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Coupled KdV Equations of Hirota-Satsuma Type

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Abstract

It is shown that the system of two coupled Korteweg-de Vries equations passes the Painlevé test for integrability in nine distinct cases of its coefficients. The integrability of eight cases is verified by direct construction of Lax pairs, whereas for one case it remains unknown.

1 Introduction

Recently, Karasu [1] proposed a Painlevé classification of coupled KdV equations. More recently, we found that the classification of Karasu missed at least two systems which possessed the Painlevé property and Lax pairs [2]. In the present paper, we give our version of the singularity analysis of coupled KdV equations.

We study the following class of nonlinear systems of partial differential equations:

$$\begin{aligned} &u_{xxx} + auu_x + bvu_x + cuv_x + dvv_x + mu_t + nv_t = 0, \\ &v_{xxx} + euu_x + fvu_x + guv_x + hvv_x + pu_t + qv_t = 0, \end{aligned} \tag{1}$$

where a, b, c, d, e, f, g, h, m, n, p and q are arbitrary constants, and

$$mq \neq np. \tag{2}$$

The condition (2) allows every system (1) to be resolved with respect to u_t and v_t, therefore our class (1) coincides with the class of all "nondegenerate KdV systems" studied by Karasu [1]. Since the "degenerate KdV systems" [1] are in fact some over-determined systems [3] reducible to lower-order ones, we do not classify them as coupled Korteweg-de Vries equations.

The system (1) is "soft" in the sense that its coefficients can be changed by the transformation

$$u' = y_1 u + y_2 v, \qquad v' = y_3 u + y_4 v, \qquad t' = y_5 t, \tag{3}$$

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where y_1, y_2, y_3, y_4 and y_5 are constants, $y_1y_4 \neq y_2y_3$, and $y_5 \neq 0$. At every stage of our analysis, we use the transformation (3) in order to simplify the system (1) as far as possible.

We perform the singularity analysis of the class (1) in accordance with the Weiss-Kruskal algorithm, which is based on the Weiss-Tabor-Carnevale singular expansions [4], Ward's requirement not to analyze solutions at their characteristics [5], Kruskal's simplifying representation of singularity manifolds [6], and which follows step by step the Ablowitz-Ramani-Segur algorithm for ordinary differential equations [7]. Since the Weiss-Kruskal algorithm is well known and widely used, we omit all unessential details of our computations.

It should be stressed, however, that the Weiss-Kruskal algorithm checks only necessary conditions for an equation to possess the Painlevé property. Moreover, as explained in Sections 2.1.1 and 2.1.4, our analysis could miss some systems with the Painlevé property. For these reasons, our work does not claim to be a "Painlevé classification", and we prefer to think that it is simply a search for those systems (1) whose analytic properties are similar to such of the Hirota-Satsuma system [8] studied in [9].

In Section 2, we study the analytic properties of the systems (1) and find, up to arbitrary transformations (3), the following nine coupled KdV equations of Hirota-Satsuma type:

$$u_t = u_{xxx} - 12uu_x + 24kvv_x, \qquad v_t = -2v_{xxx} + 12uv_x, \qquad k = -1, 0, 1, \tag{i}$$

$$u_t = u_{xxx} - 12uu_x, \qquad v_t = v_{xxx} - 6uv_x, \tag{ii}$$

$$u_t = u_{xxx} - 12uu_x, \qquad v_t = 4v_{xxx} - 6vu_x - 12uv_x, \tag{iii}$$

$$\begin{aligned} u_t &= u_{xxx} - 9v_{xxx} - 12uu_x + 18vu_x + 18uv_x, \\ v_t &= u_{xxx} - 5v_{xxx} - 12uu_x + 12vu_x + 6uv_x + 18vv_x, \end{aligned} \tag{iv}$$

$$\begin{aligned} u_t &= u_{xxx} + 9v_{xxx} - 12uu_x - 18vu_x - 18uv_x + 108vv_x, \\ v_t &= u_{xxx} - 11v_{xxx} - 12uu_x + 12vu_x + 42uv_x + 18vv_x, \end{aligned} \tag{v}$$

