and only if the corresponding monodromy matrices T_A, $T_{A'}$ are conjugate.*

Consider some particular cases:

Remark 3.9. *For the generic integrable models, the following relations hold automatically*

$$\bar{\partial}u = \partial v = [u, v] = 0.$$

Proposition 3.10. *If an integrable model satisfies $\bar{\partial}u = \partial v = [u, v] = 0$, then the zero curvature equation reduces to*

$$\bar{\partial}g = gv - vg.$$

Proof. We know that

$$A = g^{-1}ug + g^{-1}\partial g,$$
$$\bar{\partial}A = -g^{-1}\bar{\partial}gA + g^{-1}u\bar{\partial}g + g^{-1}\partial\bar{\partial}g.$$

Let us substitute our ansatz $\bar{\partial}g = gv - vg$ in the above equation:

$$\begin{aligned}\bar{\partial}A &= -g^{-1}\bar{\partial}gA + g^{-1}ugv - g^{-1}uvg + g^{-1}\partial gv - g^{-1}v\partial g\\ &= -g^{-1}\bar{\partial}gA + Av - g^{-1}uvg - g^{-1}v\partial g\\ &= -g^{-1}(gv - vg)A + Av - g^{-1}uvg - g^{-1}v\partial g.\end{aligned}$$

Since

$$g^{-1}vgA = g^{-1}vug + g^{-1}v\partial g,$$

we get back the equation of motion $\bar{\partial}A = [A, v]$. $\blacksquare$

Additionally we have

$$\partial g = gA - ug,$$

which follows from the current.

Proposition 3.11.

$$[\partial, \bar{\partial}]g = 0, \qquad \partial^2 \neq 0 \neq \bar{\partial}^2.$$

Sketch of the **Proof:** Since

$$\bar{\partial} g = gv - vg$$

, it is easy to see that $\partial\bar{\partial} g = \bar{\partial}\partial g$. ∎

Let us consider the factorization of the matrix $g(\lambda)$ such that

$$g(\lambda) = g_+(\lambda) g_-^{-1}(\lambda)$$

is the solution to the Riemann problem, where g_+ is an element of the group of all smooth functions from the unit circle S^1 to G that extend to holomorphic G-valued functions on the disk $\{\lambda : |\lambda| < 1\}$. Similarly, $g_-^{-1}(\lambda)$ is the element of the group of all smooth functions $S^1 \to G$ that extend holomorphically to the disk $\{\lambda : |\lambda| > 1\}$ and take the value 1 at infinity.

Substituting $g = g_+ g_-^{-1}$ in $\bar{\partial} g = gv - vg$ and $\partial g = gA - ug$. We obtain

$$g_-^{-1} v g_- + g_-^{-1} \bar{\partial} g_- = g_+^{-1} v g_+ + g_+^{-1} \bar{\partial} g_+,$$
$$g_-^{-1} A g_- + g_-^{-1} \partial g_- = g_+^{-1} u g_+ + g_+^{-1} \partial g_+.$$

Definition 3.12. *We define two new currents*

$$A^u = g_+^{-1} u g_+ + g_+^{-1} \bar{\partial} g_-,$$
$$A^{\bar{v}} = g_-^{-1} v g_- + g_-^{-1} \bar{\partial} g_-.$$

Consider a contour γ which consists of small circles around the points $a_j\, (j = 1, \ldots, m_1)$. Let us modify the action S by

$$S \longrightarrow \int_\gamma d\lambda S.$$

Originally, Olshanetsky [???] generalized S by introducing the kinetic term $\hbar\partial_\lambda$ in such a way that one obtains, in addition to the zero curvature equation, a new equation of motion in the form of the string equation:

$$[\partial + A(g), \hbar\partial_\lambda + v'] = \hbar.$$

Earlier, Gerasimov *et al* [8, 9] proposed a number of of programs for incorporating integrable models into the general framework of string theory. The string theory is understood as some dynamical theory on some configuration space which contains at least all the 2-dimensional field theories as its points. They argued that for the universal description of all conformal models, it is necessary to treat various Kac-Koody algebras on the same ground, through their embedding into $\hat{G}L(\infty)$ algebra. It may be described through dependence on some auxiliary variable λ. Hence this explains why λ appears in the equation.

