

Coadjoint Poisson Actions of Poisson-Lie Groups

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Abstract

A Poisson-Lie group acting by the coadjoint action on the dual of its Lie algebra induces on it a non-trivial class of quadratic Poisson structures extending the linear Poisson bracket on the coadjoint orbits.

1 Introduction

If G is a Lie group with Lie algebra \mathcal{G} then the coadjoint action of G on the dual space \mathcal{G}^* to \mathcal{G} leaves invariant the linear Poisson bracket on \mathcal{G}^* . In other words, the action map $Ad^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is a Poisson map, with the Poisson structure on G being the trivial one. If G is a Poisson-Lie group then this action, in general, is no longer Poisson unless the Poisson structure on \mathcal{G}^* could be suitably modified. In this paper we show that this is indeed possible. We construct this extension explicitly and give necessary and sufficient conditions for its existence. In particular, any Poisson structure on a finite-dimensional connected simply-connected Poisson-Lie group G coming from an r -matrix that satisfies the Classical Yang-Baxter Equation induces a Poisson structure on \mathcal{G}^* such that the coadjoint action becomes Poisson. The existence of a modification of the linear Poisson bracket for the case $G = GL(n)$ and $\mathcal{G}^* = gl(n)^*$ was shown earlier in [1] (see also the related article [2]). In [1] an equivariant quantization of the coadjoint action for the case $G = GL(2)$ and $\mathcal{G}^* = gl(2)^*$ was also constructed.

2 Main Theorem

We start with recalling some basics. Let G be a finite-dimensional connected simply-connected Poisson-Lie group. Let the Poisson structure on G be given by the skew-symmetric rank-2 contravariant tensor π^{ij} . We write the group law $G \times G \rightarrow G$ $((x, y) \mapsto f(x, y))$ in a neighbourhood of the identity as

$$f^i(x^1, \dots, x^n, y^1, \dots, y^n), \quad i = 1, \dots, n = \dim G,$$

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where f^i are assumed smooth. Let $\varphi : G \rightarrow G$ be the map of taking an inverse. We recall the properties of these maps [3]:

$$f^i(0, \dots, 0, y^1, \dots, y^n) = y^i, \quad (1)$$

$$f^i(x^1, \dots, x^n, 0, \dots, 0) = x^i, \quad (2)$$

$$f^i(x, \varphi(x)) = 0 = f^i(\varphi(x), x). \quad (3)$$

From these we deduce that

$$\frac{\partial f^i}{\partial x^j}(0, 0) = \delta_j^i = \frac{\partial f^i}{\partial y^j}(0, 0), \quad (4)$$

and

$$\delta_j^i + \frac{\partial \varphi^i}{\partial x^j}(0) = 0. \quad (5)$$

For $x \in G$ let $L_y(x) = f(y, x)$ and $R_y(x) = f(x, y)$ be the left and right actions by an element $y \in G$. The adjoint action $\text{Ad}_y(x) = R_{y^{-1}} \circ L_y(x)$ is described in local coordinates by

$$f^i(f(y, x), \varphi(y)). \quad (6)$$

Let $I \subset \mathbb{R}$ be an open interval containing the origin and let $x : I \rightarrow G$ be a curve $x(t)$ with a direction vector $\xi \in \mathcal{G} = T_e G$ at the identity $e = x(0)$. Differentiating (5) in the direction of ξ at $t = 0$ (with $x(t) = \exp(t\xi)$) we obtain a formula for the action of Ad_y on \mathcal{G} :

$$(\text{Ad}_y)_j^i \xi^j = \left. \frac{df^i}{dt} \right|_{t=0} = \frac{\partial f^i}{\partial u^k}(y, \varphi(y)) \frac{\partial f^k}{\partial v^j}(y, 0) \xi^j \equiv A_j^i(y) \xi^j. \quad (7)$$

Here ξ^i are the coorsinates of ξ and u and v refer to the first and second argument of the functions f respectively. (The Einstein convention of summation over repeated upper and lower indices is in force throughout this text.) If $\eta \in \mathcal{G}^*$ relative to the form

$$(\text{Ad}_y(\xi), \eta) = (\text{Ad}_y)_j^i \xi^j \eta_i \equiv (\xi, \text{Ad}_y^*(\eta)), \quad (8)$$

and the coadjoint action $G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is defined to be:

$$(\text{Ad}_{y^{-1}}^*)_j^i \eta_i = \frac{\partial f^i}{\partial u^k}(\varphi(y), y) \frac{\partial f^k}{\partial v^j}(\varphi(y), 0) \eta_i = A_j^i(y^{-1}) \eta_i, \quad (9)$$

where η_i are the coordinates of η .

Remark 2.1 Recall that if $\zeta \in \mathcal{G}$ is a direction vector of a curve $y(s)$ in G , differentiating (7) and (9) in the direction of ζ at $s = 0$ we obtain formulae for the maps $\text{ad}_\zeta : \mathcal{G} \rightarrow \mathcal{G}$ and $\text{ad}_\zeta^* : \mathcal{G}^* \rightarrow \mathcal{G}^*$. For example,

$$\begin{aligned} (\text{ad}_\zeta^*)_j^i &= \left[-\frac{\partial^2 f^i}{\partial u^k \partial u^s}(0, 0) \frac{\partial f^k}{\partial v^j}(0, 0) + \frac{\partial^2 f^i}{\partial u^k \partial v^s}(0, 0) \frac{\partial f^k}{\partial v^j}(0, 0) - \frac{\partial^2 f^k}{\partial u^s \partial v^j}(0, 0) \frac{\partial f^i}{\partial u^k}(0, 0) \right] \zeta^s \\ &= \left[-\frac{\partial^2 f^i}{\partial u^s \partial v^j}(0, 0) + \frac{\partial^2 f^i}{\partial u^j \partial v^s}(0, 0) \right] \zeta^s = -C_{sj}^i \zeta^s, \end{aligned}$$

and similarly for ad_ζ . Here C_{sj}^i are the structure constants of \mathcal{G} , and we have used formulae (3) and (4) and $\frac{\partial^2 f^i}{\partial u^k \partial u^s}(0, 0) = 0$.

