Application Of Radial Basis Collocation Method For Infiltration And Drainage Problem

Xinqiang Qin, Xin Deng, Xiaohong Tong, Yi Wang
School of Science
Xi'an University of Technology
Xi'an, Shaanxi China
e-mail: xqqin2007@163.com

Abstract—**The meshless method is a new kind numerical method which does not require the division of the grid. In this paper, a meshless method based on collocation method with radial basis functions is developed for the farmland drainage problems. The method is applied to solve the one-dimensional unstable flow equations that under uniform intensity of infiltration. The numerical solutions are got through iteration to the equations. A comparison of the new method with the finite element method through practical calculation examples shows that the new method can be used very well on solving the farmland drainage problems. It has high precision, good convergence and strong practicability.**

Keywords-Infiltration; Drainage; Groundwater; Radial basis; Meshless

I. INTRODUCTION

The farmland drainage under infiltration is an unstable process [1-3], and infiltration has great influence on ground-water [4]. Groundwater level fluctuates over time. It depends on the intensity of infiltration. Groundwater level rises up when infiltration amount over displacement. Groundwater level falls down when infiltration amount less than displacement. Groundwater level keeps invariant when infiltration amount equals to displacement. That is, the level converts unstable state to and from stable state. Thus, groundwater status under infiltration is relatively complicated. In agriculture production, we always seek how groundwater level will be under the influence of a given infiltration amount, rise up or fall down?

Generally, mathematical description of above phenomena can be attributed to nonlinear partial differential equation, and two means are adopted to solve the problem in practical application: analytical method and numerical method. Solving the mathematical model with analytical method can get exact solution, but it does not possess universality, particular for nonlinear differential equation, so it is difficult to find out the analytical solution of mathematical model which describes actual groundwater system, and numerical methods are widely used in current research work. The meshless method [5], which supplementing and developing the traditional numerical methods (e.g., difference method, finite element method et al.), is a new kind numerical method at present. It is an approximation method based on nodes, and doesn't need a mesh entirely or partly, then the problems can be solved without the initial partition and remeshing technique [6-9]. Not only can it ensure the precision, but also it reduces the difficulty of calculation. Therefore, in this paper the meshless based on collocation method with radial basis functions is introduced into the study of farmland drainage problem, the feasibility and accuracy of the new method is verified by practical calculation examples, and a good convergence effect has been obtained, so it has certain practical value in the study of farmland drainage problem.

II. MATHMATICAL MODEL

When the infiltration at a uniform intensity, the Boussinesq equation of one-dimensional drainage problem is,

$$\frac{\partial h}{\partial t}-\frac{k}{\mu}\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)\right]=\frac{\omega}{\mu} \tag{1}$$

with the initial condition

$$h(x,0)=H_0 \quad 0\le x\le L \tag{2α}$$

and boundary conditions

$$h(0,t)=h_0 \qquad t\ge 0 \tag{2β}$$

$$h(L,t)=h_0 \qquad t\ge 0 \tag{2χ}$$

Where h is groundwater level with x-axis as the base (m), k is permeability coefficient (m/d), μ is soil water specific yield, ω is intensity of uniform infiltration (m/d), t is time of drainage. The physical meaning of the linear model is that when infiltrating uniformly, both ditch-water level and river level maintains the same position with h_0. Groundwater flow towards drainage ditch when infiltrating, and groundwater level is influenced by drainage ditch and infiltration. In order to determine whether groundwater level fluctuates over time, we just need to study the fluctuation change of arbitrary point on groundwater level. Here, we choose the midpoint of drains spacing and study the change law of its groundwater level.

III. NUMERICAL METHOD

A truly meshless method based on collocation with radial basis functions comprises the main focus of this paper, and the radial functions are chosen to represent the solutions of the PDEs in this paper. With approach of collocation with radial basis functions, a partition of the domain is not needed and it only needs the computation of shape functions and their derivatives, while with the finite element method it requires the calculation of the relevant integrals that often lowers efficiency of computation significantly. Moreover, boundary conditions are easier to be achieved without special manipulation. This paper Gaussian function $\varphi_j(x)=\exp(-cr^2)(c>0)$ is selected for computing, where variable r_j is defined with $r_j=\|x-x_j\|$.