A string equation is a sort of "quantum deformation" of a zero curvature equation. Uptil now the holomorphic dependence of the spectral parameter is quite artifical. Moreover, the geometrical meaning of this "deformed" zero curvature equation is still lacking. We now try to give a plausible explanation of this equation.

So far we have encountered three coordinates $(z, \bar{z}, \lambda)$, where $\bar{z}$ appears along with z and $\bar{z}$ plays the role of "time". Apparantly it seems that the lack of $\bar{\lambda}$ dependence may be a drawback from the point of view of integrable systems. But this can be managed in the following way:

Let $(z, \lambda, \bar{z}, \bar{\lambda})$ be the coordinates on $\mathbf{R}^4$, which are independent and real for signature (2,2). The self dual Yang-Mills equations are the compatibility conditions for the pair of operators

$$L_0 = (D_z - \xi D_{\bar{\lambda}}), \qquad L_1 = (D_\lambda + \xi D_{\bar{z}}),$$

where $\xi \in \mathbf{C}$ is an auxiliary complex spectral parameter and D_z is the covariant derivative of some Yang-Mills connection in the direction $\partial/\partial z$. When we impose one null symmetry along $\partial/\partial \bar{z}$ and another along $\partial/\partial \bar{\lambda}$ we obtain the Lax pair:

$$L_0 = \frac{\partial}{\partial z} + A(z, \lambda), \qquad L_1 = \frac{\partial}{\partial \lambda} + B(z, \lambda).$$

Definition 3.13. *Let $g = g_1 g_2$, then the Polyakov-Wiegmann formula is defined by*

$$S_{WZW}(g_1 g_2) = S_{WZW}(g_1) + S_{WZW}(g_2) + \frac{1}{2\pi} \int d^2 z \langle g_1^{-1} \bar{\partial} g_1, \partial g_2 g_1^{-1} \rangle.$$

Hence from our previous computation we can assert:

Proposition 3.14. *For $g = g_+ g_-^{-1}$, the action becomes*

$$S = S^u(g_+) + S^{\bar{v}}(g_-) + \int d^2 z \langle A^u(g_+), A^{\bar{v}}(g_-) \rangle,$$

where

$$S^u(g_+) = 2 \int d^2 z \langle u, \bar{\partial} g_+ g_+^{-1} \rangle + S_{WZWN}(g_+),$$

$$S^{\bar{v}}(g_-) = 2 \int d^2 z \langle v, \partial g_- g_-^{-1} \rangle + S_{WZWN}(g_-^{-1}).$$

It is easy to see that $S = S(g_+ g_-^{-1})$ is a modified Polyakov-Weigmann formula. This action is gauge invariant under

$$g_+ \longrightarrow g_+ h, \qquad g_- \longrightarrow g_- h,$$

where h is independent of λ and the equation of motion is a zero curvature equation with a spectral parameter on an arbitrary Lie algebra $\mathcal{G}$:

$$\bar{\partial} A^u - \partial A^{\bar{v}} + [A^u, A^{\bar{v}}] = 0.$$

This equation does not guarantee integrability of the system. In particular, if we choose

$$A^u = A_0 + A_1 \lambda + A_2 \lambda^2 + \cdots + A_n \lambda^n,$$
$$A^{\bar{v}} = A^u \lambda^{-1},$$

then one recovers the Adler-Kostant-Symes equation, where A^u is considered to be a Lax operator L, and $A^{\bar{v}}$ is the gradient of $H = \frac{1}{2}\mathrm{tr}(L^2 \lambda^{-1})$. In fact, the hierarchy of AKS system can be recasted into this zero curvature form.

4 Applications

In this section we will present some examples. We have already stated that our Euler-Lagrange equation is a zero curvature equation, and hence does not have an immediate mechanical meaning. We show that after imposing the Cartan decomposition of Lie algebras, we obtain a family nontrivial mechanical systems associated to an arbitrary Riemannian symmetric space.

4.1 Periodic Toda Lattice

Let $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ be a system of simple roots of the affine Lie algebra $\hat{\mathcal{G}}$, where $(\alpha_1, \ldots, \alpha_n)$ are simple roots of the original finite dimensional algebra $\mathcal{G}$, and $-\alpha_0 = \sum_{j=1}^{n} a_j \alpha_j$ is the highest root.

