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On Certain Classes of Solutions of the Weierstrass-Enneper System Inducing Constant Mean Curvature Surfaces

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Abstract

Analysis of the generalized Weierstrass-Enneper system includes the estimation of the degree of indeterminancy of the general analytic solution and the discussion of the boundary value problem. Several different procedures for constructing certain classes of solutions to this system, including potential, harmonic and separable types of solutions, are proposed. A technique for reduction of the Weierstrass-Enneper system to decoupled linear equations, by subjecting it to certain differential constraints, is presented as well. New elementary and doubly periodic solutions are found, among them kinks, bumps and multi-soliton solutions.

1 Introduction

The Weierstrass-Enneper system [1] has proved to be a very useful and suitable tool in the study of minimal surfaces in $\mathbb{R}^3$. Since Weierstrass and Enneper, this subject has been investigated extensively by many authors (eg. Kenmotsu [2], Hoffman and Osserman [3], Konopelchenko [4–9]).

The original formulation by Weierstrass and Enneper [1] of a system inducing minimal surfaces can be presented briefly as follows. Let α and β be holomorphic functions that satisfy $\bar{\partial}\alpha = 0$, and $\bar{\partial}\beta = 0$ such that the equations

$$\partial w_1 = i(\alpha^2 + \beta^2), \qquad \partial w_2 = \alpha^2 - \beta^2, \qquad \partial w_3 = -2\alpha\beta,$$

hold, where the derivatives are abbreviated $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial \bar{z}$. Introduce a system of three real-valued functions $X_i(z, \bar{z})$, $i = 1, 2, 3$ which can be considered a coordinate system for a surface embedded in $\mathbb{R}^3$, defined as follows

$$\begin{aligned} X_1 &= Re\, w_1 = Re\left[i \int_C (\alpha^2 + \beta^2)\, dz'\right], \\ X_2 &= Re\, w_2 = Re\left[\int_C (\alpha^2 - \beta^2)\, dz'\right], \\ X_3 &= Re\, w_3 = -Re\left[2 \int_C \alpha\beta\, dz'\right], \end{aligned} \tag{1.1}$$

where C is any contour in the common domain of holomorphicity of the functions α and β. The X_i define a minimal surface with $z = c_1$ and $\bar{z} = c_2$ as minimal coordinate lines on this surface, respectively.

More recently, the generalized Weierstrass-Enneper (WE) representation for constant mean curvature surfaces in $\mathbb{R}^3$ has been introduced by B. Konopelchenko [7–9] and his formula is a starting point for our analysis. Namely, we consider the nonlinear system, a type of two-dimensional Dirac equation for the fields ψ_1 and ψ_2, given by

$$\partial\psi_1 = p\psi_2, \qquad \bar{\partial}\psi_2 = -p\psi_1, \qquad \bar{\partial}\bar{\psi}_1 = p\bar{\psi}_2, \qquad \partial\bar{\psi}_2 = -p\bar{\psi}_1, \tag{1.2}$$

where

$$p = |\psi_1|^2 + |\psi_2|^2,$$

in the neighbourhood of some point $(z_0, \bar{z}_0) \in \mathbb{C}$, and the bar on ψ_i denotes complex conjugation. In this paper, when we refer to the WE system, we mean the modified version (1.2) of the original Weierstrass Enneper system.

One then defines the three real-valued functions $X_i(z, \bar{z})$, $i = 1, 2, 3$, by means of the formulae

$$\begin{aligned} X_1 + iX_2 &= 2i \int_\gamma \left(\bar{\psi}_1^2\, dz' - \bar{\psi}_2^2\, d\bar{z}'\right), \\ X_1 - iX_2 &= 2i \int_\gamma \left(\psi_2^2\, dz' - \psi_1^2\, d\bar{z}'\right), \\ X_3 &= -2 \int_\gamma \left(\bar{\psi}_1\psi_2\, dz' + \psi_1\bar{\psi}_2\, d\bar{z}'\right), \end{aligned} \tag{1.3}$$

where γ is any contour in $\mathbb{C}$. On account of the system (1.2), the right hand side of (1.3) does not depend on the choice of γ. It was shown in [4, 5] that for each pair of solutions (ψ_1, ψ_2) of WE system (1.2), the formulae (1.3) determine a conformal immersion of a constant mean curvature surface in $\mathbb{R}^3$. The induced metric on the surface and its Gaussian curvature are given by [7, 14]

$$ds^2 = 4p^2\, dz\, d\bar{z}, \qquad K = -p^{-2}\partial\bar{\partial}(\ln p) \tag{1.4}$$

in isothermic coordinates, respectively. Finally, a well known property of WE system (1.2), in the context of the sigma model [10–12], is the existence of a topological charge

$$I = -\frac{i}{2\pi} \int_\gamma \frac{1}{p^2} \left(|j|^2 - p^4\right)\, dz d\bar{z}, \tag{1.5}$$

where j is an entire function[14] defined by

$$j(z) = \bar{\psi}_1 \partial \psi_2 - \psi_2 \partial \bar{\psi}_1. \tag{1.6}$$

In fact, j is a conserved quantity since

$$\bar{\partial} j = \bar{\partial} \left(\bar{\psi}_1 \partial \psi_2 - \psi_2 \partial \bar{\psi}_1 \right) = 0 \tag{1.7}$$

holds, whenever the WE system (1.2) is satisfied. The scalar function $j(z)$ is referred to as the current for WE system (1.2) [7]. Note that if the integral in (1.5) exists, then I is an integer.

In this paper we explore several different aspects of the generalized WE system (1.2). We investigate certain general characteristics of this system and propose several new approaches to the construction of its solutions. The paper is organized as follows. Section 2 contains a detailed account of the estimation of the degree of freedom of the general analytic solution to the WE system, based on Cartan's theory of systems in involution. In Section 3 we discuss the existence and uniqueness of solutions to a boundary value problem for the WE system. Section 4 and Section 5 contain several examples of potential and harmonic solutions of the WE system which include an explicit form of an algebraic multi-soliton solution. Next, in Section 6 we introduce a set of differential constraints under which the WE system can be reduced to a system of linear coupled equations and we construct several examples of solutions using this approach. Section 7 presents a certain variant of the separation of variables technique applied to the WE system which allows us to construct solitonlike solutions (bumps and kinks).

2 The estimation of the degree of indeterninancy of the general analytic solution to the Weierstrass-Enneper system

Now, using Cartan's theorem on systems in involution [13], we estimate the degree of indeterminancy of the general analytic solution of WE system (1.2). For this purpose, we perform the analysis using the apparatus of differential forms which are equivalent to the initial system. The problem is reduced to examining the Cartan numbers of these exterior forms and the number of arbitrary parameters admitted by the general solution of the system of polar equations.

For computational purposes, it is useful to introduce the following notation

$$\begin{aligned} &x = (x^1, x^2) := (\bar{z}, z), \qquad \xi = (\xi^1, \xi^2, \xi^3, \xi^4) := (\bar{\partial}\psi_1, \partial\psi_2, \partial\bar{\psi}_1, \bar{\partial}\bar{\psi}_2), \\ &u = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) := (\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2, \partial\psi_1, \bar{\partial}\psi_2, \bar{\partial}\bar{\psi}_1, \partial\bar{\psi}_2). \end{aligned} \tag{2.1}$$

