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Versal Deformations of a Dirac Type Differential Operator

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Abstract

If we are given a smooth differential operator in the variable $x \in \mathbb{R}/2\pi\mathbb{Z}$, its normal form, as is well known, is the simplest form obtainable by means of the $\mathrm{Diff}(S^1)$-group action on the space of all such operators. A versal deformation of this operator is a normal form for some parametric infinitesimal family including the operator. Our study is devoted to analysis of versal deformations of a Dirac type differential operator using the theory of induced $\mathrm{Diff}(S^1)$-actions endowed with centrally extended Lie-Poisson brackets. After constructing a general expression for tranversal deformations of a Dirac type differential operator, we interpret it via the Lie-algebraic theory of induced $\mathrm{Diff}(S^1)$-actions on a special Poisson manifold and determine its generic moment mapping. Using a Marsden-Weinstein reduction with respect to certain Casimir generated distributions, we describe a wide class of versally deformed Dirac type differential operators depending on complex parameters.

1 Introduction

Suppose we are given the linear 2-vector first order Dirac differential operator on the real axis $\mathbb{R}$:

$$L_\lambda f := -\frac{df}{dx} + l_\lambda[u, v; z]f, \qquad l_\lambda[u, v; z] := \begin{pmatrix} z-\lambda & u \\ v & \lambda - z \end{pmatrix} \tag{1.1}$$

acting on the Sobolev space $W^{(1)}_{2,loc}(\mathbb{R}; \mathbb{C}^2)$ and depending on 2π-periodic coefficients $u, v, z \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C})$ and a complex parameter $\lambda \in \mathbb{C}$. The variety of all operators (1.1), parametrized by λ, will be denoted by $\mathcal{L}_\lambda$.

Let $\mathcal{A} := \mathrm{Diff}(S^1)$ be the group of orientation preserving diffeomorphisms of the circle S^1. A group action of $\mathcal{A}$ on $\mathcal{L}_\lambda$ can be defined as follows: Fixing a parametrization of S^1, i.e., a C^∞ covering $p : \mathbb{R} \to S^1$ such that the mapping $p : [a, a + 2\pi) \rightleftharpoons S^1$ is one-to-one for every real a and $p(x + 2\pi) = p(x)$ for all $x \in \mathbb{R}$, each $\phi \in \mathcal{A}$ can obviously be represented by a smooth mapping $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(\xi + 2\pi) = \phi(\xi) + 2\pi \qquad \text{and} \qquad \phi'(\xi) > 0 \tag{1.2}$$

for all $\xi \in \mathbb{R}$. Upon making the change of variables

$$x = \phi(\xi), \qquad f(\phi(\xi)) = \Phi(\xi)\tilde{f}(\xi), \tag{1.3}$$

with $\phi \in \mathcal{A}$, $\Phi \in G := C^\infty(\mathbb{R}/2\pi\mathbb{Z}; SL(2;\mathbb{C}))$ and $x, \xi \in \mathbb{R}$, in (1.1), it is easy to see that the differential operator L_λ transforms into $L_\lambda^{(\phi,\Phi)} : W_2^{(1)} \to W_2^{(1)}$ defined as

$$L_\lambda^{(\phi,\Phi)}\tilde{f}(\xi) := -\frac{d\tilde{f}}{d\xi} + l_\lambda^{(\phi,\Phi)}[u, v; z]\tilde{f}, \tag{1.4}$$

where

$$l_\lambda^{(\phi,\Phi)}[u, v; z] := -\Phi^{-1}(\xi)\frac{d\Phi(\xi)}{d\xi} + \phi'(\xi)\Phi^{-1}(\xi)l_\lambda[u, v; z]\Phi(\xi). \tag{1.5}$$

We assume now that the matrix $\Phi(\xi)$ is chosen so that $l_\lambda^{(\phi,\Phi)}[u, v; z] = l_\lambda[\tilde{u}, \tilde{v}; \tilde{z}]$ for all $\lambda \in \mathbb{C}$ and some mapping $(\tilde{u}, \tilde{v}; \tilde{z})^T \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2 \times \mathbb{C})$. Whence we obtain an induced nonlinear transformation $A^*(\phi, \Phi) : \mathcal{L}_\lambda \to \mathcal{L}_\lambda$, $(\phi, \Phi) \in \mathcal{A} \times G$, where

$$A^*(\phi, \Phi)l_\lambda[u, v; z] := l_\lambda^{(\phi,\Phi)}[u, v; z] \tag{1.6}$$

for all mappings in $C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2 \times \mathbb{C})$. This together with expression (1.5) determines an automorphism A^* of $\mathcal{A}$, for a fixed Φ, that we shall study in detail. We are primarily interested in describing normal forms and versal deformations of (1.1) with respect to the automorphism A^*.