Discrete the computational region with N nodes, set x as calculation point, then in the region function $h(x)$ can be described as following form:

$$h(x)\approx\tilde{h}(x)=\sum_{j=1}^{N}\alpha_j^n\phi_j(x) \tag{3}$$

Where α_j is undetermined coefficients, $\varphi_j(x)$ is radial basis function, k, μ, H_0, h_0 can be got by practical tests.

Because quadratic term in (1) is nonlinear, collocation method can not be used directly. Thus, the quadratic term is treated with second centered difference:

$$\frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x}\right]_{x=x_i}^{n}\approx\frac{1}{\Delta x}\left\{\left[h\frac{\partial h}{\partial x}\right]_{x=x_{i+\frac{1}{2}}}^{n}-\left[h\frac{\partial h}{\partial x}\right]_{x=x_{i-\frac{1}{2}}}^{n}\right\}=\frac{1}{\Delta x}\left\{\left[h_{i+\frac{1}{2}}\frac{\partial h}{\partial x}\right]_{x=x_{i+\frac{1}{2}}}^{n}-\left[h_{i-\frac{1}{2}}\frac{\partial h}{\partial x}\right]_{x=x_{i-\frac{1}{2}}}^{n}\right\} \tag{4}$$

Discrete the left side of (1) with forward difference, then

$$\left(\frac{\partial h}{\partial t}\right)_{x=x_i}^{t^n}\approx\left(\frac{h^n-h^n}{\partial t}\right)_{x=x_i}=\frac{h_i^n-h_i^{n-1}}{\partial t} \tag{5}$$

Let $\tilde{h}(x,t^n)$ be an approximation of $h(x,t^n)$ as follows:

$$\tilde{h}(x,t^n)=\sum_{j=1}^{N}\alpha_j^n\phi(\|x-x_j\|)=\sum_{j=1}^{N}\alpha_j^n\phi_j(x) \tag{6}$$

Applying collocation method to (1), thus the discrete form is given

$$\begin{cases}\sum_{j=1}^{N}\alpha_j^n\left\{\phi_j(x_i)-\frac{k}{\mu}\frac{\Delta t}{\Delta x}\left[h_{i+\frac{1}{2}}^n\left(\frac{\partial\phi(x)}{\partial x}\right)_{i+\frac{1}{2}}^n-h_{i-\frac{1}{2}}^n\left(\frac{\partial\phi(x)}{\partial x}\right)_{i-\frac{1}{2}}^n\right]\right\}=h_i^{n-1}+\Delta t f_i^n \\ \qquad i=2,3,\cdots N-1 \\ h(x,0)=H_0 \\ \sum_{j=1}^{N}\alpha_j^n\phi_j(0)=\sum_{j=1}^{N}\alpha_j^n\phi_j(L)=h_0^n\end{cases} \tag{7}$$

Where $f=\frac{\omega}{\mu}$, x_i is boundary point when $i=1,N$ and x_i is interior point when $i=2,3,\ldots,N-1$, Δx is spatial interval, Δt is time interval. Let

$$\psi_j(x_i,h_i)=\phi_j(x_i)-\frac{k}{\mu}\frac{\Delta t}{\Delta x}\left[h_{i+\frac{1}{2}}^n\left(\frac{\partial\phi(x)}{\partial x}\right)_{i+\frac{1}{2}}^n-h_{i-\frac{1}{2}}^n\left(\frac{\partial\phi(x)}{\partial x}\right)_{i-\frac{1}{2}}^n\right] \tag{8}$$

$$F_i^n=h_i^{n-1}+\Delta t f_i^n \tag{9}$$

Then (7) can be expressed in the following matrix equation:

$$HU=F \tag{10}$$

Where

$$H=\begin{bmatrix}\phi_1(x_1) & \phi_N(x_1) & \phi_2(x_1) & \cdots & \phi_{N-1}(x_1)\\ \phi_1(x_N) & \phi_N(x_N) & \phi_2(x_N) & \cdots & \phi_{N-1}(x_N)\\ \psi_1(x_2,h_2) & \psi_N(x_2,h_2) & \psi_2(x_2,h_2) & \cdots & \psi_{N-1}(x_2,h_2)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \psi_1(x_{N-1},h_{N-1}) & \psi_N(x_{N-1},h_{N-1}) & \psi_2(x_{N-1},h_{N-1}) & \cdots & \psi_{N-1}(x_{N-1},h_{N-1})\end{bmatrix}$$