It means that we interpret z and $\bar{z}$ as independent coordinates x^1 and x^2, respectively, in $\mathbb{R}^2$ space, and the coordinates $(u_1, \ldots, u_8)$ as independent variables in $\mathbb{R}^8$ space. The quantity ξ represents a vector of all first derivatives of ψ_i which do not appear in the WE system. Under notation (2.1) the system (1.2) becomes

$$u_5 = p u_2, \qquad u_7 = p u_4, \qquad u_6 = -p u_1, \qquad u_8 = -p u_3, \qquad p = u_1 u_3 + u_2 u_4, \tag{2.2}$$

and the differentiation of p with respect to z and $\bar{z}$ yields

$$\partial p = u_1 \xi^3 + u_4 \xi^2, \qquad \bar{\partial} p = u_3 \xi^1 + u_2 \xi^4, \tag{2.3}$$

whenever (1.2) holds. Note that the ξ^s enter linearly into the expressions ∂p and $\bar{\partial} p$. If we consider the variables $u = (u_1, \ldots, u_8)$ and $\xi = (\xi^1, \ldots, \xi^4)$ as unknown functions of $x = (x^1, x^2)$ then, in terms of (2.1) and (2.2), the WE system (1.2) is equivalent to the system of differential one-forms

$$\begin{aligned}
\omega_1 &= du_1 - (\xi^1\, dx^1 + p u_2\, dx^2) = 0,\\
\omega_2 &= du_2 - (-p u_1\, dx^1 + \xi^2\, dx^2) = 0,\\
\omega_3 &= du_3 - (p u_4\, dx^1 + \xi^3\, dx^2) = 0,\\
\omega_4 &= du_4 - (\xi^4\, dx^1 - p u_3\, dx^2) = 0,\\
\omega_5 &= du_5 - [u_2 \bar{\partial} p - p^2 u_1]\, dx^1 - [u_2 \partial p + p\xi^2]\, dx^2 = 0,\\
\omega_6 &= du_6 - [u_1 \bar{\partial} p + p\xi^1]\, dx^1 - [u_1 \partial p + p^2 u_2]\, dx^2 = 0,\\
\omega_7 &= du_7 - [u_4 \bar{\partial} p + p\xi^4]\, dx^1 - [u_4 \partial p - p^2 u_3]\, dx^2 = 0,\\
\omega_8 &= du_8 - [u_3 \bar{\partial} p + p^2 u_4]\, dx^1 - [u_3 \partial p + p\xi^3]\, dx^2 = 0,
\end{aligned} \tag{2.4}$$

in two independent variables x^1, x^2 which form some local coordinate system in the real space $\mathbb{R}^2$. System (2.4) can be written in the abbreviated form

$$\omega_s = du_s - G_{s\mu}(x, \xi, u)\, dx^\mu, \qquad s = 1, \ldots, 8, \quad \mu = 1, 2, \tag{2.5}$$

where the functions $G_{s\mu}$ depend only on (x, ξ, u) and where ξ enters linearly into $G_{s\mu}$, due to (2.3). We are interested in the evaluation of the degree of freedom of the most general analytic solution of (2.4) which can be expressed by

$$u_s = u_s(x^1, x^2), \qquad \xi^r = \xi^r(x^1, x^2), \qquad s = 1, \ldots, 8, \quad r = 1, \ldots, 4.$$

According to Cartan's Theorem on systems in involution [13], we can formulate the following.

Proposition 1. *If the system of differential one-forms (2.4) is in involution at a regular point (x_0, ξ_0, u_0) and if it is an analytic system in some neighbourhood of (x_0, ξ_0, u_0), then the general analytic solution of (2.4) with independent variables x^1, x^2 exists in some neighbourhood of a regular point (x_0, ξ_0, u_0) and it depends on four arbitrary real analytic functions of one real variable.*

Proof. Under notation (2.1) and relations (2.3), the exterior differentiation of system (2.4) leads to the following 2-form system whenever system (2.4) holds

$$\begin{aligned}
\Omega_1 &\equiv d\omega_1 = dx^1 \wedge d\xi^1 - \left[u_2 \bar{\partial} p - p^2 u_1\right]\, dx^1 \wedge dx^2,\\
\Omega_2 &\equiv d\omega_2 = -\left[u_1 \partial p + p^2 u_2\right]\, dx^1 \wedge dx^2 + dx^2 \wedge d\xi^2,\\
\Omega_3 &\equiv d\omega_3 = \left[u_4 \partial p - p^2 u_3\right]\, dx^1 \wedge dx^2 + dx^2 \wedge d\xi^3,\\
\Omega_4 &\equiv d\omega_4 = dx^1 \wedge d\xi^4 + \left[u_3 \bar{\partial} p + p^2 u_4\right]\, dx^1 \wedge dx^2,\\
\Omega_5 &\equiv d\omega_5 = -u_2 u_3\, d\xi^1 \wedge dx^1 + u_2^2 dx^1 \wedge d\xi^4 - u_1 u_2 d\xi^3 \wedge dx^2 - (p + u_2 u_4)\, d\xi^2 \wedge dx^2\\
&\quad + [u_1 u_2 (u_4 \partial p - p^2 u_3) - (p + u_2 u_4)(u_1 \partial p + p^2 u_2)\\
&\quad - u_2 u_3 (u_2 \bar{\partial} p - p^2 u_1) + u_2^2((u_3 \bar{\partial} p + p^2 u_4)]\, dx^1 \wedge dx^2,
\end{aligned}$$

$$\begin{aligned}
\Omega_6 \equiv d\omega_6 &= -(p+u_1u_3)\,dx^1\wedge d\xi^1 - u_1u_2\,dx^1\wedge d\xi^4 - u_1u_4\,dx^2\wedge d\xi^2\\
&\quad -u_1^2\,dx^2\wedge d\xi^3 + [-u_1^2(u_4\partial p - p^2u_3) + u_1u_4(u_1\partial p + p^2u_2)\\
&\quad -u_1u_2(u_3\bar\partial p + p^2u_4) + (p+u_1u_3)(u_2\bar\partial p - p^2u_1)]\,dx^1\wedge dx^2,\\
\Omega_7 \equiv d\omega_7 &= u_3u_4\,dx^1\wedge d\xi^1 + (p+u_2u_4)\,dx^1\wedge d\xi^4 + u_4^2\,dx^2\wedge d\xi^2\\
&\quad +u_1u_4\,dx^2\wedge d\xi^3 + [u_1u_4(u_4\partial p - p^2u_3) - u_4^2(u_1\partial p + p^2u_2)\\
&\quad -u_3u_4(u_2\bar\partial p - p^2u_1) + (p+u_2u_4)(u_3\bar\partial p + p^2u_4)]\,dx^1\wedge dx^2,\\
\Omega_8 \equiv d\omega_8 &= -u_3^2\,dx^1\wedge d\xi^1 - u_2u_3\,dx^1\wedge d\xi^4 - u_3u_4\,dx^2\wedge d\xi^2\\
&\quad -(p+u_1u_3)\,dx^2\wedge d\xi^3 + [u_3u_4(u_1\partial p + p^2u_2) - (p+u_1u_3)(u_4\partial p - p^2u_3)\\
&\quad +u_3^2(u_2\bar\partial p - p^2u_1) - u_2u_3(u_3\bar\partial p + p^2u_4)]\,dx^1\wedge dx^2.
\end{aligned} \tag{2.6}$$

In this case, using (2.5), all 2-forms (2.6) can be clearly expressed in the form

$$\Omega_s \equiv \sum_{r=1}^{4}\frac{\partial G_{s\mu}}{\partial \xi^r}\,d\xi^r\wedge dx^\mu + \left(\sum_{l=1}^{8}\left(G_{l\nu}\frac{\partial G_{s\mu}}{\partial u^l}\right) + \frac{\partial G_{s\mu}}{\partial x^\nu}\right)dx^\nu\wedge dx^\mu, \quad s=1,\dots,8. \tag{2.7}$$

Let Y_μ be linearly independent vector fields defined on $\mathbb{R}^{14}$

$$Y_\mu = \left(a_\mu^1\partial_{x^1}, a_\mu^2\partial_{x^2}, b_\mu^1\partial_{\xi^1},\dots,b_\mu^4\partial_{\xi^4}, c_\mu^1\partial_{u^1},\dots,c_\mu^8\partial_{u^8}\right), \qquad \mu=1,2 \tag{2.8}$$

such that they annihilate systems (2.4) and (2.6), composed of the 1-forms ω_s and the 2-forms Ω_s, respectively, that is

$$\langle \omega_s \lrcorner Y_\mu\rangle = 0, \qquad \langle \Omega_s \lrcorner Y_1, Y_2\rangle = 0, \qquad s=1,\dots,8, \quad \mu=1,2 \tag{2.9}$$

at some regular point $(x,\xi,u)\in\mathbb{R}^{14}$. The above system is called a system of polar equations [13]. The set of vector fields Y_μ satisfying this system depends on a certain number N of free parameters. In our case, the solution of (2.9) is given by

$$\begin{aligned}
Y_1 &= \partial_{x^1} + \sum_{r=1}^{4} b_1^r\partial_{\xi^r} + \xi^1\partial_{u_1} - pu_1\partial_{u_2} + pu_4\partial_{u_3} + \xi^4\partial_{u_4} - [u_2\bar\partial p - p^2u_1]\partial_{u_5}\\
&\quad +[u_1\bar\partial p + p\xi^1]\partial_{u_6} - [u_4\bar\partial p + p\xi^4]\partial_{u_7} + [u_3\bar\partial p + p^2u_4]\partial_{u_8},
\end{aligned}$$

and

$$\begin{aligned}
Y_2 &= \partial_{x^2} + \sum_{r=1}^{4} b_2^r\partial_{\xi^r} + pu_2\partial_{u_1} + \xi^2\partial_{u_2} + \xi^3\partial_{u_3} - pu_3\partial_{u_4} - [u_2\partial p + p\xi^2]\partial_{u_5}\\
&\quad +[u_1\partial p + p^2u_2]\partial_{u_6} - [u_4\partial p - p^2u_3]\partial_{u_7} + [u_3\partial p + p\xi^3]\partial_{u_8},
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
b_1^2 &= -(u_1\partial p + p^2u_2), & b_1^3 &= u_4\partial p - p^2u_3,\\
b_2^1 &= u_2\bar\partial p - p^2u_1, & b_2^4 &= -(u_3\bar\partial p + p^2u_4).
\end{aligned}$$