In addition, the group G is assumed to be equipped with a Poisson-Lie structure [4]. That is, a rank-2 contravariant skew-symmetric Poisson tensor π^{kl} exists which is compatible with the group multiplication $G \times G \rightarrow G$ (π is a group 1-cocycle). The Poisson bracket between two smooth functions f and g can be defined in three equivalent ways:

$$\{f, g\} = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \varrho^{ij} \mathcal{D}_i f \mathcal{D}_j g = \sigma^{ij} \mathcal{D}'_i f \mathcal{D}'_j g,$$

where $\mathcal{D}_i = dL_i^s(x) \frac{\partial}{\partial x^s}$ and $\mathcal{D}'_i = dR_i^s(x) \frac{\partial}{\partial x^s}$ are left and right invariant vector fields and $dL(x)$ and $dR(x)$ are the derivatives of the maps L_x and R_x . The tensors π , ϱ , and σ are related by

$$\pi(x) = \varrho(x) dL(x) dL(x) = \sigma(x) dR(x) dR(x). \quad (10)$$

The Poisson structure on $G \times G$ is taken to be the product Poisson structure and thus the multiplication map $m : G \times G \rightarrow G$ must be Poisson, that is, $\{f, g\}_G(xy) = \{m^* f, m^* g\}_{G \times G}(x, y)$. This implies that π^{ij} must satisfy the 1-cocycle functional equation

$$\pi^{ij}(xy) = dR_k^i(y) dR_l^j(y) \pi^{kl}(x) + dL_k^i(x) dL_l^j(x) \pi^{kl}(y), \quad (11)$$

which is equivalent to

$$\sigma^{ij}(xy) = \sigma^{ij}(x) + dR_k^i((yx)^{-1}) dR_l^j((yx)^{-1}) A_p^k(x) A_q^l(x) dR_s^p(yx) dR_t^q(yx) \sigma^{st}(y), \quad (12)$$

where $A(x) = dR(x^{-1}) dL(x) = dL(x) dR(x^{-1})$ is the matrix of the adjoint representation. Let again $y : I \rightarrow G$ be a one-parameter curve passing through the identity with direction vector ξ . Differentiating (12) in the direction of ξ at $y = 0$ we obtain the system of differential equations

$$dL_k^s(x) \frac{\partial \sigma^{ij}}{\partial x^s} = A_s^i(x) \alpha_k^{sp} A_p^j(x), \quad (13)$$

where $\alpha_k^{ij} = \frac{\partial \sigma^{ij}}{\partial x^k}(0) = \frac{\partial \pi^{ij}}{\partial x^k}(0)$ are the components comprising a map $\alpha : \mathcal{G} \rightarrow \mathcal{G} \wedge \mathcal{G}$. The integrability conditions for (13) after use of the Maurer-Cartan equations and evaluation at $x = 0$ lead to

$$\alpha_s^{kl} C_{ij}^s = \alpha_j^{ml} C_{im}^k + \alpha_j^{km} C_{im}^l - \alpha_i^{ml} C_{jm}^k - \alpha_i^{km} C_{jm}^l, \quad (14)$$

and therefore $\alpha : \mathcal{G} \rightarrow \mathcal{G} \wedge \mathcal{G}$ being a 1-cocycle is a necessary and sufficient condition for the existence of a solution of (13). In particular, as is well known [4], if $\alpha = \delta r$ is a coboundary, where $r \in \mathcal{G} \wedge \mathcal{G}$, then

$$\alpha_n^{ij} = C_{ns}^i r^{sj} + C_{ns}^j r^{is}. \quad (15)$$

In this case (14) is identically satisfied and the equations (13) can be trivially integrated yielding the solution

$$\sigma^{ij}(x) = A_s^i(x) r^{sp} A_p^j(x) + r_0^{ij}, \quad (16)$$

where r_0^{ij} is a constant skew-symmetric matrix. (A formula of type (78) below, with dL instead of dR , is used in the proof of this.) Substituting (16) back into the functional equation (12), and using the fact that the left and right actions commute, we deduce that

it is a solution of the original 1-cocycle equation if and only if $r_0^{ij} = -r^{ij}$. All of the above is standard in the theory of Poisson-Lie groups. We recalled the relevant facts which we shall need in the sequel.

Our goal is to make the action $\text{Ad}^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ Poisson. Therefore we need to construct (covariant) Poisson tensors ω_{ij} on \mathcal{G}^* compatible with the coadjoint action. Here again the space $G \times \mathcal{G}^*$ is equipped with the product Poisson structure and thus the map $\text{Ad}^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is required to be Poisson. This condition is equivalent to the condition that locally ω_{ij} must satisfy the following system of functional equations

$$\omega_{ij}(B(y, \eta)) = \frac{\partial B_i}{\partial \eta_k} \frac{\partial B_j}{\partial \eta_l} \omega_{kl}(\eta) + \frac{\partial B_i}{\partial y^k} \frac{\partial B_j}{\partial y^l} \pi^{kl}(y),$$

where we have introduced $B_i(y, \eta) := A_i^j(y^{-1})\eta_j$, or equivalently

$$\omega_{ij}(A(y^{-1})\eta) = A_i^k(y^{-1})A_j^l(y^{-1})\omega_{kl}(\eta) + \frac{\partial A_i^s}{\partial y^k}(y^{-1})\frac{\partial A_j^p}{\partial y^l}(y^{-1})\eta_s\eta_p\pi^{kl}(y). \quad (17)$$

In order to construct solutions we pass to a system of differential equations which is the infinitesimal part of (17). Differentiation of the above equations in the directions of the coordinate axes yields

$$\begin{aligned} \frac{\partial \omega_{ij}}{\partial \eta_s} \frac{\partial A_s^l}{\partial y^n} \eta_l &= \frac{\partial A_i^k}{\partial y^n} A_j^l \omega_{kl}(\eta) + A_i^k \frac{\partial A_j^l}{\partial y^n} \omega_{kl}(\eta) + \frac{\partial A_i^s}{\partial y^k} \frac{\partial A_j^p}{\partial y^l} \frac{\partial \pi^{kl}}{\partial y^n} \eta_s \eta_p \\ &+ \frac{\partial^2 A_i^s}{\partial y^k \partial y^n} \frac{\partial A_j^p}{\partial y^l} \pi^{kl}(y) \eta_s \eta_p + \frac{\partial A_i^s}{\partial y^k} \frac{\partial^2 A_j^p}{\partial y^l \partial y^n} \pi^{kl}(y) \eta_s \eta_p. \end{aligned} \quad (18)$$

Evaluating at the identity $y = 0$ we obtain

$$C_{sn}^l \eta_l \frac{\partial \omega_{ij}}{\partial \eta_s} = C_{in}^k \omega_{kj} + C_{jn}^l \omega_{il} - C_{ik}^s C_{jl}^p \alpha_n^{kl} \eta_s \eta_p, \quad (19)$$

where $\alpha_n^{kl} = \frac{\partial \pi^{kl}}{\partial y^n}(0)$ and we have used formulae $A_j^i(0) = \delta_j^i$, $\frac{\partial A_j^i}{\partial y^n}(0) = C_{jn}^i$, and $\pi^{kl}(0) = 0$. We now seek solutions of the above system of differential equations.