Let $(s_0, s_1, \ldots, s_n)$ be a set of non-negative integers without a common divisor, and $N = \sum_{j=1}^{n} a_j s_j$ be the order of σ, where $\sigma^N = 1$.

Definition 4.1. *A grading is a decomposition $\mathcal{G} = \oplus_{j \in Z} \mathcal{G}_j$ of the Lie algebra $\mathcal{G}$ into a direct sum of subspaces $\mathcal{G}_j$, such that*

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \quad mod\, N,$$

$$\sigma \mathcal{G}_k = \epsilon^k \mathcal{G}_k, \qquad \epsilon = e^{\frac{2\pi i}{N}}.$$

This automorphism is called Coxeter automorphism.

An invariant subalgebra is a direct sum

$$\mathcal{G}_0 = \mathbf{R} \oplus \cdots \oplus \mathbf{R} \oplus \mathcal{G}(k),$$

where $\mathcal{G}(k)$ is a semi-simple subalgebra generated by simple roots $(\alpha_{j_1}, \ldots, \alpha_{j_k})$ for which

$$s_{j_1} = \cdots = s_{j_k} = 0.$$

Definition 4.2. *When $s_0 = s_1 = \cdots = s_n = 1$, $\mathcal{G}_0 \cong \mathcal{H}$ is a Cartan subalgebra and $N = \sum_{j=0}^{n} a_j = h$ is the Coxeter number.*

Let (H_j, E_j, F_j) $\forall\, j = 0, \ldots, n$ be the Cartan Weyl basis of $\hat{\mathcal{G}}$. We define the following action of σ:

$$\sigma E_j = \epsilon^{s_j} E_j, \qquad \sigma H_j = H_j, \qquad \sigma F_j = \epsilon^{-s_j} F_j.$$

Let us consider the zero curvature equation again for

$$A^u = A_0 + A_1 \lambda, \qquad A^{\bar{v}} = A_{-1} \lambda^{-1}.$$

Then the zero curvature equation decomposes into

$$(1) \quad \bar{\partial} A_1 = 0,$$

$$(2) \quad \partial A_{-1} = [A_0, A_{-1}],$$

$$(3) \quad \bar{\partial} A_0 + [A_1, A_{-1}] = 0.$$

One can readily identify

$$A_0 \in \mathcal{G}_0, \qquad A_1 \in \mathcal{G}_1, \qquad A_{-1} \in \mathcal{G}_{N-1}.$$

Moreover $A_1 \in \mathcal{G}_1$ is a constant matrix in $\mathcal{G}_1$.

Proposition 4.3. *Let $\eta \in \mathcal{G}_{N-1}$ and $A_0 = q^{-1}\partial q$. Then the above system of equations reduces to the periodic Toda lattice equation.*

Sketch of the **Proof:** From the first equation we let $A_1 = \eta$ be a constant matrix in $\mathcal{G}_1$ and, if $\xi \in \mathcal{G}_{N-1}$, then from the second equation we obtain $A_{-1} = q^{-1}\xi q$ and $A_0 = -q^{-1}\partial q$. Finally from the third equation we obtain

$$\bar{\partial} q^{-1}\partial q - [\eta, q^{-1}\xi^{-1}q] = 0,$$

by the substitution of $q = \exp(\phi)$, where $\phi \in \mathcal{H}$ yields the periodic Toda lattice equation. ∎

4.2 Hermitian Symmetric Spaces and Integrablity

A Riemannian manifold M is called a globally symmetric Riemannian space, if every point $p \in M$ is a fixed point of involutative isometry of M which takes any geodesic through p into itself as a curve but reverses its paramatrization.

Let G be a semi-simple Lie group and $\mathcal{G}$ its Lie algebra. Let M be a homogeneous space of G such that M is a differentiable manifold on which G acts transitively. There is a homeomorphism of the coset space G/H onto M for some isotropy subgroup H at a point of M. Let $\mathcal{H}$ be the Lie algebra of H and $\mathcal{G}$ satisfy