To simplify formulae (2.10) we have used notation (2.3). Solution (2.10) contains four arbitrary parameters b_1^1, b_1^4, b_2^2, b_2^3, hence we have

$$N = 4. \tag{2.11}$$

According to the definition of the first Cartan character [13], we have

$$s_1 = \max\ \mathrm{rank}_{X=(X^1,X^2)\in\mathbb{R}^2} \begin{vmatrix} \frac{\partial G_{1\mu}}{\partial \xi^1} X^\mu, & \cdots & \frac{\partial G_{1\mu}}{\partial \xi^4} X^\mu \\ \vdots & & \vdots \\ \frac{\partial G_{8\mu}}{\partial \xi^1} X^\mu, & \cdots & \frac{\partial G_{8\mu}}{\partial \xi^4} X^\mu \end{vmatrix}$$

at a regular point $(x, \xi, u) \in \mathbb{R}^{14}$. The nonvanishing elements of the 8×4 matrix $(a_{sr}) = \left(\frac{\partial G_{s\mu}}{\partial \xi^r} X^\mu\right)$ are

$$\begin{aligned} &a_{11} = X^1, \quad a_{22} = X^2, \quad a_{33} = X^2, \quad a_{44} = X^1, \quad a_{51} = u_2u_3X^1, \\ &a_{52} = (p + u_2u_4)X^2, \quad a_{53} = u_1u_2X^2, \quad a_{54} = u_2^2X^1, \quad a_{61} = -(p + u_1u_3)X^1, \\ &a_{62} = -u_1u_4X^2, \quad a_{63} = -u_2^2X^2, \quad a_{64} = -u_1u_2X^1, \quad a_{71} = u_3u_4X^1, \\ &a_{72} = u_4^2X^2, \quad a_{73} = u_1u_4X^2, \quad a_{74} = (p + u_2u_4)X^1, \quad a_{81} = -u_3^2X^1, \\ &a_{82} = -u_2u_3X^1, \quad a_{83} = -(p + u_1u_3)X^2, \quad a_{84} = -u_3u_4X^2, \end{aligned}$$

since the function $G_{s\mu}$ depends linearly on ξ. Hence, the maximal rank of the matrix (a_{sr}) is

$$s_1 = 4.$$

In that case, the second Cartan character is given by

$$s_2 = n - s_1 = 0,$$

where $n = 4$ is the number of coordinates ξ. Taking into account the definition [13] of the Cartan number Q, we have

$$Q = s_1 + 2s_2 = 4. \tag{2.12}$$

Thus, from (2.11) and (2.12), we get

$$Q = N = 4$$

and, according to Cartan's Theorem, system (2.4) is in involution at the regular point (x_0, ξ_0, u_0). So, its general analytic solution exists in some neighbourhood of this regular point and depends on four arbitrary real analytic functions of one real variable. ■

Let us note that, since system (2.4) is equivalent to WE system (1.2), then Proposition 1 implies the existence of the general analytic solution of (1.2). This solution depends on two arbitrary complex analytic functions of one complex variable and their complex conjugate functions (since we interpret z and $\bar{z}$ as coordinates on $\mathbb{C}$ and ψ_i and $\bar{\psi}_i$ as complex conjugate functions on $\mathbb{C}$).

3 On the boundary value problem for the Weierstrass-Enneper system

We start by considering the connection between the structure of certain conserved quantities associated with WE system (1.2) and the possibility of the construction of some classes of potential solutions.

From the conservation law associated with system (1.2)

$$\partial(\psi_1)^2 + \bar{\partial}(\psi_2)^2 = 0, \qquad \bar{\partial}(\bar{\psi}_1)^2 + \partial(\bar{\psi}_2)^2 = 0, \tag{3.1}$$

it follows that there exists a potential function $g(z,\bar{z}) : \mathbb{C} \to \mathbb{C}$, such that the functions ψ_i can be expressed in terms of the first derivatives of the function g

$$\begin{aligned} &\psi_1 = e^{in\pi}(\bar{\partial} g)^{1/2}, \qquad \psi_2 = ie^{ik\pi}(\partial g)^{1/2}, \\ &\bar{\psi}_1 = e^{-in\pi}(\partial \bar{g})^{1/2}, \qquad \bar{\psi}_2 = -ie^{-ik\pi}(\bar{\partial}\bar{g})^{1/2}, \qquad n,k \in \mathbb{Z}. \end{aligned} \tag{3.2}$$

Substituting (3.2) into WE system (1.2) one obtains

$$\begin{aligned} &\partial\bar{\partial} g = 2ie^{i(k-n)\pi}\left[(\bar{\partial} g)(\partial\bar{g})^{1/2}(\partial g)^{1/2} + (\bar{\partial} g)^{1/2}(\partial g)(\bar{\partial}\bar{g})^{1/2}\right], \\ &\bar{\partial}\partial\bar{g} = -2ie^{-i(k-n)\pi}\left[(\partial\bar{g})(\bar{\partial} g)^{1/2}(\bar{\partial}\bar{g})^{1/2} + (\partial\bar{g})^{1/2}(\bar{\partial}\bar{g})(\partial g)^{1/2}\right]. \end{aligned} \tag{3.3}$$

This result can be summarized as follows.

Proposition 2. *If a complex-valued function g of the class C^2 is a solution of system (3.3), then the complex valued functions ψ_i defined by (3.2) are solutions of WE system (1.2).*

In the next section we show some examples of these types of solutions.

Let us now establish the existence and uniqueness of the potential solution to the boundary value problem (BVP) for WE system (1.2). The BVP for this system consists in finding a class of solutions ψ_i in some open bounded simply connected region Ω in $\mathbb{C}$ for prescribed values of the functions ψ_i along the boundary $\partial\Omega$

$$\begin{aligned} &\partial\psi_1 = (|\psi_1|^2 + |\psi_2|^2)\psi_2, \qquad \bar{\partial}\psi_2 = -(|\psi_1|^2 + |\psi_2|^2)\psi_1, \qquad \text{in} \quad \Omega, \\ &\psi_1(z,\bar{z}) = e^{in\pi}(\bar{\partial} g)^{1/2}|_{\partial\Omega}, \qquad \psi_2 = ie^{ik\pi}(\partial g)^{1/2}|_{\partial\Omega}, \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.4}$$

We show now how a certain class of differentiable solutions of this problem can be obtained with the help of the conservation law (1.7). Substituting (3.2) into (1.6), we get

$$\partial^2 g - \frac{\partial g}{\partial \bar{g}}\partial^2\bar{g} + 2ie^{i(n-k)\pi} j(z)\left(\frac{\partial g}{\partial\bar{g}}\right)^{1/2} = 0. \tag{3.5}$$

The condition for the existence of the entire function $j(z)$ requires that

$$\bar{\partial}\left[\frac{(\partial\bar{g})^{1/2}}{(\partial g)^{1/2}}\partial^2 g - \frac{(\partial g)^{1/2}}{(\partial\bar{g})^{1/2}}\partial^2\bar{g}\right] = 0. \tag{3.6}$$

It should be noted that condition (3.6) is identically satisfied whenever equations (3.3) hold. This fact simplifies considerably the process of solving the BVP for the WE system, since the potential solutions of WE system (1.2) have to satisfy only conditions (3.3). Under the above considerations, we can formulate the following.