As is well known (see [1, 2, 5]), a normal form of the operator (1.1) is the simplest (in some sense) representative of its orbit under the group action of $\mathcal{A}$ on the space $\mathcal{L}_\lambda$. A versal deformation of (1.1) is a normal form for a stable parametric infinitesimal family including (1.1). As will be shown below, all such deformations can be described by means of Lie-algebraic analysis of this group action on $\mathcal{L}_\lambda$ and an associated momentum mapping reduced on certain invariant subspaces.

2 Lie-algebraic structure of the $\mathcal{A}$-action

Let us consider the loop group $G := G_{S^1}(SL(2;\mathbb{C}))$ of all smooth mappings $S^1 \to SL(2;\mathbb{C})$ and its corresponding group $\mathcal{A}$-action on a functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^3)$, which is assumed to be equivariant; that is, the diagram

$$\begin{array}{ccc} M & \xrightarrow{l} & \mathcal{G}^* \\ A_\Phi \downarrow & & \downarrow Ad^*_{\Phi^{-1}} \\ M & \xrightarrow{l} & \mathcal{G}^* \end{array} \tag{2.1}$$

commutes for all l in the adjoint $\mathcal{G}^*$ of the loop Lie algebra and $\Phi \in G$. Whence we can define on M a natural Poisson structure that induces the following canonical Lie-Poisson structure on $\mathcal{G}^*$: for any $\gamma, \mu \in D(\mathcal{G}^*)$,

$$\{\gamma, \mu\} := (l, [\nabla\gamma(l), \nabla\mu(l)]). \tag{2.2}$$

Here $(\cdot,\cdot)$ is the usual Killing type nondegenerate, symmetric, invariant scalar product on the loop Lie algebra $\mathcal{G} = C_{S^1}(sl(2;\mathbb{C}))$, i.e. for any $a, b \in \mathcal{G}$,

$$(a, b) := \int_0^{2\pi} dx\ Sp(ab) \tag{2.3}$$

and $\nabla : D(\mathcal{G}^*) \to \mathcal{G}$ is defined as $(\nabla\gamma(l), \delta l) := \frac{d}{d\epsilon}\gamma(l+\epsilon\delta l)\mid_{\epsilon=0}$ for any $\delta l \in \mathcal{G}^*, \gamma \in D(\mathcal{G}^*)$.

In order to address the problems posed in Section 1, we need to centrally extend the group action $A_\Phi : M \to M$, $\Phi \in G$, as follows: for $\hat{\Phi} := (\Phi, c) \in \hat{G} := G \times \mathbb{C}$ the corresponding action $A_{\hat{\Phi}} : M \to M$ is defined so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\hat{l}} & \hat{\mathcal{G}}^* \\ A_{\hat{\Phi}} \downarrow & & \downarrow Ad^*_{\hat{\Phi}^{-1}} \\ M & \xrightarrow{\hat{l}} & \hat{\mathcal{G}}^* \end{array} \tag{2.4}$$

commutes for all $\hat{\Phi} \in \hat{G}$ and $\hat{l} = (l, c) \in \hat{\mathcal{G}}^*$. This leads to the following (unique!) choice of the extended Ad^*-action in (2.4):

$$Ad^*_{\hat{\Phi}^{-1}} : (l, c) \in \mathcal{G}^* \to \left(\phi'(\xi) Ad_{\Phi^{-1}} l(x) - c\Phi^{-1}\frac{d\Phi}{d\xi}, c\right) \tag{2.5}$$

for all $\hat{\Phi} \in \hat{G}$, $l \in \mathcal{G}^*$ at $\xi \in \mathbb{R}$, $x = \phi(\xi)$ and $c \in \mathbb{C}$. This expression follows from the fact that the loop Lie algebra $\mathcal{G}$ admits only the central extension $\hat{\mathcal{G}} \oplus \mathbb{C}$. As the homology groups $H^1(\mathcal{G}) = 0$ and $H^2(\mathcal{G}) = 1$, it is represented as

