# Representations of the Infinite Unitary Group from Constrained Quantization

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### Abstract

We attempt to reconstruct the irreducible unitary representations of the Banach Lie group $U_0(\mathcal{H})$ of all unitary operators $U$ on a separable Hilbert space $\mathcal{H}$ for which $U - \mathbb{I}$ is compact, originally found by Kirillov and Ol'shanskii, through constrained quantization of its coadjoint orbits. For this purpose the coadjoint orbits are realized as Marsden-Weinstein quotients. The unconstrained system, given as a Weinstein dual pair, is quantized by a corresponding Howe dual pair. Constrained quantization is then performed in replacing the classical procedure of symplectic reduction by the $C^*$-algebraic method of Rieffel induction. Reduction and induction have to be performed with respect to either $U(M)$, which is straightforward, or $U(M, N)$. In the latter case one induces from holomorphic discrete series representations, and the desired result is obtained if one ignores half-forms, and induces from a representation, 'half' of whose highest weight is shifted relative to the naive orbit correspondence. This is only possible when $\mathcal{H}$ is finite-dimensional.

## 1    Introduction

### 1.1    Representations from quantized symplectic reduction

Constrained quantization [8, 23] is a very useful method that often allows one to reduce nonlinear problems in mathematical physics to linear ones. Such a reduction is possible if a given nonlinear (symplectic) space may be written as the reduced ('physical') phase space relative to a linear phase space with certain constraints defined on it. The goal of this paper is to quantize the coadjoint orbits of a certain infinite-dimensional Lie group, which are highly nonlinear infinite-dimensional symplectic manifolds, by a mathematically rigorous version of this method. The group in question (defined below) has been chosen because it is one of the few infinite-dimensional Lie groups for which the correspondence between its irreducible unitary representations and its coadjoint orbits is known. Thus it

forms an ideal testing ground for the constrained quantization (as well as for more general constructions in mathematical physics) of infinite-dimensional phase spaces.

Let $U_0(\mathcal{H})$ be the Banach Lie group of all unitary operators $U$ on a separable Hilbert space $\mathcal{H}$ for which $U - \mathbb{I}$ is compact, equipped with the uniform operator (i.e., norm) topology. The continuous unitary representations of $U_0(\mathcal{H})$ were classified by Kirillov [19] and Ol'shanskii [32]. Their classification simultaneously applies to the Fréchet Lie group $U(\mathcal{H})$ consisting of all unitary operators on $\mathcal{H}$, equipped with the strong operator topology, because all representations of $U_0(\mathcal{H})$ are also strongly continuous, and can therefore be extended to $U(\mathcal{H})$. Moreover, $U(\mathcal{H})$ re-topologized with the uniform topology has the same irreducible representations on separable Hilbert spaces as the same group equipped with the strong topology (whose irreducible representation spaces are automatically separable) [36]. (The representation theory of $U(\infty)$ equipped with the inductive limit topology is much more complicated [35, 6] and will not be discussed here.)

A remarkable aspect of the Kirillov-Ol'shanskii classification is that all irreducible representations of $U_0(\mathcal{H})$ may be thought of as the geometric quantization of certain of its coadjoint orbits. However, only the geometric quantization of orbits corresponding to positive eigenvalues may actually be found in the literature [5]; even this special case is already fairly involved. It is this quantization that we venture to redo, and much simplify, by regarding the orbits as Marsden-Weinstein quotients, and performing a certain constrained quantization procedure.

Our work was triggered by Montgomery's observation [30] (also cf. [25]) that for finite-dimensional $\mathcal{H} = \mathbb{C}^k$ certain coadjoint orbits of $U(k)$ (namely those characterized by positive eigenvalues) are Marsden-Weinstein quotients of $\mathcal{H} \otimes \mathbb{C}^M$ with respect to $U(M)$, for suitable $M$ (which depends on the orbit). The left-action of $U(k)$ and the right-action of $U(M)$ on $\mathbb{C}^k \otimes \mathbb{C}^M$ combine to form a Weinstein dual pair $U(k) \to \mathbb{C}^k \otimes \mathbb{C}^M \leftarrow U(M)$ [18, 40, 44].

The simplest reduced space thus obtained (viz. for $M = 1$) is the projective space $\mathbb{P}\mathbb{C}^k$; as in the general case, three relevant symplectic structures, namely its standard form as a Kähler manifold, its Lie-Poisson form as a coadjoint orbit, and finally its Marsden-Weinstein form as a symplectic quotient, all coincide.

We extend Montgomery's result to the situation where the eigenvalues may be of either sign, and also to the case where $\mathcal{H}$ is infinite-dimensional. The Weinstein dual pair then becomes $U_0(\mathcal{H}) \to \mathcal{H} \otimes \mathbb{C}^M \otimes \overline{\mathbb{C}}^N \leftarrow U(M, N)$, so that one reduces with respect to the non-compact group $U(M, N)$. Note that $M$ and $N$ are finite even in the infinite-dimensional case.

The quantization of the 'unconstrained system' $U_0(\mathcal{H}) \to \mathcal{H} \otimes \mathbb{C}^M \otimes \overline{\mathbb{C}}^N \leftarrow U(M, N)$ is trivially done by Fock space techniques, yielding a Howe dual pair that quantizes the classical Weinstein dual pair in question. To quantize the Marsden-Weinstein reduction process that led to the classical coadjoint orbits, we employ a relatively new method [22, 23], which is based on the $C^*$-algebraic technique of Rieffel induction [39, 10, 23]. As explained in [22, 23], this method in principle quantizes a symplectic reduction procedure vastly more general than the Marsden-Weinstein one [29, 47, 22, 23], and improves on more traditional constrained quantization techniques (such the Dirac or the BRST method) in cases where the quantized constraints fail to have a joint eigenvalue zero. In the context of the present paper, this means that for $N = 0$, where one classically reduces with respect to the compact group $U(M)$, other techniques would apply as well, whereas for $N > 0$ these

would break down.

For $N = 0$, the induction procedure is easily carried out on the basis of Weyl's classical results on tensor products and the symmetric group [45, 14]. The case $N > 0$, where the coadjoint orbit one quantizes is characterized by eigenvalues of arbitrary sign, is considerably more complicated. The quantization of the unconstrained system $S = \mathcal{H} \otimes \mathbb{C}^M \otimes \overline{\mathbb{C}}^N$ is known explicitly at least for finite-dimensional $\mathcal{H} = \mathbb{C}^k$: it is the $k$-fold tensor product of the metaplectic (or 'oscillator', or 'Segal-Shale-Weil') representation [11], restricted from $Sp(2(N + M), \mathbb{R})$ to its subgroup $U(M, N)$ (see [41, 40, 4]).

This tensor product has been decomposed by Kashiwara and Vergne [17], also cf. Howe [12]. The decomposition of the Hilbert space quantizing $S = \mathbb{C}^k \otimes \mathbb{C}^M \otimes \overline{\mathbb{C}}^N$ under $U(k)$ and $U(M, N)$ does not reflect the decomposition of $S$ under these group actions if $k > M + N$ (which is the case of relevance to us, as we are eventually interested in $k = \infty$), cf. [2]. This fascinating complication implies that for generic coadjoint orbits our method only works when $\mathcal{H}$ is finite-dimensional.

## 1.2   Rieffel induction for group actions

We briefly review how Rieffel induction [39, 10, 23] specializes to the present context. One starts from a strongly Hamiltonian right-action of a connected Lie group $H$ on a symplectic manifold $S$, with accompanying equivariant momentum map $J : S \to \mathbf{h}^*$. We assume that the reduced space $S^\mu \equiv J^{-1}(\mathcal{O}_\mu)/H$ is a manifold.

If a Lie group $G$ acts symplectically on $S$ in such a way that its action commutes with the $H$-action, the reduced space $S^\mu$ becomes a symplectic $G$-space in the obvious way; the well-known 'symplectic induction' procedure [18, 23] is a special case of this construction (it is obtained by taking $H \subset G$ and $S = T^*G$).