$$u_t = u_{xxx} - 12uu_x, \qquad v_t = v_{xxx} - 12uv_x, \tag{vi}$$

$$u_t = u_{xxx} - 12uu_x, \qquad v_t = v_{xxx} - 6vu_x - 6uv_x, \tag{vii}$$

$$u_t = u_{xxx} - 12uu_x, \qquad v_t = -2v_{xxx} + 12vu_x + 12uv_x, \tag{viii}$$

$$u_t = u_{xxx} - 12uu_x, \qquad v_t = ku_{xxx} + v_{xxx} - 12vu_x - 12uv_x, \qquad k = 0, 1. \tag{ix}$$

In Section 3, we find transformations between some of the systems (i)–(ix), and then prove the integrability of (i)–(ix) except (v) by direct construction of Lax pairs.

In Section 4, we compare our results with the results of Karasu [1] and give some unsolved problems.

2 Singularity analysis

2.1 Generic branches

2.1.1 Leading exponents

Starting the Weiss-Kruskal algorithm, we determine that a hypersurface $\phi(x,t) = 0$ is non-characteristic [3] for the system (1) if $\phi_x \neq 0$; without loss of generality we take $\phi_x = 1$, which simplifies computations and excludes characteristic singularity manifolds from our consideration. Since (1) is a normal system [3], its general solution must contain six arbitrary functions of one variable. The substitution of

$$u = u_0(t)\phi^{\alpha} + \cdots + u_r(t)\phi^{r+\alpha} + \cdots, \qquad v = v_0(t)\phi^{\beta} + \cdots + v_r(t)\phi^{r+\beta} + \cdots \tag{4}$$

with $u_0 v_0 \neq 0$ into the system (1) determines branches (i.e. the sets of admissible α, β, u_0 and v_0) and positions r of resonances for those branches. We require that the system (1) admits at least one singular generic branch, where $\alpha < 0$ or $\beta < 0$, the number of resonances is six, and $r \geq 0$ for five of them. This requirement seems to be more restrictive than the Painlevé property itself, therefore some coupled KdV equations possessing the Painlevé property may be missed.

Throughout Section 2.1, we analyze only the singular generic branches; all other branches are considered in Section 2.2. Since the number of resonances is to be six, the set of leading terms of (1) must include u_{xxx} and v_{xxx}, and the admissible values of α and β are as follow, depending on which of the other terms of (1) are leading and which of the coefficients are zero:

- $\alpha = -2$ and β is arbitrary;
- $\alpha = -4$ and $\beta = -6$;
- and the same with $\alpha \rightleftarrows \beta$, which can be omitted due to (3).

We reject systems (1) with $\alpha = -4$ and $\beta = -6$ because of bad positions of resonances, $r = -1, 6, 8, 12$ and $\frac{1}{2}(11 \pm i\sqrt{159})$, and proceed to systems (1) with $\alpha = -2$ and any integer β.

2.1.2 Systems with $\beta > -2$

If $\alpha = -2$ and $\beta > -2$, then $e = 0$ in (1); moreover, $p = 0$ if $\beta > 0$. Using (3), we make $a = -12$ in (1), and then find that $u_0 = 1$ and

$$(\beta(\beta-1)(\beta-2) - 2f + \beta g)v_0 = 2\delta_{0\beta} p \phi_t. \tag{5}$$

Positions of resonances are -1, 4, 6, r_1, r_2 and r_3, which correspond to the possible arbitrariness of ϕ, u_4, u_6, v_{r_1}, v_{r_2} and v_{r_3}, where r_1, r_2 and r_3 are three roots of

$$r^3 + 3(\beta-1)r^2 + (3\beta^2 - 6\beta + 2 + g)r + 2\delta_{0\beta} p \phi_t v_0^{-1} = 0. \tag{6}$$

It follows from (6) that $r_1 + r_2 + r_3 = 3(1-\beta)$. Since r_1, r_2 and r_3 are to be distinct non-negative integers, it is necessary that $\beta \leq 0$. Thus, we have $\{r_1, r_2, r_3\} = \{0, 1, 2\}$ and $p = 0$ for $\beta = 0$, $\{r_1, r_2, r_3\} = \{0, 1, 5\}$ or $\{0, 2, 4\}$ for $\beta = -1$, and $f = \frac{1}{2}\beta((\beta-1)(\beta-2)+g)$ from (5).