$$[(a, \alpha), (b, \beta)] := ([(a, b)], (a, db/dx)) \tag{2.6}$$

for any $a, b \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$. Taking c to be unity and defining an appropriate diffeomorphism $x \to \phi(x) = \xi$ of $\mathbb{R}$, it is easy to see that $Ad^*_{\hat{\Phi}^{-1}}$ has the same structure element as that of the action $A^*(\phi, \Phi)$ on $\mathcal{L}_\lambda$ defined above. Whence it is clear that our Lie-algebraic analysis is intimately connected with the structure of the G-orbits induced by the diffeomorphism group $\mathcal{A} = \mathrm{Diff}(S^1)$.

We define a natural Lie-Poisson bracket on the adjoint space $\hat{\mathcal{G}}^*$ as follows: for any $\gamma, \mu \in D(\hat{\mathcal{G}}) \subset \hat{\mathcal{G}}^*$,

$$\{\gamma, \mu\}_0 := (l, [\nabla\gamma(l), \nabla\mu(l)]) + \left(\nabla\gamma(l), \frac{d\nabla\mu(l)}{dx}\right), \tag{2.7}$$

and deform it into a brackets pencil using a constant parameter $\lambda \in \mathbb{C}$ via

$$\{\gamma, \mu\}_0 \xrightarrow{\lambda} \{\gamma, \mu\}_\lambda := (\nabla\gamma(l), \frac{d}{dx}\nabla\mu(l)) + (l + \lambda J, [\nabla\gamma(l), \nabla\mu(l)]), \tag{2.8}$$

where $J \in sl^*(2;\mathbb{C})$ is chosen here to be the constant matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.9}$$

The following compatibility condition is almost obvious [8, 10].

Lemma 2.1. *A pencil of brackets (2.8) is a Poisson brackets pencil for each* $\lambda \in \mathbb{C}$ *and* $J \in sl^*(2;\mathbb{C})$, *i.e. it is compatible.*

Proof. It is well known that the Lie derivative of a Poisson bracket is also a Poisson bracket if and only if

$$\{\gamma, \mu\}_1 := \mathfrak{L}_K\{\gamma, \mu\}_0 - \{\mathfrak{L}_K\gamma, \mu\}_0 - \{\gamma, \mathfrak{L}_K\mu\}_0 \tag{2.10}$$

satisfies the Jacobi identity for all $\gamma, \mu \in D(\mathcal{G}^*)$, where $\mathfrak{L}_K$ is the Lie derivative with respect to a vector field $K : \mathcal{G}^* \to T(\mathcal{G}^*)$. Choosing $K(l) := J$, it is easy to verify that the bracket (2.10) satisfies the Jacobi identity and is the usual Poisson bracket on $\mathcal{G}^*$. Consequently, the Poisson bracket (2.10) is also a Poisson bracket along a generic orbit of the vector field $dl/d\lambda = J$, hence the deformation (2.8) is also Poisson, as was to be proved.

3 Casimir functionals and reduction problem

A Casimir functional $h \in I_\lambda(\hat{\mathcal{G}}^*)$ is defined, as usual, as a functional $h \in D(\hat{\mathcal{G}}^*)$ that is invariant with respect to the following λ-deformed $Ad^*_{\hat{\Phi}^{-1}}$-action:

$$Ad^*_{\hat{\Phi}^{-1}} : (l, 1) \in \hat{\mathcal{G}}^* \to \left(Ad^*_{\Phi^{-1}}(l + \lambda J) - \Phi^{-1}\frac{d\Phi}{dx}, 1 \right) \tag{3.1}$$

for any $\Phi \in G$, $l \in \mathcal{G}^*$ and $\lambda \in \mathbb{C}$. It is easy to see from this definition that $h \in I_\lambda(\hat{\mathcal{G}}^*)$ if the equation

$$\frac{d\nabla h(l)}{dx} = [l + \lambda J, \nabla h(l)] \tag{3.2}$$

is satisfied for all $\lambda \in \mathbb{C}$. Assuming further that there exists an asymptotic expansion of the form

$$h(\lambda) \sim \sum_{j \in \mathbb{Z}_+} h_j \lambda^{-j} \tag{3.3}$$

as $|\lambda| \to \infty$, one can readily verify that $h_0 \in I_1(\hat{\mathcal{G}}^*)$ and that for all $j, k \in \mathbb{Z}_+$

$$\{h_j, h_k\}_0 = 0 = \{h_j, h_k\}_1, \qquad \{\gamma, h_j\}_0 = \{\gamma, h_{j+1}\}_1, \tag{3.4}$$

where $\gamma \in D(\hat{\mathcal{G}}^*)$ is arbitrary.