Remark 2.2 *If the Poisson tensor π on G were zero, then (19) reduces to*

$$C_{sn}^l \eta_l \frac{\partial \omega_{ij}}{\partial \eta_s} = C_{in}^k \omega_{kj} + C_{jn}^l \omega_{il}. \quad (20)$$

It is immediate that the linear bracket $\omega_{ij}(\eta) = C_{ij}^s \eta_s$ satisfies (20). Indeed, after substituting into (20) we obtain

$$\left(C_{ln}^s C_{ij}^l + C_{ni}^l C_{lj}^s + C_{jn}^l C_{li}^s \right) \eta_s = 0,$$

which is identically satisfied. Moreover, any tensor of the form $\omega_{ij}(\eta) = C_{ij}^s \eta_s \Theta(\eta)$ is a solution of (20) as long as Θ is a solution of the system

$$C_{sn}^l C_{ij}^k \eta_l \eta_k \frac{\partial \Theta}{\partial \eta_s} = 0. \quad (21)$$

The general solution of (19) is a linear combination of the general solution of the homogeneous system (20) and a particular solution. We look for a particular solution in the form

$$\omega_{ij}(\eta) = \beta_{ij}^{qr} \eta_q \eta_r, \quad (22)$$

where $\beta_{ij}^{qr} = -\beta_{ji}^{rq}$ are symmetric in the upper and skew-symmetric in the lower indices. Substituting into (19) we obtain

$$\left[C_{sn}^q \beta_{ij}^{sr} + C_{sn}^r \beta_{ij}^{qs} - C_{in}^s \beta_{sj}^{qr} - C_{jn}^s \beta_{is}^{qr} - \frac{1}{2} \left(C_{is}^q C_{jl}^r + C_{is}^r C_{jl}^q \right) \alpha_n^{sl} \right] \eta_q \eta_r = 0,$$

which leads to

$$C_{sn}^q \beta_{ij}^{sr} + C_{sn}^r \beta_{ij}^{qs} - C_{in}^s \beta_{sj}^{qr} - C_{jn}^s \beta_{is}^{qr} - \frac{1}{2} \left(C_{is}^q C_{jl}^r + C_{is}^r C_{jl}^q \right) \alpha_n^{sl} = 0. \quad (23)$$

Remark 2.3 The most general tensor β_{ij}^{qr} symmetric in the upper and skew-symmetric in the lower indices that one can construct out of the tensors α_n^{ij} and C_{ij}^k is

$$\beta_{ij}^{kl} = a \left[\alpha_i^{ks} C_{sj}^l - \alpha_j^{ks} C_{si}^l + \alpha_i^{ls} C_{sj}^k - \alpha_j^{ls} C_{si}^k \right], \quad (24)$$

where a is a constant scalar. However, (24) falls short of satisfying (23) by the term

$$-a C_{ij}^m [\alpha_n^{qs} C_{ms}^r + \alpha_n^{rs} C_{ms}^q], \quad (25)$$

where $a = -1/4$. The tensors α_n^{ij} and C_{ij}^k are related by (14).

In the case when α is a coboundary and given by (15) equation (23) reads

$$C_{sn}^q \beta_{ij}^{sr} + C_{sn}^r \beta_{ij}^{qs} - C_{in}^s \beta_{sj}^{qr} - C_{jn}^s \beta_{is}^{qr} - \frac{1}{2} \left(C_{is}^q C_{jl}^r + C_{is}^r C_{jl}^q \right) (C_{np}^s r^{pl} + C_{np}^l r^{sp}) = 0, \quad (26)$$

and (24) reduces to

$$\beta_{ij}^{kl} = \frac{1}{2} \left(C_{ip}^k C_{js}^l + C_{ip}^l C_{js}^k \right) r^{sp} + \frac{1}{4} C_{ij}^s \left[C_{sm}^l r^{km} + C_{sm}^k r^{lm} \right]. \quad (27)$$

It turns out that only the first half in the above formula yields a solution of (26).

Proposition 2.1 The tensor

$$\beta_{ij}^{kl} = \frac{1}{2} \left(C_{ip}^k C_{js}^l + C_{ip}^l C_{js}^k \right) r^{sp}. \quad (28)$$

is a solution of (26). Moreover, $\omega_{ij}(\eta) = \beta_{ij}^{qr} \eta_q \eta_r$ is a Poisson tensor, that is, it satisfies the Jacobi identities

$$\omega_{ij} \frac{\partial \omega_{kl}}{\partial \eta_i} + \omega_{ik} \frac{\partial \omega_{lj}}{\partial \eta_i} + \omega_{il} \frac{\partial \omega_{jk}}{\partial \eta_i} = 0, \quad (29)$$

if and only if r is a solution of the Classical Yang-Baxter Equation:

$$C_{sp}^n r^{sj} r^{pl} + C_{sp}^j r^{sl} r^{pn} + C_{sp}^l r^{sn} r^{pj} = 0, \quad (30)$$

provided that a special linear map from $\mathcal{G}^{\otimes 3}$ to $\mathcal{G}^{\otimes 3} \otimes \mathcal{G}^{*\otimes 3}$ has a zero kernel.