If $\beta = 0$, then $r = -1, 0, 1, 2, 4$ and 6, $f = 0$, and $g = 0$ due to (6). We make $b = 0$ by (3), substitute the expansions $u = \sum_{i=0}^{\infty} u_i(t)\phi^{i-2}$ and $v = \sum_{i=0}^{\infty} v_i(t)\phi^{i}$ into (1), find recursion relations for u_i and v_i, and then check compatibility conditions arising at resonances. We get $c = 0$ at $r = 4$ and $d = n = 0$ at $r = 6$, therefore this case of (1) is equivalent to two non-coupled equations.

When $\beta = -1$ and $r = -1, 0, 1, 4, 5$ and 6, we have $f = 0$ and $g = -6$. Then (3) lets us make $c = -2b$ to simplify computations. We substitute $u = \sum_{i=0}^{\infty} u_i(t)\phi^{i-2}$ and $v = \sum_{i=0}^{\infty} v_i(t)\phi^{i-1}$ into (1), and compatibility conditions at resonances give us $h = p = b = n = 0$, $d(m + 2q) = 0$ and $(m + 2q)(m - q) = 0$. Eliminating free parameters by (3), we get (i) and (ii).

When $\beta = -1$ and $r = -1, 0, 2, 4, 4$ and 6, we have $f = -\frac{3}{2}$ and $g = -3$, and make $c = -2b$ by (3) again. Then compatibility conditions at resonances give us $d = \frac{2}{3}h(4h - b)$, $q = \frac{1}{6}p(b - 4h) + \frac{1}{4}m$, $n = 2bq$, $hz = pz = 0$ and $h^2 = b^2$, where $z = m + \frac{2}{3}bp - \frac{16}{9}hp$. Here we have three distinct cases, which, after simplification by (3), become (iii), (iv) and (v).

2.1.3 Systems with $\beta = -2$

When $\alpha = \beta = -2$, u_0 and v_0 are determined by the system

$$\begin{aligned} &12u_0 + au_0^2 + (b + c)u_0v_0 + dv_0^2 = 0, \\ &12v_0 + eu_0^2 + (f + g)u_0v_0 + hv_0^2 = 0. \end{aligned} \tag{7}$$

If both u_0 and v_0 are some fixed constants (no resonances with $r = 0$), then (3) lets us make $\beta > -2$, but we have already considered those cases in Section 2.1.2. The assumption that both u_0 and v_0 are arbitrary and independent (two resonances with $r = 0$) leads to a contradiction with (7). Therefore one of the variables u_0 and v_0 should be arbitrary (one resonance with $r = 0$), and another one should be a function of it. According to (7), this is possible only if $d = e = 0$. Then, using (3), we make $a = -12$ and $c = -b$, and (7) gives us $h = 0$, $g = -12 - f$, $u_0 = 1$, $\forall v_0(t)$. Positions of resonances turn out to be $r = -1, 0, 4, 6$, r_1 and r_2, where r_1 and r_2 are two roots of

$$r^2 - 9r + 14 - f + bv_0(t) = 0. \tag{8}$$

Positions of resonances should not depend on $v_0(t)$, therefore we take $b = 0$, and (8) gives us $r_2 = 9 - r_1$ and $f = r_1^2 - 9r_1 + 14$, where $r_1 = 1, 2, 3$ or 4 because $0 < r_1 < r_2$.

When $r_1 = 1$, compatibility conditions at resonances lead to $m = n = q = 0$, which violates (2), and (1) is not a system of evolution equations in this case.

When $r_1 = 2$, compatibility conditions at resonances give us $n = p = 0$ and $q = m$. Simplifying (1) by (3), we get (vi).

When $r_1 = 3$, we find at resonances that $n = q = 0$, which is prohibited by (2).