Proposition 3. *The solution of the boundary value problem (3.4) for WE system on a simply connected region Ω exists and is unique, provided that there exists a C^2 complex valued function g such that the first order derivatives of the function g satisfy equations (3.3) on $\partial\Omega$.*

Proof. Indeed, if the values of the derivatives ∂g and $\bar{\partial} g$ are given on the boundary $\partial\Omega$, such that equations (3.3) hold, then the functions ψ_i, defined by (3.2), satisfy WE system (1.2). This fact follows from Proposition 2. The conservation law (3.5) implies that the current j is an entire function determined by

$$j(z) = \frac{i}{2} e^{i(k-n)\pi} \frac{(\partial\bar{g})^{1/2}}{(\partial g)^{1/2}} \left[\partial^2 g - \frac{\partial g}{\partial \bar{g}} \partial^2 \bar{g}\right] \qquad \text{on} \quad \partial\Omega.$$

Liouville's Theorem ensures that the values of the entire function $j(z)$ are uniquely defined on the whole simply connected region $\Omega \subset \mathbb{C}$. This means that there exists a one-to-one correspondence between functions ψ_i prescribed on $\partial\Omega$ and the entire function $j(z)$. Hence, the values of the solutions ψ_i of WE system (1.2) at the point $z \in \Omega$ depend only on the values of the functions ψ_i on the boundary $\partial\Omega$. This being so, the functions ψ_i defined by equations (3.2), with the property that the first derivatives of the function g satisfy (3.3) on $\partial\Omega$, are the unique solutions of the BVP (3.4) for WE system in the region Ω. ■

4 Potential solutions of the Weierstrass-Enneper system

Proposition 2 provides us with a tool for constructing particular classes of potential solutions to WE system (1.2). We now present a couple of examples of such solutions.

1. A class of rational solutions of system (3.3) is given by

$$g(z,\bar{z}) = -\frac{z^m}{1+|z|^{2m}}, \qquad m \in \mathbb{Z}. \tag{4.1}$$

Substituting the function g into relations (3.2), one obtains an explicit solution of WE system (1.2)

$$\psi_1 = e^{in\pi} m^{1/2} \frac{|z|^m}{1+|z|^{2m}} z^{(m-1)/2}, \qquad \psi_2 = e^{ik\pi} m^{1/2} \frac{z^{(m-1)/2}}{1+|z|^{2m}}. \tag{4.2}$$

It is interesting to note that this very simple and direct method yields the same result that was obtained by a much more complex approach via the $SU(2)$ sigma model [14]. For every fixed m, solution (4.2) belongs to a given topological sector. The solutions are double valued for all even m. Each solution (4.2) corresponds to a particular constant mean curvature surface which is covered n times as z runs over the complex plane $\mathbb{C}$. This surface is obtained by the parametrization (1.3)

$$X_1 + iX_2 = 2iz^{-m}\left(\frac{1-|z|^{2m}}{1+|z|^{2m}}\right), \tag{4.3.1}$$

$$X_1 - iX_2 = -2i\bar{z}^{-m}\frac{1-|z|^{2m}}{1+|z|^{2m}}, \tag{4.3.2}$$

$$X_3 = \frac{4}{1+|z|^{2m}}. \tag{4.3.3}$$

Solving expression (4.3.3) for $|z|^m$ in terms of X_3 and substituting the result into the expressions (4.3.1) and (4.3.2), one arrives at the equation of a surface which is obtained by revolving the curve

$$X_2 = 2\,(2X_3 - 1)\left(\frac{X_3}{1-X_3}\right)^{1/2}$$

around the X_3 axis. This surface has a conic point at $(0,0,2)$, and the corresponding Gaussian curvature is $K = 1$.

2. Let us discuss now the construction of an algebraic multi-soliton solution to WE system (1.2). First, we look for a particular class of rational solutions g of (3.3) admitting two simple poles only. This leads us to the following solution

$$\begin{aligned} g(z,\bar{z}) = {} & \frac{(a-b)^2(z-a)^2}{(-(2z-a-b)\bar{z} + (z-a)a + (z-b)b)(2z-a-b)} \\ & + \frac{(a-b)^2}{2(2z-a-b)}, \qquad a,b \in \mathbb{R}. \end{aligned} \tag{4.4}$$

The corresponding solution of WE system (1.2) takes the form

$$\psi_1 = e^{in\pi}(a-b)\frac{z-a}{|z-a|^2+|z-b|^2}, \qquad \psi_2 = e^{ik\pi}(a-b)\frac{\bar{z}-b}{|z-a|^2+|z-b|^2}. \tag{4.5}$$

This type of solution is known in the literature [10], and represents a one-soliton solution. The associated surface can be computed from equations (1.3), namely

$$\begin{aligned} X_1 + iX_2 &= 2i(a-b)^2\left(-\frac{(\bar{z}-a)^2}{\bar{D}} + \frac{(z-b)^2}{D}\right), \\ X_1 - iX_2 &= 2i(a-b)^2\left(-\frac{(\bar{z}-b)^2}{\bar{D}} + \frac{(z-a)^2}{D}\right), \\ X_3 &= -2(a-b)^2\left(-\frac{(\bar{z}-a)(\bar{z}-b)}{\bar{D}} - \frac{(z-a)(z-b)}{D}\right), \end{aligned} \tag{4.6}$$

where the denominator D is given by

$$D = ((2z-a-b)\bar{z} - a(z-a) - b(z-b))(2z-a-b),$$

and its respective complex conjugate is

$$\bar{D} = ((2\bar{z}-a-b)z - a(\bar{z}-a) - b(\bar{z}-b))(2\bar{z}-a-b).$$

Finally, we solve (4.6) for X_1 and X_2 and write the X_j, $(j = 1,2,3)$ in terms of $z = x+iy$ and $\bar{z} = x - iy$. This gives us the parametric forms for the X_j in terms of x and y. Eliminating the variables x and y from these equations, one can write an expression for the surface just in terms of the coordinates X_j as follows,

$$X_2^3 + X_1^2X_2 + X_2X_3^2 + 4(b-a)\left(X_1^2+X_2^2+X_3^2\right) + 4(a-b)^2X_2 = 0. \tag{4.7}$$

The curvature for this surface is given by $K = (a-b)^{-2}$. Thus, formula (4.7) represents an Enneper type surface.

We now consider a more general case when the solution g of (3.3) admits arbitrary number of simple poles. Under this assumption we have

$$\partial g = -\frac{1}{\left(1+\prod_{j=1}^{N}|\frac{z-a_j}{z-b_j}|^2\right)^2}\left(\sum_{s=1}^{N}\frac{1}{(z-b_s)}\left(\prod_{\substack{j=1\\ j\neq s}}^{N}\frac{(z-a_j)}{(z-b_j)}-\prod_{j=1}^{N}\frac{(z-a_j)}{(z-b_j)}\right)\right),$$

$$\partial\bar{g} = \frac{\prod_{j=1}^{N}\frac{\bar{z}-a_j}{\bar{z}-b_j}}{\left(1+\prod_{j=1}^{N}|\frac{z-a_j}{z-b_j}|^2\right)^2}\left(\sum_{s=1}^{N}\frac{1}{(z-b_s)}\left(\prod_{\substack{j=1\\ j\neq s}}^{N}\frac{(z-a_j)}{(z-b_j)}-\prod_{j=1}^{N}\frac{(z-a_j)}{(z-b_j)}\right)\right),\quad a_j,b_j\in\mathbb{R}. \tag{4.8}$$

Substituting (4.8) into (3.2) we determine explicitly the corresponding form of an algebraic multi-soliton solution of WE system (1.2)

$$\psi_1 = e^{in\pi}\frac{\prod_{j=1}^{N}\frac{z-a_j}{z-b_j}}{1+\prod_{j=1}^{N}|\frac{z-a_j}{z-b_j}|^2}\left(\sum_{s=1}^{N}\frac{1}{(\bar{z}-b_s)}\left(\prod_{\substack{j=1\\ j\neq s}}^{N}\frac{(\bar{z}-a_j)}{(\bar{z}-b_j)}-\prod_{j=1}^{N}\frac{(\bar{z}-a_j)}{(\bar{z}-b_j)}\right)\right)^{1/2},$$

$$\psi_2 = \frac{e^{ik\pi}}{1+\prod_{j=1}^{N}|\frac{z-a_j}{z-b_j}|^2}\left(\sum_{s=1}^{N}\frac{1}{(z-b_s)}\left(\prod_{\substack{j=1\\ j\neq s}}^{N}\frac{(z-a_j)}{(z-b_j)}-\prod_{j=1}^{N}\frac{(z-a_j)}{(z-b_j)}\right)\right)^{1/2}. \tag{4.9}$$

Note that the topological charge (1.5) for each of the instanton solutions (4.5) entering into the superposition corresponds to an integer $I = e^{in\pi}N$. It is interesting to note that the constant mean curvature surface corresponding to (4.9) is also determined by (4.7).