Proof: The proof is a straightforward calculation. First, substituting β_{ij}^{kl} from (28) into the left hand side of (26) and rearranging terms we obtain:

$$\begin{aligned} & \left[C_{ip}^s C_{sn}^q + C_{ni}^s C_{sp}^q + C_{pn}^s C_{si}^q \right] C_{jl}^r r^{lp} + \left[C_{ip}^s C_{sn}^r + C_{ni}^s C_{sp}^r + C_{pn}^s C_{si}^r \right] C_{jl}^q r^{lp} \\ & + \left[C_{jl}^s C_{sn}^r + C_{nj}^s C_{sl}^r + C_{ln}^s C_{sj}^r \right] C_{ip}^q r^{lp} + \left[C_{jl}^s C_{sn}^q + C_{nj}^s C_{sl}^q + C_{ln}^s C_{sj}^q \right] C_{ip}^r r^{lp}, \end{aligned}$$

which is identically equal to zero. To prove that $\omega_{ij}(\eta) = \beta_{ij}^{qr} \eta_q \eta_r$ is Poisson we note that the Jacobi identities (after symmetrization) are equivalent to the following identities for the components of β :

$$\beta_{si}^{qr} \beta_{jk}^{sm} + \beta_{si}^{rm} \beta_{jk}^{sq} + \beta_{si}^{mq} \beta_{jk}^{sr} + \text{cyclic}(i, j, k) = 0. \quad (31)$$

After a lengthy calculation, with β given by (28), and using only the fact that the r-matrix is skew-symmetric, $r^{kl} = -r^{lk}$, and the identities

$$C_{ip}^s C_{sn}^q + C_{ni}^s C_{sp}^q + C_{pn}^s C_{si}^q = 0 \quad (32)$$

for the structure constants C_{ij}^k of the group, we obtain from (31) the equations:

$$\begin{aligned} & \left[C_{js}^q (C_{ku}^m C_{iw}^r + C_{ku}^r C_{iw}^m) + \text{cyclic}(q, m, r) \right] \\ & \times [C_{vt}^s r^{uv} r^{wt} + C_{vt}^w r^{sv} r^{ut} + C_{vt}^u r^{wv} r^{st}] = 0. \end{aligned} \quad (33)$$

Let us sketch the major steps of the calculation. Substitution of (28) into the left hand side of (31) results in a sum of 36 terms which we group into 9 groups each consisting of 4 summands:

$$C_{sv}^q C_{jt}^s [C_{iu}^r C_{kw}^m r^{uv} r^{wt} + C_{iu}^m C_{kw}^r r^{uv} r^{wt} - C_{kw}^r C_{iu}^m r^{wv} r^{ut} - C_{kw}^m C_{iu}^r r^{wv} r^{ut}] \quad (34)$$

$$+ C_{sv}^q C_{kw}^s [C_{iu}^r C_{jt}^m r^{uv} r^{wt} + C_{iu}^m C_{jt}^r r^{uv} r^{wt} - C_{jt}^r C_{iu}^m r^{uw} r^{vt} - C_{jt}^m C_{iu}^r r^{uw} r^{vt}] \quad (35)$$

$$+ C_{sv}^r C_{jt}^s [C_{iu}^q C_{kw}^m r^{uv} r^{wt} + C_{iu}^m C_{kw}^q r^{uv} r^{wt} - C_{kw}^q C_{iu}^m r^{wv} r^{ut} - C_{kw}^m C_{iu}^q r^{wv} r^{ut}] \quad (36)$$

$$+ C_{sv}^r C_{kw}^s [C_{iu}^q C_{jt}^m r^{uv} r^{wt} + C_{iu}^m C_{jt}^q r^{uv} r^{wt} - C_{jt}^q C_{iu}^m r^{uw} r^{vt} - C_{jt}^m C_{iu}^q r^{uw} r^{vt}] \quad (37)$$

$$+ C_{sv}^m C_{jt}^s [C_{iu}^r C_{kw}^q r^{uv} r^{wt} + C_{iu}^q C_{kw}^r r^{uv} r^{wt} - C_{kw}^q C_{iu}^r r^{wv} r^{ut} - C_{kw}^r C_{iu}^q r^{wv} r^{ut}] \quad (38)$$

$$+ C_{sv}^m C_{kw}^s [C_{iu}^r C_{jt}^q r^{uv} r^{wt} + C_{iu}^q C_{jt}^r r^{uv} r^{wt} - C_{jt}^q C_{iu}^r r^{tv} r^{wu} - C_{jt}^r C_{iu}^q r^{tv} r^{wu}] \quad (39)$$

$$+ C_{sv}^q C_{iw}^s [C_{ju}^r C_{kt}^m r^{uv} r^{wt} + C_{ju}^m C_{kt}^r r^{uv} r^{wt} - C_{kt}^r C_{ju}^m r^{tv} r^{wu} - C_{kt}^m C_{ju}^r r^{tv} r^{wu}] \quad (40)$$

$$+ C_{sv}^r C_{iw}^s [C_{ju}^q C_{kt}^m r^{uv} r^{wt} + C_{ju}^m C_{kt}^q r^{uv} r^{wt} - C_{kt}^q C_{ju}^m r^{tv} r^{wu} - C_{kt}^m C_{ju}^q r^{tv} r^{wu}] \quad (41)$$

$$+ C_{sv}^m C_{iw}^s [C_{ju}^r C_{kt}^q r^{uv} r^{wt} + C_{ju}^q C_{kt}^r r^{uv} r^{wt} - C_{kt}^q C_{ju}^r r^{tv} r^{wu} - C_{kt}^r C_{ju}^q r^{tv} r^{wu}]. \quad (42)$$