To quantize the reduced space $S^\mu$ and the associated induced representation of $G$, we assume that a quantization of the unconstrained system as well as of the constraints are given. Hence we suppose we have firstly found a Hilbert space $\mathcal{F}$, which may be thought of as the (geometric) quantization of $S$. Secondly, a unitary right-action (i.e., anti-representation) $U_R(H)$ on $\mathcal{F}$ should be given, which is the quantization of the symplectic right-action of $H$ on $S$. Thirdly, we require a unitary representation $U_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, which 'quantizes' the coadjoint action of $H$ on the coadjoint orbit $\mathcal{O}_\mu$ This is only possible if the orbit is 'quantizable'; for $H = U(M)$ there is a bijective correspondence between such orbits and unitary representations, and for $U(M, N)$ one obtains at least all unitary highest weight modules by 'quantizing' such orbits [3, 43]. (In the latter case the concept of quantization has to be stretched somewhat to incorporate the derived functor technique to construct representations.)

First assuming that $H$ is compact, we construct the induced space $\mathcal{H}^\chi$ from these data as the subspace of $\mathcal{F} \otimes \mathcal{H}_\chi$ on which $U_R^{-1} \otimes U_\chi$ acts trivially (here $U_R^{-1}$ is the representation of $H$ defined by $U_R^{-1}(h) = U_R(h^{-1})$). If $H$ is only locally compact (and assumed unimodular for simplicity) with Haar measure $dh$, one has to find a dense subspace $L \subset \mathcal{F}$ such that the integral $\int_H dh \, ((U_R^{-1} \otimes U_\chi)(h)\Psi, \Phi) \equiv (\Psi, \Phi)_0$ is finite for all $\Psi, \Phi \in L \otimes \mathcal{H}_\chi$. This defines a sesquilinear form $(\cdot, \cdot)_0$ on $L \otimes \mathcal{H}_\chi$ which can be shown to be positive semi-definite under suitable conditions [22]. The induced space $\mathcal{H}^\chi$ is then defined as the completion of the quotient of $L \otimes \mathcal{H}_\chi$ by the null space of $(\cdot, \cdot)_0$; its inner product is, of course, given by the quotient of $(\cdot, \cdot)_0$. For $H$ compact the integral exists for all $\Psi, \Phi \in \mathcal{F}$ and

$(\Psi, \Phi)_0 = (P_0 \Psi, P_0 \Phi)$, where $P_0$ is the projector onto the subspace of $\mathcal{F} \otimes \mathcal{H}_\chi$ carrying the trivial representation of $H$, so that we recover the first description of $\mathcal{H}^\chi$.

We now assume that a group $G$ acts on $\mathcal{F}$ through a unitary representation $U_L$; it is required that this action commute with $U_R(H)$. The induced representation $U^\chi(G)$ on $\mathcal{H}^\chi$ is now defined as follows. For $H$ compact, $U^\chi$ is simply the restriction of $U_L \otimes \mathbb{I}$ to $\mathcal{H}^\chi \subset \mathcal{F} \otimes \mathcal{H}_\chi$; this is well defined because $U_L \otimes \mathbb{I}$ commutes with $U_R^{-1} \otimes U_\chi$. In the general case, one has to assume that $U_L$ leaves $L$ stable; then $U^\chi$ is essentially defined as the quotient of the action of $U_L \otimes \mathbb{I}$ (on $L \otimes \mathcal{H}_\chi$) to $\mathcal{H}^\chi$ as defined above (cf. [22] for technical details pertinent to the general case). The Mackey induction procedure for group representations is recovered by assuming that $H \subset G$, and taking $\mathcal{F} = L^2(G)$, cf. [39, 10, 23] for details in the original setting of Rieffel induction, and [22, 23] for the above setting.

# 2    Representations from Rieffel induction

In subsections 2.1 to 2.3 we take $\mathcal{H}$ to be an infinite-dimensional separable Hilbert space, unless explicitly stated otherwise. All results (sometimes with self-explanatory modifications) are equally well valid in the finite-dimensional case, which is considerably easier to handle; we leave this to the reader. We start with the simplest case, the defining representation.

## 2.1    The quantization of $\mathbb{P}\mathcal{H}$

One can realize $\mathbb{P}\mathcal{H}$ as a Marsden-Weinstein quotient with respect to the group $U(1)$ [1, 23]. Firstly, $\mathcal{H}$ carries a symplectic form $\omega$, expressed in terms of the standard inner product (taken linear in the first entry) by $\omega(\psi, \varphi) = -2 \operatorname{Im}(\psi, \varphi)$. Secondly, $U(1)$ (identified with the unit circle in the complex plane) acts on $\mathcal{H}$ by $z : \psi \to z\psi$; this action is symplectic, and yields an equivariant momentum map [1] $J : \mathcal{H} \to \mathbf{u}(\mathbf{1})^* \equiv \mathbb{R}$ given by $J(\psi) = (\psi, \psi)$. Then $\mathbb{P}\mathcal{H} \simeq J^{-1}(1)/U(1)$.

The quantization of this type of reduced space using Rieffel induction was outlined in the Introduction. We first need a quantization of the 'unconstrained' system $\mathcal{H}$, which we take to be the symmetric (bosonic) Fock space $\mathcal{F} = \exp(\mathcal{H})$ (this is the direct sum of all symmetrized tensor products $\mathcal{H}^{\otimes n}$ ($n = 0, 1, \ldots$) of $\mathcal{H}$ with itself). This quantization is so well-established that we will not motivate it here; cf. [11, 37] for mathematical aspects, and [46] for a derivation in geometric quantization.

The (anti) representation $U_R$ of $U(1)$ on $\mathcal{F}$ is obtained by 'quantization' of the right action on $\mathcal{H}$. We choose $U_R$ as the second quantization of this right action. Labelling this choice $U_{R,\mathrm{sq}}$, this yields $U_{R,\mathrm{sq}}(z) \upharpoonright \mathcal{H}^{\otimes n} = z^n \mathbb{I}$. Similarly, the defining representation $U_1$ of $G = U(\mathcal{H})$ (the group of all unitary operators on $\mathcal{H}$) on $\mathcal{H}_1 = \mathcal{H}$ yields a symplectic action on $\mathcal{H}$. This is 'second' quantized by the representation $U_{L,\mathrm{sq}}$ on $\mathcal{F}$, whose restriction $U_n$ to each subspace $\mathcal{H}^{\otimes n} \subset \mathcal{F}$ is the symmetrized $n$-fold tensor product of $U_1$ with itself. The representations $U_{R,\mathrm{sq}}(U(1))$ and $U_{L,\mathrm{sq}}(U(\mathcal{H}))$ obviously commute with each other. Hence $\mathcal{F}$ has a central decomposition under $U_{L,\mathrm{sq}}(U(\mathcal{H})) \otimes U_{R,\mathrm{sq}}^{-1}(U(1))$, which is explicitly given

by

$$\exp(\mathcal{H}) \overset{\mathrm{sq}}{\simeq} \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{U(\mathcal{H})} \otimes \overline{\mathcal{H}}_n^{U(1)}. \tag{1}$$

Here $\mathcal{H}_n^{U(\mathcal{H})}$ coincides with $\mathcal{H}^{\otimes n}$, now regarded as the carrier space of the representation $U_n(U(\mathcal{H}))$, which is, in fact, irreducible for all $n$ [19, 32] (also cf. subsection 3.3 below). Also, $\mathcal{H}_n^{U(1)}$ is just $\mathbb{C}$, but regarded as the carrier space of $U_n(U(1))$, defined by $U_n(z) = z^n$; $\overline{\mathcal{H}}$ stands for the carrier space of the conjugate representation.