Let us now consider the action (2.1) at a fixed $l = l[u, v; z] \in \hat{\mathcal{G}}^*$. It is easy to see that this action does not necessarily preserve the form of the element l. Thus we must reduce the initial $\hat{G}$-action on $\hat{\mathcal{G}}^*$ to an appropriate subgroup; for this we develop the reduction procedure employed in [8–10].

Define the distribution

$$D_1 := \left\{ K \in T(\hat{\mathcal{G}}^*) : K(l) = [J, \nabla\gamma(l)], l \in \hat{\mathcal{G}}^*, \gamma \in D(\hat{\mathcal{G}}^*) \right\}. \tag{3.5}$$

D_1 is integrable, that is $[D_1, D_1] \subset D_1$, since the bracket $\{\cdot,\cdot\}_1$ is Poisson. Now define another distribution

$$D_0 := \left\{ K \in T(\hat{\mathcal{G}}^*) : K(l) = [l - \frac{d}{dx}, \nabla h_0], h_0 \in I_1(\hat{\mathcal{G}}^*) \right\}, \tag{3.6}$$

which is clearly also integrable on $\hat{\mathcal{G}}^*$, since $[D_0, D_0] \subset D_0$. The set of maximal integral submanifolds of (3.6) generates the foliation $\hat{\mathcal{G}}^*_J \backslash D_0$ whose leaves are the intersections of fixed integral submanifolds $\hat{\mathcal{G}}^*_J \subset \hat{\mathcal{G}}^*$ of the distribution D_1 passing through an element $l[u,v;z] \in \hat{\mathcal{G}}^*$. If the foliation $\hat{\mathcal{G}}^*_J \backslash D_0$ is sufficiently smooth, one can define the quotient manifold $\hat{\mathcal{G}}^*_{\rm red} := \hat{\mathcal{G}}^*_J / (\hat{\mathcal{G}}^*_J \backslash D_0)$ with its associated projection mapping $\hat{\mathcal{G}}^*_J \to \hat{\mathcal{G}}^*_{\rm red}$. To continue this line of reasoning, we shall obtain explicit constructions of the objects introduced.

D_1 is obviously generated by the vector fields

$$\frac{dl}{dt} = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}, \qquad \nabla\gamma(l) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{3.7}$$

where t is a complex evolution parameter and $l \in \hat{\mathcal{G}}^*_J$, where $\hat{\mathcal{G}}^*_J \subset \hat{\mathcal{G}}^*$ is the isotropy Lie subalgebra of the element $J \in \hat{\mathcal{G}}^*$. Hence the integral submanifold $\hat{\mathcal{G}}^*_J$ consists of orbits of an element $l = l[u,v;z] \in \hat{\mathcal{G}}^*$, with $z \in \mathbb{C}$, with respect to the vector fields (3.7). The distribution D_0 on $T(\hat{\mathcal{G}}^*)$ is generated by the vector fields

$$\frac{dl}{d\tau} = \begin{pmatrix} -\chi_x & -2u\chi \\ 2v\chi & \chi_x \end{pmatrix}, \qquad \nabla h_0(l) = \begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix}, \tag{3.8}$$

where τ is a complex evolution parameter and $l = l[u,v;z] \in \hat{\mathcal{G}}^*$.