$$U=\left[\alpha_1^n,\alpha_N^n,\alpha_2^n,\cdots,\alpha_{N-1}^n\right]^T$$

$$F=\left[h_0^n,h_0^n,F_2^n,\cdots,F_{N-1}^n\right]^T$$

$$h_{i\pm\frac{1}{2}}=\frac{1}{2}\left(h_i+h_{i\pm1}\right)$$

IV. LINEARIZATION OF NONLINEAR PROBLEM

The nonlinear equation (10) is resolved with predictor-corrector scheme iteratively. Predictor-corrector scheme can be divided into prediction and correction. Prediction is to use value of step *n-1*(h_i^{n-1}) as value of h_i^n, then get $h_i^{n(1/2)}$ by calculating $h_{i-\frac{1}{2}}^{n-1}$, $h_{i+\frac{1}{2}}^{n-1}$ and plugging them into the iterative form. Correction is to get h_i^n by using $h_{i-\frac{1}{2}}^{n(1/2)}$ and $h_{i+\frac{1}{2}}^{n(1/2)}$, which they can be calculated with $h_i^{n(1/2)}$. The error estimate of nonlinear iteration should less than the maximum permissible error, that is:

$$\max_i \left| \frac{h_i^{n+1(q)} - h_i^{n+1(q-1)}}{h_i^{n+1(q-1)}} \right| \le e \quad (j = 1, 2, \cdots, N) \tag{11}$$

Where q is the number of iterations, e is the maximum permissible error (its value depends on precision of calculation, in this paper, let e=0.0001). Then $h_i^{n+1(q)}$ can be considered as the final correction value of $h_i^{n+1}(i = 1, 2, \cdots, N)$, that is, it is the numerical solution that we find.

V. NUMERICAL IMPLEMENTATION

Example1. For the infiltration-drainage problem as follows:

$$\frac{\partial h}{\partial t} - \frac{k}{\mu}\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)\right] = \frac{\omega}{\mu} \tag{12}$$

with the initial condition

$$h(x,0) = H_0 \quad 0 \le x \le 1 \tag{13α}$$

and boundary conditions

$$h(0,t) = h_0 \qquad t \ge 0 \tag{13β}$$

$$h(L,t) = h_0 \qquad t \ge 0 \tag{13χ}$$

Here, $h(x, t)=x(1-x)t$ is the exact solution, let k/μ=1.0, H_0, h_0 and ω/μ are determined by analytic solution. Gaussian function $\varphi(r)=\exp(-cr^2)(c>0)$ is selected as the radial basis function, time step Δt = 0.01 and spatial step h=0.01, parameter c=2.4. The error estimated based on L_2 Norms. The maximum permissible error of nonlinear iteration is 0.001. Fig.1 shows the comparison of numerical solution with exact solution, where t=0.2 and error is 2.5699e-004.

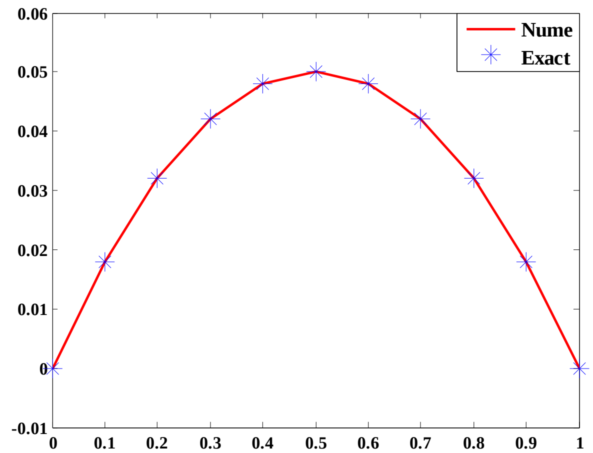

Figure 1. Schematic diagram of the drainage district (t=0.2)

Fig.1 shows that the new method has higher accuracy. For time step Δt=0.01 and different spatial step and different parameters c, Table 1 presents the CPU time and relative errors of the new method when t=0.2.