The general context for decompositions of the type (1) is the theory of Howe dual pairs [15, 13]. In the present instance, this applies to $\mathcal{H} = \mathbb{C}^k$, with $U(k)$ and $U(1)$ being the dual pair in $Sp(2k, \mathbb{R})$. (Cf. [35] for the theory of these pairs in the infinite-dimensional setting.)

The construction of the induced space $\mathcal{F}^1$ is effortless in this case. The fact that Marsden-Weinstein reduction took place at $J = 1$ means that the orbit of $U(1)$ in question is the point $1 \in \mathbf{u}(\mathbf{1})^*$. This orbit is quantized by the defining representation $U_1$ of $U(1)$ on $\mathcal{H}_1 = \mathbb{C}$. By construction, $\mathcal{F}^1$ is the subspace of $\mathcal{F} \otimes \mathcal{H}_1 = \mathcal{F}$ which is invariant under the representation $U_R^{-1} \otimes U_1$. Hence (1) implies that $\mathcal{F}^1 = \mathcal{H}$. The induced representation $U^1(U(\mathcal{H}))$ on $\mathcal{F}^1$ is simply the restriction of $U_{L,\mathrm{sq}}(U(\mathcal{H}))$ to this space, so that $U^1 \simeq U_1$. In other words, we have recovered the defining representation.

So far, so good, but unfortunately there is a subtlety if one derives $U_R$ and $U_L$ from geometric quantization. Using the 'uncorrected' formalism (as described, e.g., in Ch. 9 of [46]), exploiting the existence of an invariant positive totally complex polarization, viz. the anti-holomorphic one, one finds that $\mathcal{F}$ is realized as the space of holomorphic functions on $\mathcal{H}$. The quantization $\pi_{\mathrm{qua}}$ of the momentum maps $J_R$ for $U(1)$ and $J_L$ for $U(\mathcal{H})$ (with respect to their respective actions on $\mathbb{C}^k$) then reproduces the second quantizations $U_{R,\mathrm{sq}}$ and $U_{L,\mathrm{sq}}$, respectively.

If, however, one is too sophisticated and incorporates the half-form correction to geometric quantization [46, Ch. 10], one obtains extra contributions: for $\mathcal{H} = \mathbb{C}^k$, $\pi_{\mathrm{qua}}(J_R)$ is replaced by $\pi_{\mathrm{qua}}(J_R) + k/2$, whereas $\pi_{\mathrm{qua}}(J_L)$ acquires an additional constant $\frac{1}{2}$ (times the unit matrix). These Lie algebra representations exponentiate to unitary representations of double covers $\tilde{U}(k)$ and $\tilde{U}(1)$, which we denote by $U_{L,\mathrm{hf}}$ and $U_{R,\mathrm{hf}}$, respectively. Under $U_{L,\mathrm{hf}}(\tilde{U}(k)) \otimes U_{R,\mathrm{hf}}^{-1}(\tilde{U}(1))$ we then find the central decomposition

$$\exp(\mathcal{H}) \overset{\mathrm{hf}}{\simeq} \bigoplus_{n=0}^{\infty} \mathcal{H}_{(n+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})}^{\tilde{U}(k)} \otimes \overline{\mathcal{H}}_{n+\frac{1}{2}k}^{\tilde{U}(1)}. \tag{2}$$

Here $\mathcal{H}_{(n+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})}$ carries the representation of $\tilde{U}(k)$ with highest weight $(n + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$; this is the tensor product of $\mathcal{H}_n$ and the square-root of the determinant representation. One observes that the inclusion of half-forms is awkward for Rieffel induction – we defer a discussion of this point to Chapter 3.

## 2.2   The coadjoint orbits of $U_0(\mathcal{H})$ as reduced spaces

The Lie algebra $\mathbf{g} = \mathbf{u}_0(\mathcal{H}) = i\mathfrak{K}(\mathcal{H})_{\mathrm{sa}}$ of $G = U_0(\mathcal{H})$ consists of all skew-adjoint compact operators on $\mathcal{H}$ with the norm topology. The dual $\mathbf{g}^* = \mathbf{u}_0(\mathcal{H})^*$ is the space of all self-

adjoint trace-class operators on $\mathcal{H}$, with topology induced by the trace norm $\| \rho \|_1 = \operatorname{Tr} |\rho|$ (this coincides with the weak* topology). The pairing is given by $\langle \rho, X \rangle = i \operatorname{Tr} \rho X$.

The coadjoint action of $U_0(\mathcal{H})$ on $\mathbf{u}_0(\mathcal{H})^*$ is given by $U_{\text{co}}(U)\rho = U\rho U^*$. We are interested in those coadjoint orbits which are 'quantizable' in the sense of geometric quantization, since their quantization should produce all irreducible representations of $U_0(\mathcal{H})$ [19, 20]. Each such orbit is labeled by a pair $(\mathsf{m}, \mathsf{n})$, where $\mathsf{m}$ is an ordered $M$-tuple of positive integers satisfying $m_1 \geq m_2 \geq \ldots m_M > 0$, and $\mathsf{n}$ is a similar $N$-tuple $(M, N < \infty)$. The coadjoint orbit $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ consists of all elements of $\mathbf{u}_0(\mathcal{H})^*$ with eigenvalues $m_1, m_2, \ldots, m_M, 0^\infty, -n_N, \ldots, -n_1$. The degeneracy of each numerical eigenvalue $m_i$ (or $-n_j$) is simply the number of times it occurs in this list. The explicit quantization of the orbits $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ is not discussed in [19, 20]; the case where either $\mathsf{m}$ or $\mathsf{n}$ is empty is done in [5] using geometric quantization.

For finite-dimensional $\mathcal{H}$, it was shown by Montgomery [30] that $\mathcal{O}_{\mathsf{m},0}$ can be written as a Marsden-Weinstein reduced space with respect to the natural right-action of $U(M)$ on $\mathcal{H} \otimes \mathbb{C}^M$. This is a special instance of the theory of dual pairs. With $\mathcal{H} = \mathbb{C}^k$, the groups $U(\mathcal{H})$ and $U(M)$ form a Howe dual pair inside the symplectic group $Sp(2kM, \mathbb{R})$ [15, 40, 13], and the momentum maps $J_R$ and $J_L$ introduced below build a Weinstein dual pair, cf. [18, 44]. General theorems on the connection between coadjoint orbits of one group and Marsden-Weinstein reduced spaces with respect to the other group in a dual pair are given in [25]. We will now generalize the special case mentioned above to infinite-dimensional $\mathcal{H}$, and general orbits $\mathcal{O}_{\mathsf{m},\mathsf{n}}$.

We take $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$, which we regard as a Hilbert manifold in the obvious way. We choose the canonical basis $\{e_i\}_{i=1,\ldots,M+N}$ in $\mathbb{C}^{M+N}$. The symplectic form $\omega$ on $S$ is taken as (we put $\hbar = 1$)

$$\omega(\psi, \varphi) = -2 \operatorname{Im} \left( \sum_{i=1}^{M} (\psi_i, \varphi_i) - \sum_{i=M+1}^{M+N} (\psi_i, \varphi_i) \right), \tag{3}$$

where we have expanded $\psi = \sum_i \psi_i \otimes e_i$ and similarly for $\varphi$. It is convenient to introduce an indefinite sesquilinear form on $\mathbb{C}^{M+N}$ by putting $(e_i, e_j) = \pm\delta_{ij}$, with a plus sign for $i = 1, \ldots, M$ and a minus sign for $i = M+1, \ldots, M+N$. Together with the inner product on $\mathcal{H}$ this induces an indefinite form $(\cdot, \cdot)_S$ on $S$ in the obvious (tensor product) way. The right-hand side of (3) then simply reads $-2 \operatorname{Im} (\psi, \varphi)_S$. A simple trick shows that $S$ is strongly symplectic: we can regard $S$ as a Hilbert space $\mathcal{H} \otimes \mathbb{C}^M \oplus \overline{\mathcal{H} \otimes \mathbb{C}^N}$, with inner product $(\psi, \varphi)_{\text{trick}} = \sum_{i=1}^{M} (\psi_i, \varphi_i) + \sum_{i=M+1}^{M+N} (\varphi_i, \psi_i)$. Then $\omega(\psi, \varphi) = -2 \operatorname{Im} (\psi, \varphi)_{\text{trick}}$, and the claim follows from the well-known fact that Hilbert spaces are strongly symplectic [1].