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \qquad \text{and} \qquad [\mathcal{H}, \mathcal{H}] \subset \mathcal{H},$$

where $\mathcal{M}$ is a vector space complement of $\mathcal{H}$. Furthermore, if $\mathcal{H}$ and $\mathcal{M}$ satisfy $[\mathcal{H}, \mathcal{M}] \subset \mathcal{M}$ then G/H is called the reductive homogeneous space. We can associate to these spaces a canonical connection with curvature and torsion. Curvature and torsion at a fixed point $p \in G/H$ are given purely in terms of the Lie bracket:

$$(R(X,Y)Z)_p = -[[X,Y]_{\mathcal{H}}, Z] \qquad \forall\, X, Y, Z \in \mathcal{M},$$
$$T(X,Y)_p = -[X,Y]_{\mathcal{M}} \qquad \forall\, X, Y \in \mathcal{M}.$$

Definition 4.4. *A Hermitian symmetric space is a coset space G/H for Lie groups whose associated Lie algebras are $\mathcal{G}$ and $\mathcal{H}$, with the decomposition*

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$$

which satisfy the commutation relations

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \qquad [\mathcal{H}, \mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{H}.$$

For Hermitian symmetric spaces the curvature satisfies

$$(R(X,Y), Z)_p = -[[X,Y], Z] \qquad \forall\, X, Y, Z \in \mathcal{M},$$

here $[X,Y] \in \mathcal{H}$ is satisfied automatically due to $[\mathcal{M}, \mathcal{M}]$.

Let k be an element in the Cartan subalgebra of $\mathcal{G}$, whose centralizer in $\mathcal{G}$ is

$$\mathcal{H} = \{l \in \mathcal{G} : [k, l] = 0\}.$$

Let $j = \mathrm{ad}\ k = [k, *]$ be a linear map

$$j : T^*(G/H) \longrightarrow T^*(G/H)$$

satisfying $j^2 = -1$ or $[k, [k, m]] = -m$ for $m \in M$.

Let us consider again the zero curvature equation

$$\bar{\partial} A^u - \partial A^{\bar{v}} + [A^u, A^{\bar{v}}] = 0.$$

At this stage we project the zero curvature equation into the real plane and treat $\bar{\partial} = \partial/\partial x$ and $\partial = \partial/\partial t$. We assume that A^u is the orbit L through $u = \lambda^3 A$, where $A = i\ \mathrm{diag}\ (1, -1)$ is an Cartan element of $su(2)$. Let us derive the orbit via the coadjoint action

$$L = B^{-1}(\lambda^3 A)B,$$

where

$$B = \prod_{i=1}^{4} (b_i \lambda^{-i}, e^{\beta_i} \lambda^{-i}).$$

Here b_i's are central elements and $\beta_i \in \mathcal{M}$. After an elaborated computation we obtain

$$L = \lambda^3 A + \lambda^2 Q + \lambda \left(P - \frac{i}{2}[Q_-, Q_+] \right) + T + [S, Q],$$

where

$$Q = [A, \beta_1],$$
$$P = [A, \beta_2] + \frac{1}{2}[Q, \beta_1],$$
$$T = [A, \beta_3] + \frac{1}{2}[[A, \beta_1], \beta_2] + \frac{1}{2}[[A, \beta_2], \beta_1] + \frac{1}{6}[[Q, \beta_1], \beta_1],$$
$$S = \frac{i}{2}[P_+ - P_-] + cQ.$$

If we assume $H = -\frac{1}{8}\mathrm{tr}(L^2 \lambda^{-2})$, then

$$A^{\bar{v}}(= \pi_+ \ \mathrm{grad}\ H) = -\pi_+ \frac{1}{4} L \lambda^{-2} = -\frac{1}{4}(A\lambda + Q).$$

Proposition 4.5. *Let $(\mathcal{O}_u, \omega_u)$ be the symplectic orbit, where ω_u is the Killing two form on the orbit. Then the Hamiltonian equations of motion, corresponding to $H(L) = -\frac{1}{8}\mathrm{tr}(L^2 \lambda^{-2})$, generates the system of third order partial differential equations in $\mathbf{R}^n$.*