5 Harmonic solutions of the Weierstrass-Enneper system

We discuss now the existence of a class of harmonic solutions to the WE system (1.2) which can be obtained by applying certain composition transformations.

Proposition 4. *Suppose that the complex valued functions f_i and $\bar{f}_i$ are solutions of the following system of differential equations,*

$$\begin{aligned} &pf_1''(v)f_1(v)f_2(v)\\ &\qquad+\left[-|f_1|^2(f_1'(v))^2+|f_2|^2\bar{f}_2'(\bar{v})\frac{f_1'(v)f_2'(v)}{\bar{f}_1'(\bar{v})}\right]f_2(v)-pf_1(v)f_1'(v)f_2'(v)=0,\\ &pf_2''(v)f_1(v)f_2(v)\\ &\qquad+\left[|f_1|^2\frac{f_1'(v)f_2'(v)}{\bar{f}_2'(\bar{v})}\bar{f}_1'(\bar{v})-|f_2|^2(f_2'(v))^2\right]f_1(v)-pf_2(v)f_1'(v)f_2'(v)=0, \end{aligned} \tag{5.1}$$

with respect to the relevant variables v and $\bar{v}$. Then the compositions of the functions f_i with any harmonic function $v = h(z, \bar{z})$, defined in a simply connected region Ω,

$$\psi_i = f_i(h(z, \bar{z})), \qquad i = 1, 2$$

and their respective complex conjugates

$$\bar{\psi}_i = \bar{f}_i(\bar{h}(\bar{z}, z)), \qquad i = 1, 2$$

constitute solutions of the WE system (1.2).

Proof. Substituting the composed functions ψ_i and $\bar{\psi}_i$ into (1.2), one obtains

$$\begin{aligned} & f_1'(v)\,\partial h(z, \bar{z}) = p f_2(h(z, \bar{z})), \qquad f_2'(v)\,\bar{\partial} h(z, \bar{z}) = -p f_1(h(z, \bar{z})), \\ & p = |f_1|^2 + |f_2|^2. \end{aligned} \tag{5.2}$$

Differentiating the first equation in (5.2) with respect to $\bar{\partial}$ and making use of the equation $\bar{\partial}\partial h = 0$, one obtains

$$f_1''(v)\bar{\partial} h \partial h = (\bar{\partial} p) f_2 + p f_2'(v)\,\bar{\partial} h. \tag{5.3}$$

Similarly, differentiating the second equation of (5.2) with respect to ∂, and taking into account the relation $\bar{\partial}\partial h = 0$, one gets

$$f_2''(v)\,\partial h\,\bar{\partial} h = -(\partial p) f_1 - p f_1'(v)\,\partial h. \tag{5.4}$$

Differentiating p and using (1.2), we have

$$\partial p = \psi_1 \partial \bar{\psi}_1 + \bar{\psi}_2 \partial \psi_2, \qquad \bar{\partial} p = \bar{\psi}_1 \bar{\partial} \psi_1 + \psi_2 \bar{\partial} \bar{\psi}_2. \tag{5.5}$$

Solving (5.2) for the derivatives ∂h, $\bar{\partial} h$ and substituting these derivatives and the expressions (5.5) into equations (5.3) and (5.4), one obtains the system of differential equations (5.1). Thus, equations (5.1) are equivalent to (1.2) whenever v is a harmonic function. ■

Let us consider a simple example to illustrate Proposition 4. A special class of exponential solutions of (5.1) has the form

$$f_1 = -ie^{iv}, \qquad f_2 = ae^{iv}, \qquad a \in \mathbb{C}.$$

If we choose a specific harmonic form of the function $v = q(z^2 + \bar{z}^2)$, $q \in \mathbb{R}$, then the compositions of the functions f_i and v give particular solutions of the WE system

$$\psi_1 = -ie^{iq(z^2 - \bar{z}^2)}, \qquad \psi_2 = ae^{iq(z^2 - \bar{z}^2)}, \qquad |a|^2 = 1. \tag{5.6}$$

The corresponding constant mean curvature surface is determined by relations (1.3)

$$\begin{aligned} & X_1 + iX_2 = -i\pi^{1/2} \left(\frac{\operatorname{erf}(\xi z)}{\exp(2iq\bar{z}^2)\xi} + \frac{\bar{a}^2 \exp(2iqz^2) \operatorname{erf}(\eta \bar{z})}{\eta} \right), \\ & X_1 - iX_2 = i\pi^{1/2} \left(\frac{a^2 \operatorname{erf}(\xi z)}{\exp(2iq\bar{z}^2)\xi} + \frac{\exp(2iqz^2) \operatorname{erf}(\eta \bar{z})}{\eta} \right), \\ & X_3 = -i\pi^{1/2} \left(\frac{\operatorname{erf}(\xi z)}{\exp(2iq\bar{z}^2)\xi} + \frac{\bar{a} \exp(2iqz^2) \operatorname{erf}(\eta \bar{z})}{\eta} \right), \end{aligned}$$

where erf is the error function, $\xi = (-2iq)^{1/2}$ and $\eta = (2iq)^{1/2}$. The elimination of the quantities $\mathrm{erf}\,(\xi z)$ and $\exp(2iqz^2)$ from the above expressions leads to the formula which represents a constant mean curvature surface describing a catenoide

$$4\left(1-a_r^2\right)X_1^2 + 4a_r^2X_2^2 + 8a_r\left(1-a_r^2\right)^{1/2}X_1X_2 + 4X_3^2 = 0, \qquad a_r = Re\,a. \tag{5.7}$$

The Gaussian curvature is $K = 1 - a_r^2$.

6 Reduction of the Weierstrass-Enneper system to a linear system

Now we discuss the case when the WE system is subjected to a single differential constraint. This allows us to reduce this system to a system of linear coupled PDEs.

Proposition 5. *The overdetermined system composed of the WE system (1.2) and the first order differential constraint*

$$\psi_1\partial\bar{\psi}_1 - \epsilon\bar{\psi}_1\bar{\partial}\psi_1 + \bar{\psi}_2\partial\psi_2 - \epsilon\psi_2\bar{\partial}\bar{\psi}_2 = 0, \qquad \epsilon = \pm 1 \tag{6.1}$$

is consistent if the conditions

$$|\psi_1|^2 + |\psi_2|^2 = p(z + \epsilon\bar{z}), \tag{6.2}$$

and

$$\ddot{p} - \frac{\dot{p}^2}{p} + \epsilon\frac{A}{p} - \epsilon p^3 = 0, \qquad A \in \mathbb{R}^+, \tag{6.3}$$

hold. The general analytic solution of the above overdetermined system depends on one arbitrary complex analytic function of one complex variable and on its complex conjugate function.

Proof. Indeed, from equations (5.5), taking into account the differential constraint (6.1), one obtains

$$(\partial - \epsilon\bar{\partial})p = \psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2 - \epsilon\bar{\psi}_1\bar{\partial}\psi_1 - \epsilon\psi_2\bar{\partial}\bar{\psi}_2 = 0. \tag{6.4}$$

This means that, under the assumption (6.1), the quantity p is a real valued function of the argument $s = (z + \epsilon\bar{z})$. Hence, p is a conserved quantity. Therefore, condition (6.2) holds. Differentiating equation (6.2) with respect to z and $\bar{z}$, one obtains

$$\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2 = \dot{p}, \qquad \bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2 = \epsilon\dot{p}, \tag{6.5}$$

where we introduced the notation $\dot{p} = dp/ds$. Solving equation (6.5) with respect to $\partial\psi_2$ and next substituting this term into (1.6), one obtains

$$\partial\bar{\psi}_1 = \frac{\bar{\psi}_1}{p}\left(\dot{p} - \frac{\bar{\psi}_2}{\bar{\psi}_1}j(z)\right). \tag{6.6}$$

The complex conjugate of equation (6.6) is given by

$$\bar{\partial}\psi_1 = \frac{\psi_1}{p}\left(\epsilon\dot{p} - \frac{\psi_2}{\psi_1}\bar{j}(\bar{z})\right).$$