The Lie group $H = U(M, N)$ (which is $U(M)$ or $U(N)$ for $\mathsf{n}$ or $\mathsf{m}$ empty) acts on $S$ from the right in the obvious way, i.e., by $U \to \mathbb{I} \otimes U^T$. This action is symplectic, with anti-equivariant momentum map $J_R : S \to (\mathbf{h}^*)^-$. If we identify $X \in \mathbf{h}$ with a generator in the defining representation of $H$ on $\mathbb{C}^{M+N}$, we obtain (cf. [18, p. 501])

$$\langle J_R(\psi), X \rangle = i(\mathbb{I} \otimes X^T \psi, \psi)_S. \tag{4}$$

On a suitable Cartan subalgebra $\mathfrak{t}$ of $\mathbf{h}$, which we identify as the set of imaginary diagonal operators on $\mathbb{C}^{M+N}$, with basis $H_j = -iE_{jj}$, this simply reads $\langle J_R(\psi), H_j \rangle = \pm(\psi_j, \psi_j)$ with a plus sign for $j = 1, \ldots, M$ and a minus sign for $j = M+1, \ldots, M+N$.

We now identify $(\mathsf{m}, \mathsf{n})$ with an element of $\mathbf{h}^*$ by the pairing $\langle (\mathsf{m}, \mathsf{n}), X \rangle = i \mathrm{Tr}\, D_{(\mathsf{m},\mathsf{n})} X$, where $D_{(\mathsf{m},\mathsf{n})}$ is the diagonal matrix in $M_{M+N}(\mathbb{C})$ with entries $m_1, \ldots, m_M, -n_N, \ldots, -n_1$. This means that $(\mathsf{m}, \mathsf{n})$ defines a dominant integral weight on $\mathfrak{t}$, and vanishes on its complement. The subset $J_R^{-1}((\mathsf{m}, \mathsf{n}))$ of $S$ consists of those vectors $\psi = \sum_i \psi_i \otimes e_i$ for which $(\psi_i, \psi_i) = m_i$ for $i = 1, \ldots, m$, and $(\psi_{M+j}, \psi_{M+j}) = n_{N+1-j}$ for $j = 1, \ldots, n$, with the $\psi_k$'s mutually orthogonal. The normalizations come from $J_R$ evaluated on $\mathfrak{t}$, and the orthogonality derives from the constraint that $J_R$ vanish on its complement. *Note that the integrality of the $m_i$ and $n_j$ plays no role in this subsection.*

**Lemma 1** $J_R^{-1}((\mathsf{m}, \mathsf{n}))$ *is a submanifold of $S$.*

*Proof.* According to the theorem on p. 550 of [7], we need to show that $J_R : J_R^{-1}((\mathsf{m}, \mathsf{n})) \to \mathbf{h}^*$ is a submersion, which is the case if at any point $\psi \in J_R^{-1}((\mathsf{m}, \mathsf{n})) \subset S$ the derivative $(J_R)_* \equiv J_R^{(1)} : T_\psi S \to T_{J_R(\psi)} \mathbf{h}^* \simeq \mathbf{h}^*$ is surjective and has a complementable kernel. The former is equivalent to the statement that $\psi$ is a regular value of the momentum map [1]. The derivative at $\psi \in S$ follows from (4) as

$$\langle (J_R^{(1)})_\psi(\xi), X \rangle = 2 \mathrm{Re}\, (\mathbb{I} \otimes i X^T \xi, \psi)_S. \tag{5}$$

This formula shows that $J_R^{(1)}$ is continuous, so that its kernel is closed. The complementability of this kernel is then immediate, since $S$ is a Hilbert manifold. The surjectivity of $J_R^{(1)}$ follows from (5) by inspection, but it is more instructive to derive it from Prop. 2.11 (due to Smale) in [27]. This states that $\psi$ is a regular value of the momentum map iff the stability group $H_\psi \subseteq H$ of $\psi$ is discrete. Now, as pointed out earlier, $\psi = \sum_i \psi_i \otimes e_i \in J_R^{-1}((\mathsf{m}, \mathsf{n}))$ implies that all $\psi_i$ are nonzero are orthogonal, so that $H_\psi$ is just the identity. ∎

The action of $H$ on $S$ is not proper unless $\mathsf{m}$ or $\mathsf{n}$ is empty (in which case $H$ is compact). However:

**Lemma 2** *The action of $H$ on $J_R^{-1}((\mathsf{m}, \mathsf{n}))$ is proper.*

*Proof.* Let $\psi^{(n)} \to \psi$ in $S$; equivalently, $\psi_i^{(n)} \to \psi_i$ in $\mathcal{H}$ for all $i$. If $\{U^{(n)}\}$ is a sequence in $H$ and $U^{(n)} \psi^{(n)}$ converges, the fact that for each $n$ all $\psi_i^{(n)}$ are nonzero and orthogonal implies that $\{U_{ij}^{(n)} e_j\}$ must converge in $\mathbb{C}^{M+N}$ for each $i$. Since convergence in the topology on $U(M, N)$ is given by convergence of all matrix elements in the defining representation, this implies that $\{U^{(n)}\}$ must converge in $H$. ∎

By the standard theory of Marsden-Weinstein reduction [26, 1], these lemmas imply that the reduced space

$$S^{(\mathsf{m},\mathsf{n})} = J_R^{-1}((\mathsf{m}, \mathsf{n}))/H_{(\mathsf{m},\mathsf{n})} \tag{6}$$

(where $H_{(\mathsf{m},\mathsf{n})}$ is the stability group of $(\mathsf{m}, \mathsf{n}) \in \mathbf{h}^*$ under the coadjoint action) is a smooth symplectic manifold. We will proceed to show that it is symplectomorphic to the coadjoint orbit $\mathcal{O}_{\mathsf{m},\mathsf{n}} \in \mathbf{g}^*$, where $G = U_0(\mathcal{H})$, as explained above. The required diffeomorphism is given by a quotient of the momentum map $J_L : S \to \mathbf{g}^*$ defined from the natural left-action of $G$ on $S$, which action is evidently symplectic. Identifying $\mathbf{g}$ with the space

of compact skew-adjoint operators $Y$ on $\mathcal{H}$, one easily finds that this momentum map is given by

$$-i\langle J_L(\psi), Y \rangle = (Y \otimes \mathbb{I}\psi, \psi)_S = \sum_{i=1}^{M}(Y\psi_i, \psi_i) - \sum_{i=M+1}^{M+N}(Y\psi_i, \psi_i). \tag{7}$$

Since the left-$G$ action and the right-$H$ action commute, $J_L$ is invariant under $H$ (i.e., $J_L(\psi U) = J_L(\psi)$ for all $U \in H$ and $\psi \in \mathcal{H}$), so that $J_L$ (restricted to $J_R^{-1}((\mathsf{m},\mathsf{n})))$ quotients to a well-defined map $\tilde{J}_L : S^{(\mathsf{m},\mathsf{n})} \to \mathcal{O}_{\mathsf{m},\mathsf{n}}$. Once we have shown that $\tilde{J}_L$ is a diffeomorphism, it will follow that it is symplectic, because of the definition of the symplectic structure on $S^{(\mathsf{m},\mathsf{n})}$ and the fact that $J_L$ is equivariant.