In this case the zero curvature equation is

$$\frac{dL}{dt} = [A\lambda + Q, L] + \frac{1}{4}(A\lambda + Q)_x.$$

Setting various coefficients of λ^m equal to zero we obtain:

$$\dot{Q} = [A, P] - \frac{i}{2}[A, [Q_-, Q_+]].$$

We now apply the group decomposition properties of Hermitian symmetric spaces. Observe that $[Q_-, Q_+] \in h$ and that A is a constant matrix. Hence we obtain

$$P = -\frac{i}{2}(\dot{Q}_+ - \dot{Q}_-).$$

Similarly

$$T = -\frac{1}{4}\ddot{Q} + \frac{1}{4}[Q_+, [Q_-, Q_+]] - \frac{1}{4}[Q_-, [Q_-, Q_+]],$$

$$S = \frac{1}{4}(\dot{Q}_+ + \dot{Q}_-) + cQ.$$

Finally equating the λ^0 coefficient we obtain

$$\dot{T} + [S, Q]_t = [Q, T] + [Q, [S, Q]] + \frac{1}{4}Q_x.$$

If we choose

$$Q = \begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix},$$

then we get, from the zero curvature equation,

$$q_{ttt} + 6q_t|q|^2 + q_x = 0.$$

This is a coupled KdV equation.

When we consider the orbit L through $u = \lambda^2 A$ we obtain

$$L = \lambda^2 A + \lambda Q + \left(P - \frac{i}{2}[Q_-, Q_+]\right).$$

A similar calculation yields the nonlinear Schrödinger equation

$$q_{tt} + iq_x + 2q|q|^2 = 0.$$

4.3 Geometric Action and Virasoro Group

The geometric action of the Virasoro group has the form of Polyakov's 2d quantum gravity

$$S_{\text{grav}} = \int d^2z \frac{\bar{\partial}F}{\partial F}\left(\frac{\partial^3 F}{\partial F} - 2\left(\frac{\partial^2 F}{\partial F}\right)^2\right), \qquad (F \in \text{ Diff } (S^1)).$$

Let S^1 be the circle parametrized by $x : 0 \leq x \leq 2\pi$ and Diff (S^1) be the group of all orientation preserving C^∞ diffeomorphisms of S^1. It is natural to consider the Lie algebra of vector fields on S^1 Vect (S^1) as its Lie algebra. The dual of the Diff$(S^1)/S^1$ is identified with the space of quadratic differential forms $u(x)dx^2$ by the following pairing

$$\langle u(x), \xi\rangle = \int_0^{2\pi} u(x)\xi(x)dx,$$

where $\xi = \xi(x)\frac{d}{dx} \in \text{Vect}\,(S^1)$.

Let us consider the shift of $(u(x), c)$, induced by S^1 diffeomorphism

$$x \longrightarrow s(x) = x + \epsilon f(x),$$
$$(u(x), c) \longmapsto s'(x)^{3/2}(u(s(x)), c)s'(x)^{1/2} = (\tilde{u}(x), c),$$

where

$$\tilde{u}(x) = s'(x)^2 u(s(x)) + \frac{1}{2}\left(\frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2\right).$$

The last term is known as Schwarzian $\mathcal{S}(s)$. After redefining, or adjusting, the coefficients we can define the current

$$A = u(F)(\partial F)^2 - \frac{c}{24\pi}\mathcal{S}(F).$$

In this case we have the action in the form of $S = S^u(A) - H(A)$, where

$$S^u = -\int d^2z u(F)\partial F\bar{\partial}F + \frac{c}{48\pi}S_{\text{grav}},$$

which is linear with respect to A.

The equation of motion is

$$\bar{\partial}\,\text{Ad}^*_F(u, c) = \text{ad}^*_v\,\text{Ad}^*_F(u, c),$$

where $v = \text{grad}\,H \in \text{Vect}\,(S^1)$.

This action can be derived from the canonical action on the cotangent bundle of the group $\hat{\text{D}}\text{iff}\,S^1$. The above equation can be transformed to symmetric form by the Polyakov-Wiegmann formula for the group Diff S^1. Unfortunately this equation can not be recasted to the zero curvature equation. Nevertheless, some integrable models can be described within this approach.

5 Appendix

In this appendix we present a connection between the nonlinear Schrödinger equation and the continuous Heisenberg ferromagnetic equation. The continuous Heisenberg ferromagnetic model is an important integrable model associated to some Hermitian symmetric spaces [17, 18].

The action of the Heisenberg ferromagnetic model is given by

$$S = \int d^2z\,\text{tr}\left[2u\bar{\partial}gg^{-1} + \partial(g^{-1}kg)\partial(g^{-1}kg)\right],$$

where $k \in \mathcal{H}$ is a constant element.