It follows immediately from (3.8) that

$$\frac{dz}{d\tau} = -\chi_x, \qquad \frac{du}{d\tau} = -2u\chi \qquad \text{and} \qquad \frac{dv}{d\tau} = 2v\chi \tag{3.9}$$

for all $\tau \in \mathbb{R}$ along D_0. Eliminating the variable χ from (3.9), we obtain

$$\frac{d}{d\tau}\left[\frac{d}{dx}(\ln u) - 2z\right] = 0 = \frac{d}{d\tau}\left[\frac{d}{dx}(\ln v) + 2z\right]; \tag{3.10}$$

that is, the mapping

$$\hat{\mathcal{G}}^* \ni l = \begin{pmatrix} z & u \\ v & -z \end{pmatrix} :\xrightarrow{\nu} \begin{pmatrix} 0 & \exp(\partial^{-1}\alpha) \\ \exp(\partial^{-1}\beta) & 0 \end{pmatrix} \to \hat{\mathcal{G}}^*_{\rm red}, \tag{3.11}$$

where

$$\alpha := u_x u^{-1} - 2z, \qquad \beta := v_x v^{-1} + 2z, \tag{3.12}$$

explicitly determines the reduction $\nu : \hat{\mathcal{G}}^* \to \hat{\mathcal{G}}^*_{\rm red}$ discussed above. We are now in a position to compute the bracket (2.8) reduced upon the submanifold $\hat{\mathcal{G}}^*_{\rm red}$ by defining the functionals $\lambda, \mu \in D(\hat{\mathcal{G}}^*)$ to be constant along the distribution D_0, that is

$$\gamma := \tilde{\gamma} \circ \nu, \qquad \mu := \tilde{\mu} \circ \nu, \tag{3.13}$$

for any $\tilde{\gamma}, \tilde{\mu} \in D(\hat{\mathcal{G}}^*_{\rm red})$. From (3.12) one readily obtains the expressions

$$\nabla\gamma(l)|_{l\in\hat{\mathcal{G}}^*_{\rm red}} = \begin{pmatrix} \frac{\delta\tilde{\gamma}}{\delta\beta} - \frac{\delta\tilde{\gamma}}{\delta\alpha} & -\frac{1}{v}\left(\frac{\delta\tilde{\gamma}}{\delta\beta}\right)_x \\ -\frac{1}{u}\left(\frac{\delta\tilde{\gamma}}{\delta\alpha}\right)_x & \frac{\delta\tilde{\gamma}}{\delta\alpha} - \frac{\delta\tilde{\gamma}}{\delta\beta} \end{pmatrix},$$
$$\nabla\mu(l)|_{l\in\hat{\mathcal{G}}^*_{\rm red}} = \begin{pmatrix} \frac{\delta\tilde{\mu}}{\delta\beta} - \frac{\delta\tilde{\mu}}{\delta\alpha} & -\frac{1}{v}\left(\frac{\delta\tilde{\mu}}{\delta\beta}\right)_x \\ -\frac{1}{u}\left(\frac{\delta\tilde{\mu}}{\delta\alpha}\right)_x & \frac{\delta\tilde{\mu}}{\delta\alpha} - \frac{\delta\tilde{\mu}}{\delta\beta} \end{pmatrix}, \tag{3.14}$$

which satisfy the desired identities

$$(\nabla\gamma(l), dl/d\tau) = 0 = (\nabla\mu(l), dl/d\tau) \tag{3.15}$$

for all $l \in \hat{\mathcal{G}}^*_{\rm red} \subset \hat{\mathcal{G}}^*$. Substituting now (3.14) into (2.8), we obtain

$$\{\tilde{\gamma}, \tilde{\mu}\}_\lambda := \{\gamma, \mu\}_\lambda\,|_{l\in\hat{\mathcal{G}}^*_{\rm red}} = (\nabla\tilde{\gamma}, (\eta + \lambda\theta)\nabla\tilde{\mu}), \tag{3.16}$$

where we have used the obvious relationship

$$\{\tilde{\gamma}, \tilde{\mu}\}_\lambda \circ \nu = \{\tilde{\gamma}\circ\nu, \tilde{\mu}\circ\nu\}_\lambda, \tag{3.17}$$

and where

$$\eta := \begin{pmatrix} 2\partial & -\partial\exp[-\partial^{-1}(\alpha+\beta)]\partial^2 - 2\partial - \partial\cdot\beta\exp[-\partial^{-1}(\alpha+\beta)]\cdot\partial \\ -\partial\exp[-\partial^{-1}(\alpha+\beta)]\partial^2 - 2\partial - \partial\cdot\alpha\exp[-\partial^{-1}(\alpha+\beta)]\cdot\partial & 2\partial \end{pmatrix}, \tag{3.18}$$

$$\theta := \begin{pmatrix} 0 & 2\partial\exp[-\partial^{-1}(\alpha+\beta)]\partial \\ -2\partial\exp[-\partial^{-1}(\alpha\beta)]\partial & 0 \end{pmatrix}.$$

It is straightforward to verify that these integro-differential, implectic (=co-symplectic= Poisson) operators are compatible [4] (see also [11] for a general theory of iso-symplectic structures on functional manifolds) on the reduced submanifold $\hat{\mathcal{G}}^*_{\rm red}$ and define a bi-Hamiltonian structure on it.