Generalizing a standard result in the root and weight theory for compact Lie groups, see e.g. [21], we first note that the the stability group of $(\mathsf{m},\mathsf{n}) \in \mathbf{h}^*$ under the coadjoint action is $H_{(\mathsf{m},\mathsf{n})} = \prod_l U(l)$, where $\sum l = M + N$, and the product is over the multiplicities within either $\mathsf{m}$ or $\mathsf{n}$ in $(\mathsf{m},\mathsf{n})$; this is a subgroup of $U(M,N)$ in the obvious block-diagonal form. (For example, if $(\mathsf{m},\mathsf{n}) = ((2,1,1),(2,2,2))$ the stability group is $U(1) \times U(2) \times U(3)$.) It then follows from (7) that $\tilde{J}_L$ is a bijection onto $\mathcal{O}_{\mathsf{m},\mathsf{n}}$.

**Proposition 3** $\tilde{J}_L$ *is smooth.*

*Proof.* The manifold structure of $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ is defined by its embedding in $\mathbf{g}^*$, which is a Banach space in the trace-norm topology (cf. the beginning of this section). The smoothness of $\tilde{J}_L$ then follows from that of $J_L : J_R^{-1}((\mathsf{m},\mathsf{n})) \to \mathbf{g}^*$, since the Lie group $H$ acts smoothly, freely, and properly on $J_R^{-1}((\mathsf{m},\mathsf{n}))$.

1. *Continuity of $J_L$.* We prove continuity on all of $S$. As a map between separable metric spaces ($S$ is separable because $\mathcal{H}$ is by assumption, and $\mathbf{g}^*$ is separable because the finite-rank operators are dense in it), $J_L$ is continuous if $\psi^{(n)} \to \psi$ in $S$ implies $J_L(\psi^{(n)}) \to J_L(\psi)$ in $\mathbf{g}^*$. The topology on $\mathbf{g}^*$ coincides with the weak$^*$-topology, so the desired continuity follows from (7), the boundedness of $Y$, and Cauchy-Schwartz.

2. *Existence and continuity of $J_L^{(1)}$.* The derivative of $J_L$ at $\psi$ is given by

$$\langle (J_L^{(1)})_\psi(\xi), Y \rangle = 2\mathrm{Re}\left( \sum_{i=1}^{M}(iY\xi_i, \psi_i) - \sum_{i=M+1}^{M+N}(iY\xi_i, \psi_i) \right). \tag{8}$$

By the same reasoning as in the previous item, $(J_L^{(1)})_\psi$ lies in $\mathcal{L}(S, \mathbf{g}^*)$ and is continuous.

The second derivative $J_L^{(2)} : S \times S \to \mathbf{g}^*$ can be read off from (8); its existence and continuity are established as before. Higher derivatives vanish. ∎

**Proposition 4** $\tilde{J}_L^{-1}$ *is smooth.*

*Proof.* We pick an arbitrary point $\rho_0 \in \mathcal{O}_{\mathsf{m},\mathsf{n}}$, with stability group $G_0$. Let $\mathcal{H} = \oplus_l \mathcal{H}_l$ be the decomposition of $\mathcal{H}$ under which $\rho_0$ is diagonal (the dimension of each $\mathcal{H}_0$ is the degeneracy of the corresponding eigenvalue; this dimension is finite unless the eigenvalue is 0). Then $G_0 = \oplus_l U_0(\mathcal{H}_l)$, in self-evident notation. The Lie algebra $\mathbf{g}_0$ of $G_0$ is given by those operators in $\mathbf{g} = i\mathfrak{K}(\mathcal{H})_{\mathrm{sa}}$ which commute with $\rho_0$. The manifold $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ is modelled on $\mathbf{g}/\mathbf{g}_0$. This has the quotient topology inherited from $\mathbf{g}$, i.e., the trace-norm topology determined by $\| A \|_1 = \mathrm{Tr}\,|A|$.

We define a neighbourhood $V_0 \subset \mathcal{O}_{\mathsf{m},\mathsf{n}}$ of $\rho_0$ as follows. Since $G$ is a Banach-Lie group, by [24] there exists a neighbourhoud $V$ of $0 \in \mathbf{g}$ such that exp is a diffeomorphism on $V$ into $\mathbf{g}$. We put $V_0 = \{U_{\mathrm{co}}(\exp(A))\rho_0 | A \in V\}$ (recall that the coadjoint action is given by $U_{\mathrm{co}}(U)\rho = U\rho U^*$). To define a chart on $V_0$, we first show that $\mathbf{g}$ (equipped with the trace-norm topology) admits a splitting $\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{m}_0$. Here $\mathbf{m}_0$ consists of those operators $A$ in $\mathbf{g}$ whose matrix elements $(A\psi, \varphi)$ vanish if both $\psi$ and $\varphi$ lie in the same space $\mathcal{H}_l$, for all $l$. It is clear that $\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{m}_0$ as a set, and it quickly folows that each summand is closed: since $\| A \| \leq \| A \|_1$, the uniform topology is weaker than the trace-norm one, so that closedness in the former implies the corresponding property in the latter topology. As to the uniform closedness of $g_0$, one has $\| [A, \rho_0] \| \leq 2 \| A \| \| \rho_0 \|$, so that $\mathbf{g}_0 \ni A_n \to A$ implies that $A \in \mathbf{g}_0$. On $\mathbf{m}_0$ an even more elementary inequality does the job. Thus $\mathbf{g}/\mathbf{g}_0 \simeq \mathbf{m}_0$, and we may use $\mathbf{m}_0$ as a modelling space for $\mathcal{O}_{\mathsf{m},\mathsf{n}}$.

We define a chart on $V_0$ by $\varphi_0 : V_0 \to \mathbf{m}_0$, given by $\varphi_0(U_{\mathrm{co}}(\exp(A))\rho_0) = A_0$, where $A_0$ is the component of $A \in \mathbf{g}$ in $\mathbf{m}_0$. We would like to model $S^{(\mathsf{m},\mathsf{n})}$ on $\mathbf{m}_0$ as well, but this is not directly possible because it has the wrong topology. Hence the following detour. Take a $\psi_0 \in J_R^{-1}((\mathsf{m}, \mathsf{n})) \subset S$ for which $J_L(\psi_0) = \rho_0$. Using the fact that $J_L$ is a bijection, we model $S^{(\mathsf{m},\mathsf{n})} = J_R^{-1}((\mathsf{m}, \mathsf{n}))/H_{(\mathsf{m},\mathsf{n})}$ on the closed linear subspace of $S$ given by $M_0 = \{A \otimes \mathbb{I}\psi_0 | A \in \mathbf{m}_0\}$, equipped with the relative topology of $S$. Put $W_0 = \{\exp(A) \otimes \mathbb{I}\psi_0 | A \in \mathbf{m}_0\} \subset S$. If $pr : J_R^{-1} \to J_R^{-1}((\mathsf{m}, \mathsf{n}))/H_{(\mathsf{m},\mathsf{n})}$ is the canonical projection, we have a chart on the neighbourhood $pr(W_0)$ of $pr(\psi_0)$ defined by $\phi_0 : pr(W_0) \to M_0$ given by $\phi_0(pr(\exp(A)\psi_0)) = A\psi_0$. This procedure respects the manifold structure of $S^{(\mathsf{m},\mathsf{n})}$, which by definition is quotiented from $J_R^{-1}((\mathsf{m}, \mathsf{n})) \subset S$.