As can be seen from Table 1, the new method has small errors, good convergence and strong practicability.

Example2. Solving the infiltration-drainage problem in literature [4] by new method in this paper, with the initial condition

TABLE I. NUMERICAL SOLUTION OF THE NEW METHOD

spatial step	parameters c	Calculation time(s)	Calculation errors
0.1	2.3	0.180885	8.2953e-004
	2.4	0.369573	2.5699e-004
	2.5	0.137337	6.4727e-004
0.05	23	0.548021	2.0812e-004
	24	0.595426	1.5597e-004
	25	0.570836	1.9521e-004
0.025	149	2.023082	5.1587e-005
	150	2.019652	4.8772e-005
	151	2.081779	5.0778e-005

$$h(x,0) = 3.0 \quad 0 \le x \le 40 \tag{14α}$$

and boundary conditions

$$h(0,t) = 2.0 \qquad t \ge 0 \tag{14β}$$

$$h(40,t) = 2.0 \qquad t \ge 0 \tag{14χ}$$

Here, let k=1.0, μ=0.05, l=20, α=1.0(an integrate drainage ditch), T=5.0, ω=0.01, time step Δt = 0.01d, c_0=0.1, N=21. For the changes of groundwater level of midpoint over 0~7d, Table 2 and Fig.2 show the comparison between the numerical solution of the new method, finite element method and exact solution in literature [4].

TABLE II. SOLUTION OF THE GROUDWATER LEVEL RESOLVED BY DIFFERENT METHODS OVER 0-7D

	theoretical solution	**numerical solution**	**FEM solution**
t=0	3.00	3.00	3.00
t=1	3.1258	3.0829	3.0814
t=2	3.0631	3.0245	3.0175
t=3	3.0005	2.9750	2.9645
t=4	2.9535	2.9397	2.9278
t=5	2.9189	2.9148	2.8996
t=6	2.8935	2.8972	2.8821
t=7	2.8749	2.8847	2.8735

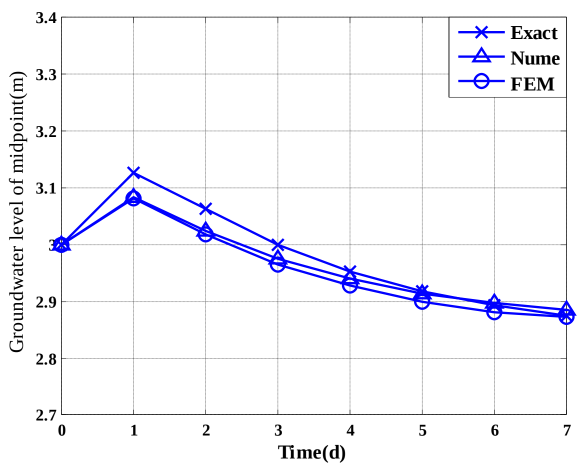

Figure 2. Groundwater level changes over time

For this example, the data given in Table 2 shows that: the coefficient of determination determined by the new method and theoretical method is 0.9996, the relative error is 0.0080, the root mean squared error is 0.0238; the coefficient of determination determined by the finite element method and theoretical method is 0.9995, the relative error is 0.0098, the root mean squared error is 0.0291. According to the Table 2 and Fig.2, the approximate solution by new method is closer to the exact solution when compared with the finite element method, the new method do a pretty good job of simulating the drain problem under infiltration.

VI. CONCLUSION

In this paper, the radial basis collocation method was established to solve the farmland drainage problems. The numerical examples showed that the method can simulate the actual situation, and can improve computational efficiency. Compared to the traditional finite element method, the new method shows obvious benefits in improved computing efficiency and accuracy. Through numerical analysis, we found that the selection of the interpolation nodes, the radial basis functions and the free parameter c are the factors which influence the accuracy of the solution.

It should be noted that the intensity of infiltration in this paper is assumed as a constant in time. Also, intensity of infiltration can be calculated phase-by-phase if it changes in separated time periods.

ACKNOWLEDGMENT

This paper is supported by Research Fund of Shaanxi Province (2012JM1008).The authors are very grateful to other staffs for their suggestions which have improved the paper.

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