Let us define $Q := g^{-1}kg$, which immediately leads to

Lemma 5.1.

$$\partial Q = [Q, g^{-1}\partial g], \qquad \bar{\partial}Q = [Q, g^{-1}\bar{\partial}g].$$

The equation of motion is

$$\bar{\partial}Q + \partial[Q, \partial Q] = 0.$$

Lemma 5.2. *When* $Q = g^{-1}kg \in \mathcal{M}$, *then* $\bar{\partial}Q + \partial[Q, \partial Q] = 0$ *is gauge equivalent to*

$$\partial(\partial g g^{-1}) - [k, \bar{\partial} g g^{-1}] = 0.$$

Sketch of the **Proof:**

$$[Q, \partial Q] = [Q, [Q, g^{-1}\partial g]] = g^{-1}[k, [k, \partial g g^{-1}]g = -g^{-1}(\partial g g^{-1})g = -g^{-1}\partial g.$$

It is easy to prove

$$\partial(g^{-1}\partial g) = g^{-1}\partial(\partial g g^{-1})g$$

. Then the result follows from these two. ■

Additionally we have an identity

$$[\bar{\partial} + \bar{\partial} g g^{-1}, \partial + \partial g g^{-1}] = 0.$$

There is a natural splitting

$$\bar{\partial} g g^{-1} = (\bar{\partial} g g^{-1})_{\mathcal{M}} + (\bar{\partial} g g^{-1})_{\mathcal{H}},$$

where the subscripts $\mathcal{M}$ and $\mathcal{H}$ refer to the component of $\bar{\partial} g g^{-1}$ in these vector subspaces.

Hence the above equation reduces to

$$[k, (\bar{\partial} g g^{-1})_{\mathcal{M}}] - \partial(\partial g g^{-1}) = 0.$$

Lemma 5.3.

$$[k, \partial(\partial g g^{-1})] = -(\bar{\partial} g g^{-1})_{\mathcal{M}}.$$

Sketch of the **Proof.** We know

$$[k, (\bar{\partial} g g^{-1})_{\mathcal{M}}] = \partial(\partial g g^{-1}),$$
$$[k, [k, (\bar{\partial} g g^{-1})_{\mathcal{M}}]] = [k, \partial(\partial g g^{-1})].$$

Lemma 5.4.

$$[k, \partial(\partial g g^{-1})] = -(\bar{\partial} g g^{-1})_{\mathcal{M}}.$$

Sketch of the **Proof.** We know

$$[k, (\bar{\partial} g g^{-1})_{\mathcal{M}}] = \partial(\partial g g^{-1}),$$
$$[k, [k, (\bar{\partial} g g^{-1})_{\mathcal{M}}]] = [k, \partial(\partial g g^{-1})],$$

where we have applied ad k on both sides. ■

Let us decompose the zero curvature equation into $\mathcal{H}$ and $\mathcal{M}$ part:

$$\partial(\bar{\partial} g g^{-1})_{\mathcal{H}} + [\partial g g^{-1}, (\bar{\partial} g g^{-1})_{\mathcal{M}}] = 0,$$

and

$$\partial(\bar{\partial}gg^{-1})_{\mathcal{M}} - \bar{\partial}g(\partial gg^{-1}) + [\partial gg^{-1}, (\bar{\partial}gg^{-1})_{\mathcal{H}}] = 0$$

respectively.

From the $\mathcal{H}$ part of the equation we obtain

$$\partial(\bar{\partial}gg^{-1})_{\mathcal{H}} - [\partial gg^{-1}, [k, \partial(\partial gg^{-1})] = 0,$$
$$\partial(\bar{\partial}gg^{-1})_{\mathcal{H}} + \frac{1}{2}\partial[\partial gg^{-1}, [\partial gg^{-1}, k]] = 0,$$

Hence we obtain $(\bar{\partial}gg^{-1})_{\mathcal{H}}$ upto some constant which we can always set to zero:

$$(\bar{\partial}gg^{-1})_{\mathcal{H}} = -\frac{1}{2}[\partial gg^{-1}, [\partial gg^{-1}, k]].$$

We finally derive the nonlinear Schrödinger equation

$$\bar{\partial}(\partial gg^{-1}) + [k, \partial^2(\partial gg^{-1})] + \frac{1}{2}[\partial gg^{-1}, [\partial gg^{-1}, [\partial gg^{-1}, k]]] = 0.$$

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