4 Diff(S^1) action, associated momentum mapping and versal deformations

Let us introduce some additional notation concerning versal deformations [1, 7]. By a deformation of the operator (1.1) we shall mean an operator of the same form with a matrix $l_\lambda(\epsilon)$ whose entries are analytic in ϵ in a neighborhood of $\epsilon = 0$ in $\mathbb{C}^n$ and satisfies $l_\lambda(0) = l_\lambda$ for all $\lambda \in \mathbb{C}$. The coordinates $\epsilon_i \in \mathbb{C}$, $1 \le i \le n$, of ϵ are called the deformation parameters and the space of these parameters is called the base of the deformation.

Two deformations $l'_\lambda(\epsilon)$ and $l''_\lambda(\epsilon)$ of a matrix l_λ will be called equivalent if there exists a deformation $A^*(\phi_\epsilon): l'_\lambda(\epsilon) \to l''_\lambda(\epsilon)$ generated by a diffeomorphism $\phi_\epsilon \in \mathrm{Diff}(S^1)$ satisfying $\phi_\epsilon |_{\epsilon=0} = id$.

From a given deformation $l_\lambda(\epsilon)$ one can obtain a new deformation $\tilde{l}_\lambda(\tilde{\epsilon})$ by setting $\tilde{l}_\lambda(\tilde{\epsilon}) := l_\lambda(\epsilon(\tilde{\epsilon}))$, where $\epsilon : \mathbb{C}^m \to \mathbb{C}^n$ is an analytic mapping in a neighborhood of $\tilde{\epsilon} = 0$ in $\mathbb{C}^m$ and satisfies the condition $\epsilon(0) = 0$. The deformation $\tilde{l}_\lambda(\tilde{\epsilon})$ is said to be induced from $l_\lambda(\epsilon)$ by the mapping $\epsilon : \mathbb{C}^m \to \mathbb{C}^n$.

A deformation $l_\lambda(\epsilon)$, $\epsilon \in \mathbb{C}^n$, is called versal if every one of its deformations $l_\lambda(\tilde{\epsilon})$, $\tilde{\epsilon} \in \mathbb{C}^m$, is equivalent to a deformation induced from it. A versal deformation is said to be universal if the induced deformation described in the definition of versality is unique.

Before we give a definition of a transversal deformation for the induced group $\hat{\mathcal{G}}_{\mathrm{red}}$ orbits, let us consider a family of smooth induced transformations $\phi_\sigma(x) \in \hat{G}_{\mathrm{red}}$, $\sigma \in \mathbb{R}$, where $\phi_\sigma(x) = 1 + O(\sigma)$ as $\sigma \to 0$. Each such transformation generates (via formula (1.5)) a new matrix $l_\lambda(\sigma)$, $\sigma \to 0$, that obviously belongs to the orbit space associated to the $\hat{G}_{\mathrm{red}}$ action. The set of matrices

$$\left.\frac{dl_\lambda(\sigma)}{d\sigma}\right|_{\sigma=0} \in \hat{\mathcal{G}}^*_{\mathrm{red}} \tag{4.1}$$

spans a linear subspace $\hat{V}_\lambda \subset \hat{\mathcal{G}}^*_{\mathrm{red}}$ of finite codimension. Consider an arbitrary deformation $l_\lambda(\epsilon)$, $\epsilon \in \mathbb{C}^n$, of a given matrix $l_\lambda \in \hat{\mathcal{G}}^*_{\mathrm{red}}$ and denote by $\hat{E}_\lambda$ the linear span in $\hat{\mathcal{G}}^*_{\mathrm{red}}$ over the matrices $\partial l_\lambda(\epsilon)/\partial\epsilon_i |_{\epsilon=0}$, $1 \le i \le n$. The above deformation is said to be transverse to the induced $\hat{G}_{\mathrm{red}}$ orbit if the subspaces $\hat{E}_\lambda$ and $\hat{V}_\lambda$ together span their ambient space, that is

$$\hat{E}_\lambda + \hat{V}_\lambda = \hat{\mathcal{G}}^*_{\mathrm{red}}. \tag{4.2}$$

The following general theorem [1] holds for versal deformations of the Dirac operator (1.1).