When $r_1 = 4$, compatibility conditions give us $n = p = q - m = 0$ or $n = q + \frac{1}{2}m = 0$, and this leads through (3) to (vii) and (viii).

2.1.4 Systems with $\beta < -2$

When $\alpha = -2$ and $\beta = -3$, we have $b = c = d = h = 0$ in (1), make $a = -12$ by (3), and get $u_0 = 1$, $f = -30 - \frac{3}{2}g$, $\forall v_0(t)$. Positions of resonances are $r = -1, 0, 4, 6$, r_1 and r_2, where $r_1 + r_2 = 12$ and $r_1 r_2 = 47 + g$. Since r_1 and r_2 correspond to the possible arbitrariness of v_{r_1} and v_{r_2}, we have to study the following five distinct cases: $r_1 = 1, 2, 3, 4$ and 5. In the cases $r_1 = 1, 2$ and 3, compatibility conditions at resonances violate (2). When $r_1 = 4$, compatibility conditions restrict arbitrary functions in expansions (4). But in the case $r_1 = 5$, when $f = g = -12$, we have only $n = 0$ and $q = m$ at resonances, and then, simplifying (1) by (3), get (ix).

When $\alpha = -2$ and $\beta = -4$, we have $b = c = d = h = 0$ in (1). In this case, nv_t is a leading term, therefore we have to consider $n = 0$ and $n \neq 0$ separately. There are seven distinct cases of positions of resonances when $n = 0$ and six such cases when $n \neq 0$. In every case of those thirteen cases, however, compatibility conditions at resonances either contradict (2) or restrict arbitrary functions in (4).

When $\alpha = -2$ and $\beta \leq -5$, we have $b = c = d = h = n = 0$ in (1). There are $\frac{1}{2} - \frac{3}{2}\beta$ distinct cases of positions of resonances if β is odd and $1 - \frac{3}{2}\beta$ such cases if β is even. Now, using the *Mathematica* computer system [10], we can check that, for $\beta = -5, -6, \ldots, -10$ and for all possible positions of resonances, compatibility conditions at resonances either contradict (2) or restrict arbitrary functions in (4). This is very suggestive that in fact no systems (1) with $\beta < -3$ possess the Painlevé property. But we were unable to prove this conjecture within the Weiss-Kruskal algorithm for all $\beta < -3$, and therefore our list (i)–(ix) may be incomplete.

2.2 Non-generic branches

The systems (i)–(ix) admit many branches. The singular generic branches have been studied in Section 2.1. All the nonsingular branches correspond to Taylor expansions governed by the Cauchy-Kovalevskaya theorem because the system (1) is written in a nonsingular Kovalevskaya form [3]. Most of admissible singular non-generic branches of (i)–(ix) correspond in fact to the singular generic expansions, where one or two arbitrary functions at resonances are taken to be zero. Therefore we have to study only the following singular non-generic branches:

- $\alpha = \beta = -2$, $u_0 = 2$, $v_0^2 = 1/k$, $r = -2, -1, 3, 4, 6$ and 8 for (i) with $k \neq 0$;
- $\alpha = \beta = -2$, $u_0 = 2$, $v_0 = -\frac{2}{3}$, $r = -2, -1, 3, 4, 6$ and 8 for (iv);
- $\alpha = \beta = -2$, $u_0 = 6$, $v_0 = \frac{10}{3}$, $r = -5, -1, 4, 6, 6$ and 8 for (iv);
- $\alpha = \beta = -2$, $u_0 = 2$, $v_0 = -\frac{2}{3}$, $r = -2, -1, 4, 5, 6$ and 6 for (v);
- $\alpha = \beta = -2$, $u_0 = 4$, $v_0 = \frac{4}{3}$, $r = -4, -1, 3, 4, 6$ and 10 for (v).

Compatibility conditions at all resonances of these branches turn out to be satisfied identically. Consequently, the systems (i)–(ix) have passed the Weiss-Kruskal algorithm well.