Each of the above 9 expressions is further transformed to an expression of only 2 summands. We describe in detail how this is done for the first of the above groups (34). Using the Jacobi identities for the structure constants of \mathcal{G}

$$C_{sv}^q C_{jt}^s + C_{sj}^q C_{tv}^s + C_{st}^q C_{vj}^s = 0 \quad \Longleftrightarrow \quad C_{sv}^q C_{jt}^s = -C_{sj}^q C_{tv}^s - C_{st}^q C_{vj}^s, \quad (43)$$

we transform (34) to

$$\begin{aligned} & \left(C_{sj}^q C_{tv}^s + C_{st}^q C_{vj}^s \right) \left[-C_{iu}^r C_{kw}^m r^{uv} r^{wt} - C_{iu}^m C_{kw}^r r^{uv} r^{wt} + C_{kw}^r C_{iu}^m r^{wv} r^{ut} + C_{kw}^m C_{iu}^r r^{wv} r^{ut} \right] \\ &= C_{sj}^q C_{kw}^m \left[-C_{tv}^s C_{iu}^r r^{uv} r^{wt} + C_{tv}^s C_{iu}^r r^{wv} r^{ut} \right] \end{aligned} \quad (44)$$

$$+ C_{sj}^q C_{kw}^r \left[-C_{tv}^s C_{iu}^m r^{uv} r^{wt} + C_{tv}^s C_{iu}^m r^{wv} r^{ut} \right] \quad (45)$$

$$+ C_{iu}^r C_{kw}^m \left[-C_{st}^q C_{vj}^s r^{uv} r^{wt} + C_{st}^q C_{vj}^s r^{wv} r^{ut} \right] \quad (46)$$

$$+ C_{kw}^r C_{iu}^m \left[-C_{st}^q C_{vj}^s r^{uv} r^{wt} + C_{st}^q C_{vj}^s r^{wv} r^{ut} \right]. \quad (47)$$

Each of the terms (44)–(47) is now transformed as follows: for (44) we have

$$\begin{aligned} & C_{sj}^q C_{kw}^m \left[-C_{tv}^s C_{iu}^r r^{uv} r^{wt} + C_{tv}^s C_{iu}^r r^{wv} r^{ut} \right] \\ &= C_{sj}^q C_{kw}^m \left[-C_{tv}^s C_{iu}^r r^{uv} r^{wt} + C_{vt}^s C_{iu}^r r^{uv} r^{wt} \right] \\ &= C_{sj}^q C_{kw}^m \left[C_{vt}^s C_{iu}^r r^{uv} r^{wt} + C_{vt}^s C_{iu}^r r^{uv} r^{wt} \right] \\ &= 2C_{sj}^q C_{kw}^m C_{vt}^s C_{iu}^r r^{uv} r^{wt}; \end{aligned} \quad (48)$$

for (45) we have

$$\begin{aligned} & C_{sj}^q C_{kw}^r \left[-C_{tv}^s C_{iu}^m r^{uv} r^{wt} + C_{tv}^s C_{iu}^m r^{wv} r^{ut} \right] \\ &= C_{sj}^q C_{kw}^r \left[-C_{tv}^s C_{iu}^m r^{uv} r^{wt} + C_{vt}^s C_{iu}^m r^{uv} r^{wt} \right] \\ &= C_{sj}^q C_{kw}^r \left[C_{vt}^s C_{iu}^m r^{uv} r^{wt} + C_{vt}^s C_{iu}^m r^{uv} r^{wt} \right] \\ &= 2C_{sj}^q C_{kw}^r C_{iu}^m C_{vt}^s r^{uv} r^{wt}; \end{aligned} \quad (49)$$

for (46) we have

$$\begin{aligned} & C_{iu}^r C_{kw}^m \left[-C_{st}^q C_{vj}^s r^{uv} r^{wt} + C_{st}^q C_{vj}^s r^{wv} r^{ut} \right] \\ &= C_{iu}^r C_{kw}^m \left[-C_{st}^q C_{vj}^s r^{uv} r^{wt} + C_{sv}^q C_{tj}^s r^{wt} r^{uv} \right] \\ &= C_{iu}^r C_{kw}^m \left[-C_{st}^q C_{vj}^s - C_{sv}^q C_{jt}^s \right] r^{uv} r^{wt} \\ &= -C_{iu}^r C_{kw}^m C_{sj}^q C_{vt}^s r^{uv} r^{wt}; \end{aligned} \quad (50)$$

and finally for (47) we have

$$\begin{aligned} & C_{kw}^r C_{iu}^m \left[-C_{st}^q C_{vj}^s r^{uv} r^{wt} + C_{st}^q C_{vj}^s r^{wv} r^{ut} \right] \\ &= C_{kw}^r C_{iu}^m \left[-C_{st}^q C_{vj}^s r^{uv} r^{wt} + C_{sv}^q C_{tj}^s r^{uv} r^{wt} \right] \\ &= C_{kw}^r C_{iu}^m \left[-C_{st}^q C_{vj}^s - C_{sv}^q C_{jt}^s \right] r^{uv} r^{wt} \\ &= -C_{kw}^r C_{iu}^m C_{sj}^q C_{vt}^s r^{uv} r^{wt}. \end{aligned} \quad (51)$$

Adding together (48)–(51) we obtain for (34) the following expression

$$(34) = C_{sj}^q C_{vt}^s C_{kw}^m C_{iu}^r r^{uv} r^{wt} + C_{sj}^q C_{vt}^s C_{kw}^r C_{iu}^m r^{uv} r^{wt}.$$

Performing analogous manipulations as above with each of the terms (35)–(42) we deduce that the left hand side of (31) is equivalent to

$$C_{sj}^q C_{vt}^s C_{kw}^m C_{iu}^r r^{uv} r^{wt} + C_{sj}^q C_{vt}^s C_{kw}^r C_{iu}^m r^{uv} r^{wt} \quad (52)$$

$$+ C_{sk}^q C_{vw}^s C_{iu}^r C_{jt}^m r^{uv} r^{wt} + C_{sk}^q C_{vw}^s C_{iu}^m C_{jt}^r r^{uv} r^{wt} \quad (53)$$

$$+ C_{sj}^r C_{vt}^s C_{kw}^m C_{iu}^q r^{uv} r^{wt} + C_{sj}^r C_{vt}^s C_{iu}^m C_{kw}^q r^{uv} r^{wt} \quad (54)$$

$$+ C_{sk}^r C_{vw}^s C_{iu}^q C_{jt}^m r^{uv} r^{wt} + C_{sk}^r C_{vw}^s C_{iu}^m C_{jt}^q r^{uv} r^{wt} \quad (55)$$

$$+ C_{sj}^m C_{iw}^r C_{tv}^s C_{ku}^q r^{uv} r^{wt} + C_{sj}^m C_{iw}^q C_{tv}^s C_{ku}^r r^{uv} r^{wt} \quad (56)$$

$$+ C_{sk}^m C_{jt}^q C_{vw}^s C_{iu}^r r^{uv} r^{wt} + C_{sk}^m C_{iu}^q C_{vw}^s C_{jt}^r r^{uv} r^{wt} \quad (57)$$