Theorem 4.1. *A deformation $l_\lambda(\epsilon)$, $\epsilon \in \mathbb{C}^n$, is versal if and only if it is transverse to the induced group $\hat{G}$ orbit.*

This theorem can be proved by applying standard perturbation theory techniques to the Dirac type operator (1.1).

We are now ready to make use of the results of Section 3 to describe the spaces $\hat{E}_\lambda$ and $\hat{V}_\lambda$ analytically. Let $\tilde{\gamma} \in D(\hat{\mathcal{G}}^*_{\mathrm{red}})$ be any smooth functional on $\hat{\mathcal{G}}^*_{\mathrm{red}}$; it generates a flow on the loop group $\hat{G}_{\mathrm{red}}$ orbit via the (σ, x)-evolutions

$$\frac{dl}{d\sigma} := \{\tilde{\gamma}, l\}_\lambda, \qquad \frac{dl}{dx} := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} l \tag{4.3}$$

with respect to the Poisson bracket (3.16). In view of (4.3), (3.16) implies that the subspace $\hat{V}_\lambda$ is isomorphic to the following subspace of vector functions in $T^*(M)$:

$$V_\lambda := \{\Lambda_\lambda \psi := (\eta + \lambda\theta)\psi : \nabla\tilde{\gamma} = \psi \in T^*(M)\}. \tag{4.4}$$

Theorem 4.1 suggests the following construction of versal deformations for the Dirac type operator (1.1): As Λ_λ is skew-symmetric, the operator $i\Lambda_\lambda$ is formally selfadjoint in the space $L_2(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2)$. Therefore, the orthogonal complement to the subspace V_λ with respect to the natural scalar product in $L_2(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2)$ consists of 2π-periodic solutions to the equation

$$\Lambda_\lambda \psi = 0. \tag{4.5}$$

Whence we have the following characterization of versal deformations of the operator (1.1).

Theorem 4.2. *The prolongation of the matrix $l_\lambda \in \hat{\mathcal{G}}^*_{red}$ defined as*

$$\bar{l}_\lambda(\epsilon) := \begin{pmatrix} \lambda & \exp(\partial^{-1}\beta) \\ \exp(\partial^{-1}\alpha) & -\lambda \end{pmatrix} + \sum_{i,j=1}^{2} \epsilon_{ij} \bar{f}_i \otimes \bar{f}_j \tag{4.6}$$

generates a versal deformation of the Dirac type operator (1.1). Here $\otimes$ is the usual Kronecker tensor product in $\mathbb{C}^2$, $\epsilon_{ij} \in \mathbb{C}, 1 \leq i,j \leq 2$, $\epsilon_{12} = -\epsilon_{21}$ are any deformation constants, and $\bar{f}_i \in W_2^{(1)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C}^2), i = 1,2,$ are two linearly independent, normalized solutions to the Dirac equations

$$\frac{d\bar{f}_i}{dx} + \bar{l}_\lambda \bar{f}_i = 0, \qquad \left\|\bar{f}_i, \bar{f}_j\right\|\big|_{x=0} = 1, \tag{4.7}$$

with spectral parameter $\lambda \in \mathbb{C}$.

Proof. It is easy to verify that the set of solutions to equation (4.5) is isomorphic to the set of functions

$$\hat{\psi} = \sum_{i,j=1}^{2} \epsilon_{ij} \bar{f}_i \otimes \bar{f}_j,$$

and these functions satisfy the canonical Casimir equation

$$\left[l_\lambda, \hat{\psi}\right] - \frac{d\hat{\psi}}{dx} = 0, \tag{4.8}$$

which is equivalent to equation (4.5). Owing to the fact that any matrix $l_\lambda \in \hat{\mathcal{G}}^*$ in (1.1) can be transformed into the expression $\bar{l}_\lambda(0) \in \hat{\mathcal{G}}^*$ with functional parameters α, β given by (3.12), this leads to the general form (4.6) for versal deformations of (1.1). This ends the proof.

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