3 Integrability

No two of the systems (i)–(ix) can be related by transformations (3), but not all of (i)–(ix) are distinct with respect to more general transformations. If we take (i) with $k = 0$ and make $v_x = w$, then we get exactly (viii) for u and w. If we make $v_x = w$ in (ii), then we get exactly (vii) for u and w. If we make $v_x = w$ in (vi), then we get (ix) with $k = 0$ for u and w. Therefore we can restrict our further study to (i)–(v) and (ix) only.

Since the systems (i)–(v) and (ix) pass the Painlevé test well, they can be expected to be integrable. Let us try to find their Lax pairs. We consider the over-determined linear system

$$\Psi_x = A\Psi, \qquad \Psi_t = B\Psi, \tag{9}$$

where A and B are some matrices and Ψ is a column, and require that the compatibility condition of (9),

$$A_t = B_x - AB + BA, \tag{10}$$

represents the system under consideration. We assume that

$$A = Pu + Qv + R \tag{11}$$

and $B = B(u,\, u_x,\, u_{xx},\, v,\, v_x,\, v_{xx})$, where P, Q and R are constant matrices. Under this assumption, we get from (10) an explicit expression for B in terms of P, Q, R and a constant matrix S, as well as a set of conditions for P, Q, R and S. Then we try to satisfy those conditions, increasing the dimension of the matrices. (This is essentially the Dodd-Fordy version [11] of the Wahlquist-Estabrook method [12].) In this way, we obtain the following results for the systems (i), (ii), (iii), (iv) and (ix):

$$A_{(\mathrm{i})} = \begin{pmatrix} 0 & u+\sigma & 0 & kv \\ 2 & 0 & 0 & 0 \\ 0 & v & 0 & u-\sigma \\ 0 & 0 & 2 & 0 \end{pmatrix}, \tag{12}$$

$B_{(\mathrm{i})} = \{\{-2u_x,\ u_{xx} - 4u^2 + 8kv^2 + 4\sigma u + 8\sigma^2,\ 4kv_x,\ -2kv_{xx} + 4kuv\},\ \{-8u + 16\sigma,\ 2u_x,\ 16kv,\ -4kv_x\},\ \{4v_x,\ -2v_{xx} + 4uv,\ -2u_x,\ u_{xx} - 4u^2 + 8kv^2 - 4\sigma u + 8\sigma^2\},\ \{16v,\ -4v_x,\ -8u - 16\sigma,\ 2u_x\}\}$;

$$A_{(\mathrm{ii})} = \begin{pmatrix} 2\sigma & u & 0 \\ 2 & -\sigma & 0 \\ v & 0 & -\sigma \end{pmatrix}, \tag{13}$$

$B_{(\mathrm{ii})} = \{\{-2u_x - 6\sigma u + 18\sigma^3,\ u_{xx} - 4u^2 + 3\sigma u_x + 9\sigma^2 u,\ 0\},\ \{-8u + 18\sigma^2,\ 2u_x + 6\sigma u - 9\sigma^3,\ 0\},\ \{v_{xx} - 4uv - 3\sigma v_x + 9\sigma^2 v,\ vu_x - uv_x + 3\sigma uv,\ -9\sigma^3\}\}$;

$$A_{(\mathrm{iii})} = \begin{pmatrix} 0 & u+\sigma & 0 \\ 2 & 0 & 0 \\ 0 & v & 0 \end{pmatrix}, \tag{14}$$

$B_{(\mathrm{iii})} = \{\{-2u_x,\, u_{xx} - 4u^2 + 4\sigma u + 8\sigma^2,\, 0\},\, \{-8u + 16\sigma,\, 2u_x,\, 0\},\, \{-8v_x,\, 4v_{xx} - 4uv + 8\sigma v,\, 0\}\}$;

$$A_{(\mathrm{iv})} = \begin{pmatrix} \sigma & u - v & \sigma(\frac{8}{3}u - 4v) \\ \frac{3}{2} & -2\sigma & u - v \\ 0 & \frac{3}{2} & \sigma \end{pmatrix}, \tag{15}$$