$$+ C_{si}^q C_{ju}^r C_{vw}^s C_{kt}^m r^{uv} r^{wt} + C_{si}^q C_{ju}^m C_{vw}^s C_{kt}^r r^{uv} r^{wt} \quad (58)$$

$$+ C_{si}^r C_{ju}^q C_{vw}^s C_{kt}^m r^{uv} r^{wt} + C_{si}^r C_{ju}^m C_{vw}^s C_{kt}^q r^{uv} r^{wt} \quad (59)$$

$$+ C_{si}^m C_{ju}^r C_{vw}^s C_{kt}^q r^{uv} r^{wt} + C_{si}^m C_{ju}^q C_{vw}^s C_{kt}^r r^{uv} r^{wt}, \quad (60)$$

where each of the pair of terms in (52)–(60) are obtained from the quadruples of terms in (34)–(42) correspondingly. We now rearrange the above 18 terms in the following 6 groups each consisting of 3 terms:

$$C_{sj}^q C_{vt}^s C_{kw}^m C_{iu}^r r^{uv} r^{wt} + C_{sk}^m C_{jt}^q C_{vw}^s C_{iu}^r r^{uv} r^{wt} + C_{si}^r C_{ju}^q C_{vw}^s C_{kt}^m r^{uv} r^{wt} \quad (61)$$

$$+ C_{sj}^q C_{vt}^s C_{kw}^r C_{iu}^m r^{uv} r^{wt} + C_{sk}^r C_{vw}^s C_{iu}^m C_{jt}^q r^{uv} r^{wt} + C_{si}^m C_{ju}^q C_{vw}^s C_{kt}^r r^{uv} r^{wt} \quad (62)$$

$$+ C_{sj}^r C_{vt}^s C_{kw}^m C_{iu}^q r^{uv} r^{wt} + C_{sk}^m C_{iu}^q C_{vw}^s C_{jt}^r r^{uv} r^{wt} + C_{si}^q C_{ju}^r C_{vw}^s C_{kt}^m r^{uv} r^{wt} \quad (63)$$

$$+ C_{sj}^r C_{vt}^s C_{iu}^m C_{kw}^q r^{uv} r^{wt} + C_{sk}^q C_{vw}^s C_{iu}^m C_{jt}^r r^{uv} r^{wt} + C_{si}^m C_{ju}^r C_{vw}^s C_{kt}^q r^{uv} r^{wt} \quad (64)$$

$$+ C_{sk}^q C_{vw}^s C_{iu}^r C_{jt}^m r^{uv} r^{wt} + C_{sj}^m C_{iw}^r C_{tv}^s C_{ku}^q r^{uv} r^{wt} + C_{si}^r C_{ju}^m C_{vw}^s C_{kt}^q r^{uv} r^{wt} \quad (65)$$

$$+ C_{sk}^r C_{vw}^s C_{iu}^q C_{jt}^m r^{uv} r^{wt} + C_{sj}^m C_{iw}^q C_{tv}^s C_{ku}^r r^{uv} r^{wt} + C_{si}^q C_{ju}^m C_{vw}^s C_{kt}^r r^{uv} r^{wt}. \quad (66)$$

The 3 terms in each of the above 6 groups we manipulate further. We show the steps for (61). Thus, for (61) we have

$$\begin{aligned} & C_{sj}^q C_{vt}^s C_{kw}^m C_{iu}^r r^{uv} r^{wt} + C_{sk}^m C_{jt}^q C_{vw}^s C_{iu}^r r^{uv} r^{wt} + C_{si}^r C_{ju}^q C_{vw}^s C_{kt}^m r^{uv} r^{wt} \\ &= C_{sj}^q C_{vt}^s C_{kw}^m C_{iu}^r r^{uv} r^{wt} + C_{wk}^m C_{js}^q C_{vt}^r C_{iu}^s r^{uv} r^{ts} + C_{ui}^r C_{js}^q C_{vt}^u C_{kw}^m r^{sv} r^{tw} \\ &= C_{sj}^q C_{vt}^s C_{kw}^m C_{iu}^r r^{uv} r^{wt} + C_{kw}^m C_{sj}^q C_{vt}^r C_{iu}^s r^{uv} r^{ts} + C_{iu}^r C_{sj}^q C_{vt}^u C_{kw}^m r^{sv} r^{tw} \\ &= C_{sj}^q C_{kw}^m C_{iu}^r [C_{vt}^s r^{uv} r^{wt} + C_{vt}^w r^{uv} r^{ts} + C_{vt}^u r^{sv} r^{tw}] \\ &= C_{sj}^q C_{kw}^m C_{iu}^r [C_{vt}^s r^{uv} r^{wt} + C_{vt}^w r^{sv} r^{ut} + C_{vt}^u r^{wv} r^{st}] \\ &= C_{sj}^q C_{ku}^m C_{iw}^r [C_{vt}^s r^{wv} r^{ut} + C_{vt}^u r^{sv} r^{wt} + C_{vt}^w r^{uv} r^{st}] \\ &= C_{js}^q C_{ku}^m C_{iw}^r [C_{vt}^s r^{wt} r^{uv} + C_{vt}^u r^{st} r^{wv} + C_{vt}^w r^{ut} r^{sv}]. \end{aligned} \quad (67)$$

In a completely similar way we obtain for (62)–(66) the following expressions:

$$(62) = C_{js}^q C_{iw}^m C_{ku}^r [C_{vt}^s r^{wt} r^{uv} + C_{vt}^u r^{st} r^{wv} + C_{vt}^w r^{ut} r^{sv}], \quad (68)$$

$$(63) = C_{js}^r C_{ku}^m C_{iw}^q [C_{vt}^s r^{wt} r^{uv} + C_{vt}^u r^{st} r^{wv} + C_{vt}^w r^{ut} r^{sv}], \quad (69)$$

$$(64) = C_{js}^r C_{iw}^m C_{ku}^q [C_{vt}^s r^{wt} r^{uv} + C_{vt}^u r^{st} r^{wv} + C_{vt}^w r^{ut} r^{sv}], \quad (70)$$

$$(65) = C_{js}^m C_{ku}^q C_{iw}^r [C_{vt}^s r^{wt} r^{uv} + C_{vt}^u r^{st} r^{wv} + C_{vt}^w r^{ut} r^{sv}], \quad (71)$$

$$(66) = C_{js}^m C_{iw}^q C_{ku}^r [C_{vt}^s r^{wt} r^{uv} + C_{vt}^u r^{st} r^{wv} + C_{vt}^w r^{ut} r^{sv}]. \quad (72)$$

Finally, adding (67)–(72) we obtain (33).