A similar analysis can be performed for $\partial\bar{\psi}_1$, in order to determine derivatives of ψ_2 in terms of ψ_i, j and p. As a result, one obtains from WE system (1.2) the following system of equations

$$
\begin{aligned}
&\partial\psi_1 = p\psi_2, && \partial\psi_2 = \frac{\psi_2}{p}\left(\dot{p} + \frac{\psi_1}{\psi_2} j(z)\right),\\
&\bar{\partial}\psi_1 = \frac{\psi_1}{p}\left(\epsilon\dot{p} - \frac{\psi_2}{\psi_1}\bar{j}(\bar{z})\right), && \bar{\partial}\psi_2 = -p\psi_1,\\
&\partial\bar{\psi}_1 = \frac{\bar{\psi}_1}{p}\left(\dot{p} - \frac{\bar{\psi}_2}{\bar{\psi}_1} j(z)\right), && \partial\bar{\psi}_2 = -p\bar{\psi}_1,\\
&\bar{\partial}\bar{\psi}_1 = p\bar{\psi}_2, && \bar{\partial}\bar{\psi}_2 = \frac{\bar{\psi}_2}{p}\left(\epsilon\dot{p} + \frac{\bar{\psi}_1}{\bar{\psi}_2}\bar{j}(\bar{z})\right).
\end{aligned}
\tag{6.7}
$$

The compatibility conditions for (6.7) require $|j|^2 = A \in \mathbb{R}^+$ and the current j is constant. This gives a differential equation for p of the form (6.3). Expressing (6.7) in the language of differential one-forms and making use of Propositions 1, one can easily show that, if the compatibility conditions (6.2) and (6.3) are satisfied, then the general analytic solution of (6.7) with j constant exists and depends on one arbitrary analytic complex function of one complex variable. ■

Now, let us discuss some classes of solutions of the differential equation (6.3) which allow one to reduce the WE system to the coupled linear PDEs. Note that equation (6.3) is of Painlevé type, PXII, having only poles for moveable singularities. The first integral is given by

$$
\dot{p}(s)^2 = (\epsilon p^4 + Kp^2 + \epsilon A), \tag{6.8}
$$

where K is an arbitrary real constant. The forms of the real solutions for p depend on the relationships between the roots of the right-hand side of the ODE (6.8). They lead to the following cases.

(i) Elementary solutions, such as constant, algebraic with one or two simple poles, trigonometric and hyperbolic solutions.

(ii) Doubly periodic solutions which can be expressed in terms of the Jacobi elliptic functions sn and cn. The moduli k of the elliptic functions are chosen in such a way that $0 < k^2 < 1$. This fact ensures that the elliptic solutions possess one real and one purely imaginary period. Consequently, for real argument s, we have $-1 \leq \mathrm{sn}\,(s,k) \leq 1$, and $-1 \leq \mathrm{cn}\,(s,k) \leq 1$.

The different classes of solutions of (6.8) are summarized in Tables 1 and 2. They lead us to twelve different types of solutions of the linear coupled WE system for which the compatibility conditions are satisfied identically.

Finally, let us discuss the case when we add multiple differential constraints compatible with our basic WE system (1.2). We show that if the conditions (6.1) for $\epsilon = +1$ and $\epsilon = -1$ are simultaneously satisfied

$$
\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2 = 0, \qquad \bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2 = 0 \tag{6.9}
$$

then WE system (1.2) can be reduced to a linear decoupled system.

Table 1: Finite real elementary and elliptic solutions $p = p(s-s_0)$ with $\dot{p}^2 = \epsilon p^4 + Kp^2 + \epsilon A$, where $A \geq 0$, $K \in \mathbb{R}$, $w \equiv \sqrt{K^2-4A}$, $a \equiv (K \pm w)^{1/2}/\sqrt{2}$, $B \equiv 1+\sqrt{2}$, $\epsilon, \epsilon_1 = \pm 1$ and $s - s_0 = z + \epsilon\bar{z} - s_0$.

No	ϵ	$p(s-s_0)$	k	g	*Range*
1	$+1$	$\epsilon_1 A^{1/4}\left(\dfrac{1+\mathrm{cn}\left(2A^{1/4}(s-s_0),\frac{1}{2}\right)}{1-\mathrm{cn}\left(2A^{1/4}(s-s_0),\frac{1}{2}\right)}\right)^{1/2}$	$\dfrac{1}{\sqrt{2}}$	—	$K=0, \quad 0 \leq p < \infty$
2	$+1$	$\epsilon_1 A^{1/4}\dfrac{B\,\mathrm{cn}\left(\frac{A^{1/4}}{g}(s-s_0),k\right)-\mathrm{sn}\left(\frac{A^{1/4}}{g}(s-s_0),k\right)}{B\,\mathrm{cn}\left(\frac{A^{1/4}}{g}(s-s_0),k\right)+\mathrm{sn}\left(\frac{A^{1/4}}{g}(s-s_0),k\right)}$	$2\left(3\sqrt{2}-4\right)^{1/2}$	$2-\sqrt{2}$	$K=0, \quad 0 \leq p \leq 1$
3	-1	$\dfrac{\epsilon_1(2A)^{1/2}}{\left[(K+w)(1-k^2\mathrm{sn}^2\left(\frac{(s-s_0)}{g},k\right)\right]^{1/2}}$	$\left(\dfrac{2w}{K+w}\right)^{1/2}$	$\dfrac{\sqrt{2}}{(K+w)^{1/2}}$	$K>0, \quad K^2-4A>0,$ $\dfrac{1}{\sqrt{2}}(K-w)^{1/2} \leq p < \dfrac{1}{\sqrt{2}}(K+w)^{1/2}$
4	-1	$\epsilon_1\left(\dfrac{1}{2}(K+w)-w\,\mathrm{sn}\left(\dfrac{s-s_0}{g},k\right)\right)^{1/2}$	$\left(\dfrac{2w}{K+w}\right)^{1/2}$	$\left(\dfrac{2}{K+w}\right)^{1/2}$	$K>0, \quad K^2-4A>0$ $\dfrac{1}{\sqrt{2}}(K-w)^{1/2} \leq p < \dfrac{1}{\sqrt{2}}(K+w)^{1/2}$
5	$+1$	$\epsilon_1\dfrac{(\lvert K\rvert \pm w)^{1/2}}{\sqrt{2}\,\mathrm{sn}\left(\frac{s-s_0}{g},k\right)}$	$\left(\dfrac{\lvert K\rvert \mp w}{\lvert K\rvert \pm w}\right)^{1/2}$	$\dfrac{\sqrt{2}}{(\lvert K\rvert \pm w)^{1/2}}$	$K<0, \quad K^2-4A>0,$ $\dfrac{1}{\sqrt{2}}(\lvert K\rvert \pm w)^{1/2} \leq p < \infty$
6	$+1$	$\dfrac{K}{\cosh\left(\sqrt{K}(s-s_0)\right)}$	—	—	$A=0,\ K>0, \quad 0<p<K$

Table 2: Singular real elementary and elliptic solutions $p = p(s - s_0)$ with $\dot{p}^2 = \epsilon p^4 + K p^2 + \epsilon A$, where $A \geq 0$, $K \in \mathbb{R}$, $\epsilon, \epsilon_1 = \pm 1$, $w \equiv \sqrt{K^2 - 4A}$ and $s - s_0 = z + \epsilon \bar{z} - s_0$.