$B_{(\mathrm{iv})} = \{\{6v_x - 3\sigma(u - 3v) - 18\sigma^3,\, -4v_{xx} + 6uv - 6v^2 + 2\sigma(u_x - 3v_x) + 6\sigma^2(u - 3v),\, \sigma(-\frac{4}{3}u_{xx} - 4v_{xx} + 6u^2 + 4uv - 18v^2)\},\, \{9v - 27\sigma^2,\, 6\sigma(u - 3v) + 36\sigma^3,\, -4v_{xx} + 6uv - 6v^2 - 2\sigma(u_x - 3v_x) + 6\sigma^2(u - 3v)\},\, \{\frac{27}{2}\sigma,\, 9v - 27\sigma^2,\, -6v_x - 3\sigma(u - 3v) - 18\sigma^3\}\}$;

$$A_{(\mathrm{ix})} = \begin{pmatrix} 0 & u + \sigma & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & v + \tau & 0 & u + \sigma \\ -2k & 0 & 2 & 0 \end{pmatrix}, \tag{16}$$

$B_{(\mathrm{ix})} = \{\{-2u_x,\, u_{xx} - 4u^2 + 4\sigma u + 8\sigma^2,\, 0,\, 0\},\, \{-8u + 16\sigma,\, 2u_x,\, 0,\, 0\},\, \{-2v_x,\, ku_{xx} + v_{xx} - 8uv + 4\tau u + 4\sigma v + 16\sigma\tau,\, -2u_x,\, u_{xx} - 4u^2 + 4\sigma u + 8\sigma^2\},\, \{8ku - 8v - 16k\sigma + 16\tau,\, 2v_x,\, -8u + 16\sigma,\, 2u_x\}\}$; where the index of A and B corresponds to the system, σ and τ are arbitrary parameters, and the cumbersome matrices B are written by rows. The invariant technique from [13] allows us to prove that the parameters σ and τ are essential, i.e. they cannot be eliminated by gauge transformations of Ψ.

The system (i) is the original Hirota-Satsuma equation [8], and the Lax pair found in [11] corresponds to our result (9) and (12) with $k = 1$ up to a necessary transformation (3) and a gauge transformation of Ψ. The system (ix) represents the two $(1+1)$-dimensional reductions [2] of the $(2+1)$-dimensional perturbed KdV equation [14], and the Lax pair (9) and (16) follows from the $(2+1)$-dimensional Lax pair [2] by reduction as well.

In the case of (v), however, this approach leads us only to matrices A containing no essential parameters, at least for dimensions from 2×2 to 5×5; probably, the assumption (11) is too restrictive. The method of truncation of singular expansions [15] gives us a similar result: though the truncation procedure turns out to be compatible for the system (v), the truncated singular expansions contain no essential parameters. Therefore it remains unknown whether the system (v) is integrable or not.

4 Conclusion

Let us make a brief comparison between our study of coupled KdV equations and the classification given in [1].

- The integrability of selected systems, equivalences between them, as well as non-generic branches were not studied in [1].
- Karasu considered only the case $\alpha = \beta = -2$ (in our notations of (4)) but did not use the condition $u_0 v_0 \neq 0$; therefore the results of [1] could, in principle, contain all our systems with $\beta \geq -2$, i.e. (i)-(viii), and should miss only (ix) with $\beta = -3$.
- Our systems (iii), (v) and (ix) are missed in the classification [1].
- Appropriate transformations (3) change Karasu's systems (13), (14), (15), (16), (17) and (19) (numbered as in [1]) into our systems (vi), (vii), (viii), (ii), (i) and (iv), respectively, eliminating all free parameters.

- The system (20) from [1] can be transformed by (3) into a system of two non-coupled equations and therefore is absent from our results.
- The system (21) from [1] does not pass the Painlevé test (even after a correction of misprints).

Below we give some unsolved problems.

- Does the system (v) possess a Lax pair with an essential parameter?
- Is there a system (1) with the Painlevé property and $\beta < -3$ in its singular generic branch?
- Can some of the systems (i)–(v) and (ix) be related to each other by Miura and Bäcklund transformations?

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