Thus, we arrive at the following dichotomy. The “if” part in the statement of the proposition follows immediately from (33). The “only if” part follows from (33), provided the linear map $\mathcal{C} : \mathcal{G}^{\otimes 3} \rightarrow \mathcal{G}^{\otimes 3} \otimes \mathcal{G}^{*\otimes 3}$ with matrix components

$$\begin{aligned} \mathcal{C}_{sj,kw,iu}^{qmr} &= C_{js}^q (C_{ku}^m C_{iw}^r + C_{ku}^r C_{iw}^m) \\ &\quad + C_{js}^r (C_{ku}^m C_{iw}^q + C_{ku}^q C_{iw}^m) + C_{js}^m (C_{ku}^q C_{iw}^r + C_{ku}^r C_{iw}^q), \end{aligned} \quad (73)$$

has a zero kernel. We do not know how to interpret geometrically this condition on the group G . On the other hand, if G is such that $\mathcal{C}_{sj,kw,iu}^{qmr} = 0$, then any skew-symmetric matrix r induces a Poisson structure on \mathcal{G}^* , given by (28). This concludes the proof. ■

Two Poisson tensors $\omega_{ij}^{(1)}$ and $\omega_{ij}^{(2)}$ are said to form a *Poisson pair* if their linear combination $a\omega_{ij}^{(1)} + b\omega_{ij}^{(2)}$ is also a Poisson tensor for arbitrary constants a and b . It is easy to see that the Poisson tensors $\omega_{ij}^{(1)}$ and $\omega_{ij}^{(2)}$ form a Poisson pair if and only if

$$\omega_{ij}^{(1)} \frac{\partial \omega_{kl}^{(2)}}{\partial \eta_i} + \omega_{ik}^{(1)} \frac{\partial \omega_{lj}^{(2)}}{\partial \eta_i} + \omega_{il}^{(1)} \frac{\partial \omega_{jk}^{(2)}}{\partial \eta_i} + \omega_{ij}^{(2)} \frac{\partial \omega_{kl}^{(1)}}{\partial \eta_i} + \omega_{ik}^{(2)} \frac{\partial \omega_{lj}^{(1)}}{\partial \eta_i} + \omega_{il}^{(2)} \frac{\partial \omega_{jk}^{(1)}}{\partial \eta_i} = 0. \quad (74)$$

Proposition 2.2 *The Poisson tensors*

$$\omega_{ij}^{(1)} = C_{ij}^s \eta_s \Theta(\eta) \quad \text{and} \quad \omega_{ij}^{(2)} = 1/2 \left(C_{ip}^k C_{js}^l + C_{ip}^l C_{js}^k \right) r^{sp} \eta_k \eta_l$$

form a Poisson pair.

Proof: After substituting $\omega_{ij}^{(1)}$ and $\omega_{ij}^{(2)}$ in (74) and collecting terms we obtain

$$\begin{aligned} & (C_{uj}^s C_{kp}^u + C_{uk}^s C_{pj}^u + C_{up}^s C_{jk}^u) C_{lq}^v r^{qp} \eta_s \eta_v \\ & + C_{jq}^v r^{qp} \eta_v \left[C_{ip}^u C_{kl}^s \eta_u \eta_s \frac{\partial \Theta}{\partial \eta_i} \right] + \text{cyclic}(j, k, l) = 0. \end{aligned} \quad (75)$$

From (21) and (32) follows that this is an identity. ■

We can thus summarize the above results in the following theorem.

Theorem 2.1 *For any finite-dimensional connected simply connected Poisson-Lie group there exists the family of Poisson structures*

$$\omega_{ij}(\eta) = C_{ij}^s \eta_s \Theta(\eta) + C_{ip}^k C_{js}^l r^{sp} \eta_k \eta_l, \quad (76)$$

on the dual of its Lie algebra such that it makes the coadjoint action Poisson. Here Θ is an arbitrary invariant function on \mathcal{G}^ .*

Proof: What remains to be proved is that the solution of the infinitesimal part of (17) obtained above is actually an invariant Poisson bracket under the coadjoint action of the group. In other words we need to show that the tensor (76) satisfies the functional equation (17). For this we rewrite equation (17) in a new equivalent form. This is done by using the following properties of the Lie group G and the map Ad . Let $x : I \rightarrow G$ be a curve passing through the identity of G . We have

$$A_j^i(f(x(t), y)) = A_l^i(x(t))A_j^l(y). \quad (77)$$

Differentiating at $t = 0$ we obtain

$$\frac{\partial A_j^i}{\partial y^k}(y) dR_s^k(y) = C_{ls}^i A_j^l(y) \implies \frac{\partial A_j^i}{\partial y^k}(y) = C_{ls}^i A_j^l(y) dR_k^s(y^{-1}). \quad (78)$$

From the identity $A_s^i(y)A_j^s(y^{-1}) = \delta_j^i$ we have also (after differentiating in the coordinate directions)

$$\frac{\partial A_j^i}{\partial y^k}(y^{-1}) = -A_s^i(y^{-1}) \frac{\partial A_p^s}{\partial y^k}(y) A_j^p(y^{-1}). \quad (79)$$

We note also the invariance of the constant tensor C_{jk}^i :

$$C_{jk}^i = A_s^i(y) C_{pq}^s A_j^p(y^{-1}) A_k^q(y^{-1}). \quad (80)$$

Now we substitute (79) into the original functional equation (17) and after an easy calculation using (78) and (80) we transform it to the equivalent equation

$$\begin{aligned} \omega_{ij}(A(y^{-1})\eta) &= A_i^k(y^{-1}) A_j^l(y^{-1}) \omega_{kl}(\eta) \\ &\quad + C_{iq}^m C_{jt}^n A_m^s(y^{-1}) A_n^p(y^{-1}) \eta_s \eta_p dR_k^q(y^{-1}) dR_l^t(y^{-1}) \pi^{kl}(y). \end{aligned} \quad (81)$$