We now define $_0\tilde{J}_L^{-1} = \phi_0 \circ \tilde{J}_L^{-1} \circ \varphi_0^{-1}$; this is a map from $\varphi_0(V_0) \subset \mathbf{m}_0$ to $\phi_0 \circ pr(W_0) \subset M_0$. Clearly, $_0\tilde{J}_L^{-1}(A) = A\psi_0$. This immediately implies that $_0\tilde{J}_L^{-1}$, and therefore $\tilde{J}_L^{-1}$, is smooth. ∎

To sum up, we have proved

**Theorem 5** *For any separable Hilbert space $\mathcal{H}$, the coadjoint orbit $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ of the group $U_0(\mathcal{H})$ (which consists of all trace-class operators on $\mathcal{H}$ with $M$ specific positive and $N$ specific negative eigenvalues) is symplectomorphic to the Marsden-Weinstein quotient $S^{(\mathsf{m},\mathsf{n})} = J_R^{-1}((\mathsf{m}, \mathsf{n}))/H_{(\mathsf{m},\mathsf{n})}$ with respect to $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$ and the natural right-action of $H = U(M, N)$.*

## 2.3   Representations induced from $U(M)$

The representations of $U_0(\mathcal{H})$ were fully classified in [19, 32, 33] (also cf. [20, 35, 6]). A remarkable fact is that $U_0(\mathcal{H})$ is a type I group, so that all its factorial representations are of the form $U \otimes \mathbb{I}$ on $\mathcal{H}_U \otimes \mathcal{H}_{\mathrm{mult}}$, where $(U, \mathcal{H}_U)$ is irreducible. Each irreducible representation corresponds to an integral weight $(\mathsf{m}, \mathsf{n})$ of the type specified above, where $M$ and $N$ are arbitrary (but finite). The carrier space $\mathcal{H}^{(\mathsf{m},\mathsf{n})}$ is of the form $\mathcal{H}^{(\mathsf{m},\mathsf{n})} = \mathcal{H}^{\mathsf{m}} \otimes \overline{\mathcal{H}}^{\mathsf{n}}$, and carries the irreducible representation $U^{(\mathsf{m},\mathsf{n})} = U^{\mathsf{m}} \otimes \overline{U}^{\mathsf{n}}$. Here $\mathcal{H}^{\mathsf{m}}$ is the subspace of $\otimes^M \mathcal{H}$ obtained by symmetrization according to the Young diagram whose $k$-th row has length $m_k$, and $\overline{\mathcal{H}}^{\mathsf{n}}$ is the conjugate space of $\mathcal{H}^{\mathsf{n}}$. The representation $U^{\mathsf{m}}$ is the one given by the restriction of the $M$-fold tensor product of the defining representation of $U_0(\mathcal{H})$ to $\mathcal{H}^{\mathsf{m}}$, etc.

This is almost identical to the theory for finite-dimensional $\mathcal{H} = \mathbb{C}^k$ [45, 48] (which

has the obvious restriction that $M, N \leq k$); the only difference is that in the infinite-dimensional case $\mathcal{H}^{\mathsf{m}} \otimes \overline{\mathcal{H}}^{\mathsf{n}}$ is already irreducible. For $k < \infty$, on the other hand, one needs to take the so-called Young product [48] of $\mathcal{H}^{\mathsf{m}}$ and $\overline{\mathcal{H}}^{\mathsf{n}}$ rather than the tensor product (this is the irreducible subspace generated by the tensor product of the highest-weight vectors in each factor); moreover, the use of conjugate spaces may be avoided in that case by tensoring with powers of the determinant representation. For example, $\mathbb{C}^k \otimes \overline{\mathbb{C}}^k$ contains the irreducible subspace $\sum_{i=1}^{k} e_i \otimes \overline{e_i}$ which does not lie in the Young product; for $k = \infty$ this subspace evidently no longer exists. For $M = 0$ or $N = 0$ there is no difference whatsoever.

We will now show how the representations $(U^{\mathsf{m}}, \mathcal{H}^{\mathsf{m}})$ can be obtained by Rieffel induction; the representations $(\overline{U}^{\mathsf{n}}, \overline{\mathcal{H}}^{\mathsf{n}})$ may then be constructed similarly. This will quantize the coadjoint orbits $\mathcal{O}_{\mathsf{m}} \equiv \mathcal{O}_{(\mathsf{m}, \emptyset)}$ and $\mathcal{O}_{\mathsf{n}}^{-} \equiv \mathcal{O}_{(\emptyset, \mathsf{n})}$, respectively. We note that $\mathcal{O}_{\mathsf{n}}^{-}$ is $\mathcal{O}_{\mathsf{n}}$ with the sign of the symplectic form changed; this relative minus sign corresponds to the passage from $\mathcal{H}$ to $\overline{\mathcal{H}}$ upon quantization.

Our starting point is Theorem 5, in which we take $S = \mathcal{H} \otimes \mathbb{C}^M$, with $H = U(M)$ acting on $S$ from the right and $G = U_0(\mathcal{H})$ acting from the left in the natural way; we call these actions $U_1^T(H)$ and $U_1(G)$, respectively. As explained in the Introduction, we first have to quantize $S$ and the group actions defined on it. We do so by taking the bosonic second quantization, or symmetric Fock space, $\mathcal{F} = \exp(S)$ over $S$ [37, 46], cf. subsection 2.1. For later use, we equivalently define this as the subspace of $\sum_{n=0}^{\infty} \otimes^n S$ on which the natural representation of the symmetric group $S_n$ on $\otimes^n S$ acts trivially for all $n$.

As in the $M = 1$ case (cf. subsection 2.1) we first investigate the representations of $U_0(\mathcal{H})$ and $U(k)$ on $\mathcal{F}$ obtained by second quantization, or equivalently, by geometric quantization without the half-form modification. This goes as follows. The groups $H$ and $G$ act on each subspace $\otimes^n S$ by the $n$-fold tensor product of their respective actions on $S$, and these actions restrict to $\mathcal{F}$. Thus the actions $U_1^T(H)$ (which we turn into a representation by taking the inverse) and $U_1(G)$ on $S$ are quantized by the unitary representations $\Gamma \overline{U}_1(H)$ ($= U_{R,\mathrm{sq}}^{-1}(H)$ in the notation of subsection 2.1, and $U_R^{-1}(H)$ in that of the Introduction) and $\Gamma U_1(G)$ ($= U_{L,\mathrm{sq}}(G)$), respectively (note that $U_1^T(h^{-1}) = \overline{U}_1(h)$). Here $\Gamma$ is the second quantization functor [37]. This setup, and the associated central decomposition of $\mathcal{F}$ under these group actions, illustrate Howe's theory of dual pairs [15, 12, 13] in an infinite-dimensional setting, cf. [35].

The fact that the coadjoint orbit $\mathcal{O}_{\mathsf{m}}$ of $G$ is (symplectomorphic to) the Marsden-Weinstein quotient of $S$ with respect to $\mathsf{m} \in \mathbf{h}^*$, cf. Theorem 5, should now be reflected, or rather quantized, by constructing the unitary representation $U^{\mathsf{m}}(G)$ (which according to Kirillov is attached to $\mathcal{O}_{\mathsf{m}}$) by Rieffel induction from the representation $U_{\mathsf{m}}(H)$ attached to the orbit through $\mathsf{m}$ in $H$. Here $U_{\mathsf{m}}(U(M))$ is simply the unitary irreducible representation given by the highest weight $\mathsf{m}$; it is realized on $\mathcal{H}_{\mathsf{m}}$, which is the subspace of $\otimes^M \mathbb{C}^M$ obtained by symmetrization according to the Young diagram whose $k$-th row has length $m_k$.

To find the carrier space of the induced representation $U^{\mathsf{m}}(G)$ we merely have to identify the subspace of $\mathcal{F} \otimes \mathcal{H}_{\mathsf{m}}$ which is invariant under $\Gamma \overline{U}_1 \otimes U_{\mathsf{m}}(H)$. This is very easy on the basis of the following well-known facts [45, 48, 14]:

1. The representations of the symmetric group $S_n$ are self-conjugate; for any irreducible representation $U_{\mathsf{l}}(S_n)$, the tensor product $U_{\mathsf{l}} \otimes U_{\mathsf{l}}$ contains the identity representation

once, and $U_\mathsf{l} \otimes U_{\mathsf{l}'}$ does not contain the identity unless $\mathsf{l} = \mathsf{l}'$. (Recall that the irreducible representations of $S_n$ are labelled by an $n$-tuple of integers $\mathsf{l} = (l_1, \ldots, l_n)$, where $l_1 \geq l_2 \geq \ldots l_n \geq 0$ and $\sum_i l_i = n$.) The collection of all such $n$-tuples $\mathsf{l}$ forms the dual $\hat{S}_n$.