No	ϵ	$p(s - s_0)$	k	g	$Range$
1	$+1$	$\epsilon_1 \dfrac{(2A)^{1/2}}{K \pm w}\, tn\left(\dfrac{s - s_0}{g}, k\right)$	$\dfrac{(2(K^2 - 4A \pm Kw))^{1/2}}{K \pm w}$	$\left(\dfrac{1}{2}(K \pm w)\right)^{-1/2}$	$K > 0, \quad K^2 - 4A > 0,$ $A^{1/2} < (K \pm w)/2,\ 0 < p$
2	$+1$	$\epsilon_1 \dfrac{\left[\lvert K\rvert \pm w + (\lvert K\rvert \mp w)\,\mathrm{sn}^2\left(\frac{s - s_0}{g}, k\right)\right]^{1/2}}{\sqrt{2}\,\mathrm{cn}\left(\frac{s - s_0}{g}, k\right)}$	$\left(\dfrac{\lvert K\rvert \mp w}{\lvert K\rvert \pm w}\right)^{1/2}$	$\dfrac{\sqrt{2}}{(\lvert K\rvert \pm w)^{1/2}}$	$K < 0, \quad K^2 - 4A > 0,$ $\dfrac{1}{\sqrt{2}}(\lvert K\rvert \pm w) < p$
3	$+1$	$a \cdot \mathrm{tn}\,(a(s - s_0), k)$	$\dfrac{(2(K^2 - 4A \pm Kw))^{1/2}}{K \pm w}$	$\dfrac{\epsilon_1 \sqrt{2}}{(K \pm w)^{1/2}}$	$K > 0, \quad A^{1/2} < \dfrac{1}{2}(K \pm w)$
4	$+1$	$\dfrac{K}{\sinh(\sqrt{K}(s - s_0))}$	—	—	$A = 0, \quad K > 0$
5	$+1$	$\sqrt{\lvert K\rvert}\sec\dfrac{\sqrt{\lvert K\rvert}}{2}(s - s_0)$	—	—	$A = 0, \quad K < 0, \quad \sqrt{\lvert K\rvert} \leq p$
6	$+1$	$\dfrac{1}{s_0 - s}$	—	—	$A = 0, \quad K = 0, \quad 0 < p$

Proposition 6. *If the functions ψ_1 and ψ_2 satisfying WE system (1.2) are subjected to differential constraints (6.9), then the WE system is reduced to a linear decoupled system of second order*

$$\bar{\partial}\partial\psi_i + p_0^2\psi_i = 0, \qquad \partial\bar{\partial}\bar{\psi}_i + p_0^2\bar{\psi}_i = 0, \qquad i = 1, 2, \tag{6.10}$$

where

$$|\psi_1|^2 + |\psi_2|^2 = p_0 \in \mathbb{R}^+. \tag{6.11}$$

Proof. In fact, taking into account differential constraints (6.9) and the derivatives of p given by (6.4), we obtain that p is a real positive constant. This means that the overdetermined system composed of WE system (1.2) and differential constraints (6.9) admits a conserved quantity (6.11). Hence, the WE system (1.2) can be decoupled into the second order system (6.10). ■

A simple example of the solution of (1.2) constructed with the use of differential constraints (6.9) was presented in [14]. This solution has the form of the plane wave or so called vacuum solution

$$\psi_1 = Ae^{i(hz+k\bar{z})}, \qquad \psi_2 = i\frac{A}{k}e^{i(hz+k\bar{z})}, \qquad hk = p_0, \qquad h, k \in \mathbb{R}, \quad A \in \mathbb{C}. \tag{6.12}$$

Proposition 6 implies that, due to the linearity of equations (6.10), a more general class of solutions can be constructed. Namely, the linear superposition of plane waves (6.12) gives

$$\begin{aligned} &\psi_1 = A_1e^{\alpha_1(z+\bar{z})} + A_2e^{\alpha_2(z-\bar{z})}, \qquad \psi_2 = B_1e^{\alpha_1(z+\bar{z})} + B_2e^{\alpha_2(z-\bar{z})}, \\ &\alpha_i \in \mathbb{R}, \qquad A_i, B_i \in \mathbb{C}, \qquad i = 1, 2, \end{aligned} \tag{6.13}$$

where $p_0 = |A_1|^2 + |A_2|^2 + |B_1|^2 + |B_2|^2$ and $A_1\bar{A}_2 + B_1\bar{B}_2 = 0$, $\bar{A}_1A_2 + \bar{B}_1B_2 = 0$, $B_1 = \alpha_1A_1/p_0$ and $B_2 = \alpha_2A_2/p_0$, and where $\alpha_1 = \pm ip_0$ and $\alpha_2 = \pm p_0$. Next, substituting (6.13) into system (1.3), we obtain a set of equations which determine a constant mean curvature surface in the parametric form

$$\begin{aligned} X_1 + iX_2 &= 2i\left(-\frac{\bar{A}_1^2}{\alpha_1}u^2 - \frac{\bar{A}_2^2}{\alpha_2}v^2 - 2\left(\frac{1}{\alpha_1+\alpha_2} + \frac{i}{\alpha_1-\alpha_2}\right)\bar{A}_1\bar{A}_2uv\right), \\ X_1 - iX_2 &= 2i\left(-\frac{A_1^2}{\alpha_1u^2} + \frac{A_2^2}{\alpha_2v^2} + 2\left(\frac{i}{\alpha_1+\alpha_2} - \frac{1}{\alpha_1-\alpha_2}\right)A_1A_2(uv)^{-1}\right), \\ X_3 &= \frac{1}{2}\left(\left(i|A_1|^2 + |A_2|^2\right)\left(\frac{\ln(u)}{\alpha_1} - \frac{\ln(v)}{\alpha_2}\right) - \frac{\bar{A}_1A_2u}{(\alpha_1-\alpha_2)v} + i\frac{A_1\bar{A}_2v}{(\alpha_1-\alpha_2)u}\right) \\ &\quad + \frac{1}{2}\left(\left(-i|A_1|^2 + |A_2|^2\right)\left(\frac{\ln(u)}{\alpha_1} + \frac{\ln(v)}{\alpha_2}\right) + i\frac{\bar{A}_1A_2u}{(\alpha_1+\alpha_2)v} + \frac{A_1\bar{A}_2v}{(\alpha_1+\alpha_2)u}\right), \end{aligned}$$

in terms of $u = \exp(-\alpha_1(z+\bar{z}))$, and $v = \exp(-\alpha_2(z-\bar{z}))$. The Gaussian curvature is $K = 1$.

7 Separation of variables

Now let us discuss the separation of variables admitted by WE system (1.2) which enables us to construct the family of solitonlike solutions. The methodological approach assumed in this section is based on the generalized method of separation of variables developed in [17]. We are looking for a special class of solutions of WE system (1.2) of the form

$$\psi_i(z,\bar z) = \varphi_i(X\cdot Y), \qquad i=1,2, \tag{7.1}$$

where $X = X(z)$, $Y = Y(\bar z)$. Its respective complex conjugate is given by

$$\bar\psi_i(z,\bar z) = \bar\varphi_i(\bar X\cdot \bar Y), \tag{7.2}$$

where $\bar X = \bar X(\bar z)$, $\bar Y = \bar Y(z)$. We assume the existence of two complex scalar functions ξ and η of z and $\bar z$, respectively, such that the differential equations

$$\frac{dX}{dz} = \xi(X), \qquad \frac{dY}{d\bar z} = \eta(Y), \qquad \frac{d\bar X}{d\bar z} = \bar\xi(\bar X), \qquad \frac{d\bar Y}{dz} = \bar\eta(\bar Y), \tag{7.3}$$

hold. This means that the complex functions X and Y are locally piecewise monotonic functions. Substituting (7.2) into WE system (1.2) and taking into account (7.3), we obtain a system of differential equations

$$\dot\varphi_1 \xi Y = \left(|\varphi_1|^2 + |\varphi_2|^2\right)\varphi_2, \qquad \dot{\bar\varphi}_1 \bar\xi \bar Y = \left(|\varphi_1|^2 + |\varphi_2|^2\right)\bar\varphi_2, \tag{7.4i}$$

$$\dot\varphi_2 \eta X = -\left(|\varphi_1|^2 + |\varphi_2|^2\right)\varphi_1, \qquad \dot{\bar\varphi}_2 \bar\eta \bar X = -\left(|\varphi_1|^2 + |\varphi_2|^2\right)\bar\varphi_1. \tag{7.4ii}$$

Let us introduce two differential operators

$$A = X\partial_X - Y\partial_Y, \qquad \bar A = \bar X\partial_{\bar X} - \bar Y\partial_{\bar Y}, \tag{7.5}$$

which are annihilators of any complex function of $s = X\cdot Y$ and $\bar s = \bar X\cdot\bar Y$, respectively. These operators commute

$$[A,\bar A] = 0. \tag{7.6}$$

We operate with the operators A and $\bar A$ on equations (7.4i) and (7.4ii), respectively. Taking into account (7.3), one obtains

$$\begin{aligned} &\dot\varphi_1(XY\xi' - Y\xi) = 0, \qquad \dot{\bar\varphi}_1(\bar X\bar Y\bar\xi' - \bar Y\bar\xi) = 0,\\ &\dot\varphi_2(X\eta - XY\eta') = 0, \qquad \dot{\bar\varphi}_2(\bar X\bar\eta - \bar X\bar Y\bar\eta') = 0, \end{aligned} \tag{7.7}$$

where dots or primes mean the derivatives of the respective functions with respect to their own arguments.