Using the relation (10) between the tensors π and σ we finally obtain

$$\omega_{ij}(A(y^{-1})\eta) = A_i^k(y^{-1}) A_j^l(y^{-1}) \omega_{kl}(\eta) + C_{iq}^m C_{jt}^n A_m^s(y^{-1}) A_n^p(y^{-1}) \eta_s \eta_p \sigma^{qt}(y). \quad (82)$$

With σ given by formula (16) it is now a straightforward calculation to verify that the tensor ω as given by formula (76) satisfies the functional equation (82). Indeed, the left hand side of (82) after substitution of (76) reads

$$C_{ij}^s A_p^p(y^{-1}) \eta_p \Theta(A(y^{-1})\eta) - C_{iu}^s C_{jv}^p r^{uv} A_s^m(y^{-1}) A_p^n(y^{-1}) \eta_m \eta_n. \quad (83)$$

The right hand side of (82) yields

$$\begin{aligned} &A_i^k(y^{-1}) A_j^l(y^{-1}) C_{kl}^s \eta_s \Theta(\eta) - A_i^k(y^{-1}) A_j^l(y^{-1}) C_{ku}^s C_{lv}^p r^{uv} \eta_s \eta_p \\ &+ C_{iq}^m C_{jt}^n A_m^s(y^{-1}) A_n^p(y^{-1}) \eta_s \eta_p A_u^q(y) r^{uv} A_v^t(y) - C_{iq}^m C_{jt}^n A_m^s(y^{-1}) A_n^p(y^{-1}) \eta_s \eta_p r^{qt}. \end{aligned} \quad (84)$$

Using

$$C_{iq}^m A_u^q(y) A_m^s(y^{-1}) \stackrel{(80)}{=} C_{qu}^s A_i^q(y^{-1}), \quad C_{jt}^n A_v^t(y) A_n^p(y^{-1}) \stackrel{(80)}{=} C_{mv}^p A_j^m(y^{-1}), \quad (85)$$

we transform the third term of (84) and after comparison of terms in the left and right hand sides we conclude that the functional equation (82) is identically satisfied. Thus, every solution r of the Classical Yang-Baxter Equation induces the Poisson structure (76) on \mathcal{G}^* making it a homogeneous Poisson space under the coadjoint action of G . This concludes the proof. \blacksquare

3 Discussion

Here we show that the newly obtained Poisson bracket on \mathcal{G}^* when specialized to the case of $G = GL(n)$ and $\mathcal{G}^* = gl(n)^*$ recovers the one obtained in [1]. Let x_i^α be the components of a matrix representation $x : GL(n) \rightarrow \text{Mat}(n)$ of $GL(n)$. The multiplication map $(x, y) \mapsto f(x, y)$ is given by

$$f_i^\alpha(x, y) = x_s^\alpha y_i^s. \quad (86)$$

Then it is easy to compute the structure constants

$$C_{i\beta\gamma}^{\alpha jk} = \left[\frac{\partial^2 f_i^\alpha}{\partial u_\beta^\gamma \partial v_k^\gamma}(0, 0) - \frac{\partial^2 f_i^\alpha}{\partial u_k^\gamma \partial v_\beta^\gamma}(0, 0) \right] = \delta_\beta^\alpha \delta_\gamma^j \delta_i^k - \delta_\gamma^\alpha \delta_\beta^k \delta_i^j. \quad (87)$$

With these structure constants and the r-matrix components $r_{ij}^{\alpha\beta} = -r_{ji}^{\beta\alpha}$ we have

$$\begin{aligned} \left\{ \eta_\beta^j, \eta_\gamma^k \right\} &= C_{i\beta\gamma}^{\alpha jk} \eta_\alpha^i \Theta(\eta) + C_{i\gamma\omega}^{\alpha km} C_{l\beta\lambda}^{\epsilon jn} \eta_\alpha^i \eta_\epsilon^l r_{nm}^{\lambda\omega} \\ &= (\delta_\beta^\alpha \delta_\gamma^j \delta_i^k - \delta_\gamma^\alpha \delta_\beta^k \delta_i^j) \eta_\alpha^i \Theta(\eta) + (\delta_\gamma^\alpha \delta_\omega^k \delta_i^m - \delta_\omega^\alpha \delta_\gamma^m \delta_i^k) (\delta_\beta^\epsilon \delta_\lambda^j \delta_l^n - \delta_\lambda^\epsilon \delta_\beta^n \delta_l^j) \eta_\alpha^i \eta_\epsilon^l r_{nm}^{\lambda\omega} \\ &= (\delta_\gamma^j \eta_\beta^k - \delta_\beta^k \eta_\gamma^j) \Theta(\eta) + (\delta_\omega^k \eta_\gamma^m - \delta_\gamma^m \eta_\omega^k) (\delta_\lambda^j \eta_\beta^n - \delta_\beta^n \eta_\lambda^j) r_{nm}^{\lambda\omega} \\ &= (\delta_\gamma^j \eta_\beta^k - \delta_\beta^k \eta_\gamma^j) \Theta(\eta) + r_{\beta\gamma}^{\lambda\omega} \eta_\omega^k \eta_\lambda^j + r_{nm}^{jk} \eta_\beta^n \eta_\gamma^m - r_{\beta m}^{\lambda k} \eta_\gamma^m \eta_\lambda^j - r_{n\gamma}^{j\omega} \eta_\omega^k \eta_\beta^n, \end{aligned} \quad (88)$$

which is in agreement with the formula obtained in [1].

Since there is a canonical isomorphism $T^*G \simeq G \times \mathcal{G}^*$ and the Poisson tensor $\omega_{ij}^{(2)}$ on \mathcal{G}^* forms a Poisson pair with the linear tensor $\omega_{ij}^{(1)}$ on \mathcal{G}^* , the corresponding tensors on T^*G will also form a Poisson pair under this isomorphism. It will be interesting to construct quantizations of the cotangent space $T^*G \simeq G \times \mathcal{G}^*$ and the group G and lift the coadjoint Poisson action to an equivariant quantum action between non-commutative spaces in the quantum case. We hope to address this problem in a future publication.

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