2. Any unitary irreducible representation $U_\mathsf{l}(U(M))$ is given by an $M$-tuple $\mathsf{l} = (l_1, \ldots, l_M)$ of positive nondecreasing integers (possibly zero), as in the preceding item, or by the conjugate $\overline{U}_\mathsf{l}$ of such a representation. Then $U_\mathsf{l} \otimes \overline{U}_\mathsf{l}$ contains the identity representation once, but the identity does not occur in any $U_\mathsf{l} \otimes U_{\mathsf{l}'}$, or in any $U_\mathsf{l} \otimes \overline{U}_{\mathsf{l}'}$ unless in the latter case $\mathsf{l} = \mathsf{l}'$.

3. The defining representation of $S_n$ on $\otimes^n \mathbb{C}^M$ commutes with the $n$-fold tensor product of the conjugate of the defining representation of $U(M)$, so that one has the central decomposition

$$\otimes^n \mathbb{C}^M \simeq \bigoplus_{\mathsf{l}' \in \hat{S}_n}{}' \mathcal{H}_\mathsf{l}^{S_n} \otimes \overline{\mathcal{H}}_\mathsf{l}^{U(M)}, \tag{9}$$

where the prime (relevant only when $M < n$) on the $\oplus$ indicates that the sum is only over those $n$-tuples $\mathsf{l}$ for which $l_{M+1} = 0$. Here $\mathcal{H}_\mathsf{l}^{S_n}$ is the carrier space of $U_\mathsf{l}(S^n)$, and $\overline{\mathcal{H}}_\mathsf{l}^{U(M)}$ is the carrier space of the conjugate of the irreducible representation of $U(M)$ obtained by making $\mathsf{l}$ an $M$-tuple by adding or removing zeros. (A simliar statement holds without the conjugation, of course.)

4. Similarly,

$$\otimes^n \mathcal{H} \simeq \bigoplus_{\mathsf{l} \in \hat{S}_n} \mathcal{H}_\mathsf{l}^{S_n} \otimes \mathcal{H}^\mathsf{m}, \tag{10}$$

under the appropriate representations of $S_n$ and $U_0(\mathcal{H})$, where $\mathcal{H}^\mathsf{m}$ was introduced at the beginning of this subsection (for $\mathcal{H} = \mathbb{C}^k$ this is equivalent to a classical result in invariant theory, see e.g. [14, 4.3.3.9]).

Now consider $\otimes^n(\mathcal{H} \otimes \mathbb{C}^M) \simeq \otimes^n \mathcal{H} \otimes \otimes^n \mathbb{C}^M$. This carries the representation $U_n^\mathcal{H} \otimes U_n^{\mathbb{C}^M}$ of $S_n$, where $U_n^\mathcal{K}(S_n)$ is the natural representation on $\otimes^n \mathcal{K}$. Applying items 4 and 3, and subsequently 1 above, we find that the subspace $\otimes_s^n(\mathcal{H} \otimes \mathbb{C}^M) \subset \otimes^n(\mathcal{H} \otimes \mathbb{C}^M)$ which is invariant under $S_n$ can be decomposed as

$$\bigotimes_s^n (\mathcal{H} \otimes \mathbb{C}^M) \simeq \bigoplus_{\mathsf{l}' \in \hat{S}_n} \mathcal{H}^\mathsf{l} \otimes \overline{\mathcal{H}}_\mathsf{l}^{U(M)}, \tag{11}$$

in the sense that the restriction $\otimes_s^n(U_1(G) \otimes \overline{U}_1(H))$ of $\Gamma U_1(G) \otimes \Gamma \overline{U}_1(H)$ (defined on $\mathcal{F} = \exp(\mathcal{H} \otimes \mathbb{C}^M)$) to $\otimes_s^n(\mathcal{H} \otimes \mathbb{C}^M) \subset \mathcal{F}$ decomposes as

$$\bigotimes_s^n (U_1(G) \otimes \overline{U}_1(H)) \simeq \bigoplus_{\mathsf{l}' \in \hat{S}_n} U^\mathsf{l}(G) \otimes \overline{U}_\mathsf{l}(H). \tag{12}$$

We then apply item 2 to conclude that the only subspace of $\mathcal{F} \otimes \mathcal{H}_{\mathsf{m}}$ which is invariant under $\Gamma \overline{U}_1 \otimes U_{\mathsf{m}}(H)$ corresponds to $n = \sum_{i=1}^{M} m_i$ (where $m_i$ are the entries of the $M$-tuple $\mathsf{m}$). Moreover, by (11) this invariant subspace is exactly $\mathcal{H}^{\mathsf{m}}$ as a $U_0(\mathcal{H})$ module. Hence we have proved

**Theorem 6** *Regard the symmetric Fock space $\mathcal{F} = \exp(\mathcal{H} \otimes \mathbb{C}^M)$ as a left-module (representation space) of $U_0(\mathcal{H})$ and a right-module of $U(M)$ under the second quantization of their respective natural actions on $\mathcal{H} \otimes \mathbb{C}^M$. Applying Rieffel induction to this bimodule, inducing from the irreducible representation $U_{\mathsf{m}}(U(M))$ (which corresponds to the highest weight $\mathsf{m} = (m_1, \ldots, m_M)$), yields the induced space $\mathcal{H}^{\mathsf{m}}$ carrying the irreducible representation $U^{\mathsf{m}}(U_0(\mathcal{H}))$.*

This, then, is the exact quantum counterpart of Theorem 5, specialized to $\mathsf{n} = \emptyset$. As remarked earlier, there exists an obvious analogue of Theorem 6 for $\mathsf{m} = \emptyset$, in which all Hilbert spaces and representations occurring in the construction are replaced by their conjugates.

To prepare for the next subsection we will now give a slight reformulation of the proof. We start with finite-dimensional $\mathcal{H} = \mathbb{C}^k$, with $k > M$. Classical invariant theory [14] then provides the decomposition of $\exp(S)$ under $\Gamma U_1(U(k)) \otimes \Gamma \overline{U}_1(U(M))$ as

$$\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \stackrel{\text{sq}}{\simeq} \bigoplus_{\mathsf{l} \in D_M} \mathcal{H}_{\mathsf{l}}^{U(k)} \otimes \overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}, \tag{13}$$

where the sum is over all Young diagrams (or tuples) $D_M$ with $M$ rows or less, including the empty diagram. (Note that it would have been consistent with our previous notation to write $(\mathcal{H}^{\mathsf{l}})^{U(k)}$ for $\mathcal{H}_{\mathsf{l}}^{U(k)}$; both stand for the irreducible representation of $U(k)$ defined by the Young diagram $\mathsf{l}$. In what follows, we will reserve the notation $\mathcal{H}^{\mathsf{l}}$ for $\mathcal{H}_{\mathsf{l}}(U_0(\mathcal{H}))$, where $\mathcal{H} = l^2$.) Eq. (13) is an illustration of the theory of Howe dual pairs [15, 12, 13]: it exhibits a multiplicity-free central decomposition of $\mathcal{F} = \exp(S)$ under the commuting actions of $U(k)$ and $U(M)$ (which form a dual pair in $Sp(2kM, \mathbb{R})$, of which $\mathcal{F}$ carries the metaplectic representation).