Let us consider separately two cases, namely the case in which $\dot\varphi_i$ does not vanish anywhere and the case in which $\dot\varphi$ is identically equally to zero. It is easy to show that in the case when $\dot\varphi_i = 0$, equations (7.4) do not admit separable solutions, since

$$|\varphi_1|^2 + |\varphi_2|^2 = 0 \tag{7.8}$$

holds.

In the second case, when $\dot{\varphi}_i \neq 0$, equations (7.7) can be integrated and their first integrals are

$$X = \gamma e^{\alpha z}, \qquad \bar{X} = \bar{\gamma} e^{\bar{\alpha}\bar{z}}, \qquad Y = \kappa e^{\beta \bar{z}}, \qquad \bar{Y} = \bar{\kappa} e^{\bar{\beta} z}, \tag{7.9}$$

where α, β, γ and κ are arbitrary complex constants. Substituting (7.9) into (7.4), we obtain a nonlinear system of ODEs

$$\frac{d\varphi_1}{ds} = \frac{p}{\alpha s}\varphi_2, \qquad \frac{d\bar{\varphi}_1}{d\bar{s}} = \frac{p}{\bar{\alpha}\bar{s}}\bar{\varphi}_2, \qquad \frac{d\varphi_2}{ds} = -\frac{p}{\beta s}\varphi_1, \qquad \frac{d\bar{\varphi}_2}{d\bar{s}} = -\frac{p}{\bar{\beta}\bar{s}}\bar{\varphi}_1. \tag{7.10}$$

System (7.10) can be integrated using the condition (1.7) for the conservation of current. Taking into account (1.2) and (7.9), we obtain

$$\alpha^2 s\left[\frac{\ddot{\varphi}_2(s)}{\varphi_2(s)} s + \frac{\dot{\varphi}_2(s)}{\varphi_2(s)}\right] - \bar{\beta}^2 \bar{s}\left[\frac{\ddot{\bar{\varphi}}_1(\bar{s})}{\bar{\varphi}_1(\bar{s})}\bar{s} + \frac{\dot{\bar{\varphi}}_1(\bar{s})}{\bar{\varphi}_1(\bar{s})}\right] = 0. \tag{7.11}$$

The variables s and $\bar{s}$ are separable if the Cauchy-Euler differential equations

$$s^2\ddot{\varphi}_2 + s\dot{\varphi}_2 - \frac{\mu}{\alpha^2}\varphi_2 = 0, \qquad \bar{s}^2\ddot{\bar{\varphi}}_1 + \bar{s}\dot{\bar{\varphi}}_1 - \frac{\mu}{\bar{\beta}^2}\bar{\varphi}_1 = 0 \tag{7.12}$$

hold, where μ is a complex separation constant. After the integration of (7.12), we obtain the solutions

$$\begin{aligned} \varphi_1 &= d_1 s^{\bar{q}} + d_2 s^{-\bar{q}}, & \varphi_2 &= c_1 s^{r} + c_2 s^{-r}, \\ \bar{\varphi}_1 &= \bar{d}_1 \bar{s}^{q} + \bar{d}_2 \bar{s}^{-q}, & \bar{\varphi}_2 &= \bar{c}_1 \bar{s}^{\bar{r}} + \bar{c}_2 \bar{s}^{-\bar{r}}. \end{aligned} \tag{7.13}$$

Here, q, $\bar{q}$, r, $\bar{r}$ and c_i, d_i, $\bar{c}_i$, $\bar{d}_i$, for $i = 1, 2$ are arbitrary complex constants of integration. Substituting (7.13) into (7.10), we obtain the equations

$$\begin{aligned} &-\alpha\bar{q}(d_1 s^{\bar{q}} - d_2 s^{-\bar{q}}) + (c_1 s^r + c_2 s^{-r})[|d_1|^2 s^{\bar{q}}\bar{s}^{q} + d_1\bar{d}_2 s^{\bar{q}}\bar{s}^{-q} + \bar{d}_1 d_2 s^{-\bar{q}}\bar{s}^{q} \\ &\qquad + |d_2|^2 s^{-\bar{q}}\bar{s}^{-q} + |c_1|^2 s^r\bar{s}^{\bar{r}} + \bar{c}_1 c_2 s^{-r}\bar{s}^{\bar{r}} + c_1\bar{c}_2 s^r \bar{s}^{-r} + |c_2|^2 s^{-r}\bar{s}^{-\bar{r}}] = 0, \\ &\beta r(c_1 s^r - c_2 s^{-r}) + (d_1 s^{\bar{q}} + d_2 s^{-\bar{q}})[|d_1|^2 \bar{s}^{q} s^{\bar{q}} + d_1\bar{d}_2 s^{\bar{q}}\bar{s}^{-q} + \bar{d}_1 d_2 s^{-\bar{q}}\bar{s}^{q} \\ &\qquad + |d_2|^2 s^{-\bar{q}}\bar{s}^{-q} + |c_1|^2 s^r\bar{s}^{\bar{r}} + \bar{c}_1 c_2 s^{-r}\bar{s}^{\bar{r}} + c_1\bar{c}_2 s^r\bar{s}^{-\bar{r}} + |c_2|^2 s^{-r}\bar{s}^{-\bar{r}}] = 0, \end{aligned} \tag{7.14}$$

and their respective complex conjugates. We require that system (7.14) is satisfied for any value of s and $\bar{s}$. This means that the coefficients of the successive powers of s and $\bar{s}$ in equations (7.14) have to vanish. As a result, we obtain a consistent system of algebraic equations for c_i and d_i, $i = 1, 2$. The solutions of WE system (1.2) in this case take the form

$$\begin{aligned} \psi_1 &= DE\frac{\exp(3\lambda(z+\bar{z})/2)\exp(\lambda(z-\bar{z})/2)}{E^2\exp(2\lambda(z+\bar{z}))+1}, \\ \psi_2 &= D\frac{\exp(\lambda(z+\bar{z})/2)\exp(\lambda(z-\bar{z})/2)}{E^2\exp(2\lambda(z+\bar{z}))+1}, \end{aligned} \tag{7.15}$$

where $D = 2(c+i)(\lambda E/(c^2+1))^{1/2}$, and c, λ and E are arbitrary real constants of integration. For $\lambda < 0$ solutions (7.15) are nonsingular and represent a bump-type solutions. The associated surface is obtained from relations (1.3) as

$$\begin{aligned} X_1 &= -2\left((c^2-1)\sin x + 2c\cos x\right)\left(1+\frac{1}{2\lambda}\right)\left(p^2+q^2\right), \\ X_2 &= 2\left(2c\sin x - (c^2-1)\cos x\right)\left(1+\frac{1}{2\lambda}\right)\left(p^2+q^2\right), \\ X_3 &= 4\left(E^2e^{4\lambda x}+1\right)^{-1}. \end{aligned} \tag{7.16}$$

Here, x is the real part of z, and p^2 and q^2 are given by

$$p^2 = \frac{4\lambda E^3 e^{6\lambda x}}{A(E^2e^{4\lambda x}+1)^2}, \qquad q^2 = \frac{4\lambda E e^{2\lambda x}}{A(E^2e^{4\lambda x}+1)^2}, \qquad A \in \mathbb{R}.$$

One can eliminate quantities p, q and x from (7.16) and obtain the following expression for the surface representing a catenoide

$$X_1^2 + X_2^2 = 4\left(\frac{\lambda}{A}\right)^2\left[(c-1)^2+4c^2\right]\left(1+\frac{1}{2\lambda}\right)^2(4-X_3)X_3.$$

The Gaussian curvature K is constant and given by

$$K = 4\frac{\lambda^2E^2}{|D|^2}.$$

8 Final remarks

We have presented a variety of new approaches to the study of the generalized WE system. They proved to be particularly effective in delivering solutions from which it was possible to derive explicit formulae for associated constant mean curvature surfaces embedded in $\mathbb{R}^3$. One of the more interesting results of our analysis is the observation that the WE system admits potential solutions. This fact made it possible, for the first time, to deal with the boundary value problem for the generalized WE system. We were also able to construct potential multi-soliton solutions.

It is worth noting that the treatments of the WE system proposed here can be applied, with necessary modifications, to more general cases of WE systems describing surfaces immersed in multi-dimensional Euclidean and pseudo-Riemannian spaces. Such generalization of the WE system has been recently presented by Konopelchenko in [18] where, in particular, the explicit formulae for minimal surfaces immersed in four-dimensional Euclidean space $\mathbb{R}^4$ and S^4 have been derived. An extention of our analysis to this case will be a subject of a future work.

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