In order to study the limit $k \to \infty$ we realize $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ (with $\mathcal{H} = l^2$ now infinite-dimesional) as an (incomplete) infinite tensor product [31] with respect to the vacuum vector $\Omega \in \exp(\mathbb{C}^M)$, that is (recalling $\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \simeq \otimes^k \exp(\mathbb{C}^M)$), $\exp(\mathcal{H} \otimes \mathbb{C}^M) \simeq \otimes_{\Omega}^{\infty} \exp(\mathbb{C}^M)$, where the right-hand side is the Hilbert space closure (with respect to the natural inner product on tensor products) of the linear span of all vectors of the type $\psi_1 \otimes \ldots \psi_l \otimes \Omega \otimes \Omega \ldots$, $\psi_i \in \exp(\mathbb{C}^M)$, in which only finitely many entries differ from $\Omega$. (The term 'incomplete' refers to the fact that only 'tails' close to an infinite product of $\Omega$'s appear.) Thus $\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \simeq \otimes^k \exp(\mathbb{C}^M)$ is naturally embedded in $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ by simply adding an infinite tail of $\Omega$'s, and this provides an embedding $\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \subset \exp(\mathbb{C}^{k+1} \otimes \mathbb{C}^M)$ as well. Clearly, $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ coincides with the closure of the inductive limit $\cup_{k=1}^{\infty} \exp(\mathbb{C}^k \otimes \mathbb{C}^M)$ defined by this embedding.

Choosing the natural basis in $\mathcal{H} = l^2$, we obtain an embedding $U(k) \subset U(k+1)$, with corresponding actions on $\mathcal{H}$; our group $U_0(\mathcal{H})$ (realized in its defining representation on $\mathcal{H}$) is the norm-closure of the inductive limit group $\cup_{k=1}^{\infty} U(k)$. Using the explicit realization of $\mathcal{H}^{\mathsf{l}}$ as a Young-symmetrized tensor product, we similarly obtain embeddings $\mathcal{H}_{\mathsf{l}}(U(k)) \subset \mathcal{H}_{\mathsf{l}}(U(k+1))$. Thus the inductive limit $\cup_{k=1}^{\infty} \mathcal{H}_{\mathsf{l}}(U(k))$ is well-defined. Using (13), we

then have that $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ is the closure of $\cup_{k=1}^{\infty} \oplus_{l \in D_M} \mathcal{H}_l^{U(k)} \otimes \overline{\mathcal{H}}_l^{U(M)}$, which in turn coincides with the closure of $\oplus_{l \in D_M} \cup_{k=1}^{\infty} \mathcal{H}_l^{U(k)} \otimes \overline{\mathcal{H}}_l^{U(M)}$. We now use the fact that the closure of $\cup_{k=1}^{\infty} \mathcal{H}_l^{U(k)}$ is $\mathcal{H}^l$ as a representation space of $U_0(\mathcal{H})$ (this is obvious given the explicit realization of these spaces, but it is a deep result that an analogous fact holds for all representations of $U_0(\mathcal{H})$ [32, 33, 35]). This yields the desired decomposition

$$\exp(\mathcal{H} \otimes \mathbb{C}^M) \overset{\text{sq}}{\cong} \bigoplus_{l \in D_M} \mathcal{H}_l \otimes \overline{\mathcal{H}}_l^{U(M)}, \tag{14}$$

under $\Gamma U_1(U_0(\mathcal{H})) \otimes \Gamma \overline{U}_1(U(M))$. This result was previously derived in [35] using a technique of holomorphic extension of representations.

Starting from (14), Theorem 6 follows immediately from item 2 on the list of ingredients of our previous proof.

To end this subsection we register how the half-form correction to geometric quantization modifies (13), cf. subsection 2.1, and in particular (2). These corrections are finite only for $\mathcal{H} = \mathbb{C}^k$, $k < \infty$, so we only discuss that case. As for $M = 1$, one finds that the half-form quantizations of the momentum maps corresponding to the $U(k)$ and $U(M)$ actions on $\mathbb{C}^k \otimes \mathbb{C}^M$ lead to Lie algebra representations that can only be exponentiated to representations $U_{L,\text{hf}}$ and $U_{R,\text{hf}}^{-1}$ of the covering groups $\tilde{U}(k)$ and $\tilde{U}(M)$ of $U(k)$ and $U(M)$, respectively, on which the square-root of the determinant is defined. A straightforward exercise leads to the decomposition

$$\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \overset{\text{hf}}{\cong} \bigoplus_{l \in D_M} \mathcal{H}_{l+\frac{1}{2}M}^{\tilde{U}(k)} \otimes \overline{\mathcal{H}}_{l+\frac{1}{2}k}^{\tilde{U}(M)} \tag{15}$$

under $U_{L,\text{hf}}(\tilde{U}(k)) \otimes U_{R,\text{hf}}^{-1}(\tilde{U}(M))$. Here $l + \frac{1}{2}M$, regarded as a highest weight, has components $(l_1 + \frac{1}{2}M, l_2 + \frac{1}{2}M, \ldots)$, and analogously for $l + \frac{1}{2}k$. Hence $\mathcal{H}_{l+\frac{1}{2}M}$ carries the tensor product of the representation of $\tilde{U}(k)$ characterized by the Young diagram $l$, and the determinant representation to the power $M/2$, etc. This will be further discussed in subsection 3.

## 2.4   Representations induced from $U(M, N)$

We are now going to attempt to 'quantize' Theorem 5 for $N \neq 0$. The first problem is finding a unitary representation of $H = U(M, N)$ that corresponds to the dominant integral weight $(\mathsf{m}, \mathsf{n})$ on $\mathfrak{t}$ (or the corresponding coadjoint orbit in $\mathbf{h}^*$, cf. subsection 2.2); this is the representation we should induce from. This problem was solved in [3], partly on the basis of the classification of all unitary highest-weight modules of $U(M, N)$ [9, 16, 34]. In the compact case, each dominant integral weight corresponds to an irreducible unitary representation with this weight as its highest weight. For $U(M, N)$ on the other hand, there are two new phenomena. Firstly, there are further conditions on the dominant integral weight $(\mathsf{m}, \mathsf{n})$, namely that all entries of $\mathsf{m}$ should be different, and that all entries of $\mathsf{n}$ should be different. Secondly, the representation corresponding to $(\mathsf{m}, \mathsf{n})$, albeit a highest weight representation, does not in fact have $(\mathsf{m}, \mathsf{n})$ as its highest weight. Rather, the highest weight corresponding to $(\mathsf{m}, \mathsf{n})$ is 'renormalized': with $m_1 > m_2 > \ldots >$

$m_M > 0$ and $n_1 > n_2 > \ldots > n_N > 0$, the highest weight (naively expected to be $(m_1, \ldots, m_M, -n_N, \ldots, -n_1)$) is in fact

$$(m_1 + \tfrac{1}{2}(N - M) + \tfrac{1}{2}, \ldots, m_i + \tfrac{1}{2}(N - M) + i - \tfrac{1}{2}, \ldots, m_M + \tfrac{1}{2}(N + M) - \tfrac{1}{2},$$

$$-(n_N + \tfrac{1}{2}(M + N) - \tfrac{1}{2}), \ldots, -(n_j + \tfrac{1}{2}(M - N) + j - \tfrac{1}{2}), \ldots, -(n_1 + \tfrac{1}{2}(M - N) + \tfrac{1}{2})).$$

Note that this highest weight is still dominant; however, it may no longer be integral, so that it defines a projective representation of $U(M, N)$ (single-valued on its double cover $\tilde{U}(M, N)$). These highest weight representations belong to the holomorphic discrete series of $U(M, N)$ [21].

The second problem is the quantization of $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$, with the corresponding actions of $G = U_0(\mathcal{H})$ and $H = U(M, N)$. One regards $U(M, N)$ as a subgroup of $Sp(2(M + N), \mathbb{R})$, so that the symplectic action of the former on $\mathbb{C}^{M+N}$ is the restriction of the action of the latter [40, 18]. Due to the special way we defined the $U(M, N)$ action in subsection 2.2 as the inverse of a right-action, the quantization of this action of $Sp(2(M + N), \mathbb{R})$ is then given by the conjugate of the metaplectic representation $U_m$ on $L^2(\mathbb{R}^{M+N}) \equiv \mathcal{L}$, cf. [17, 41, 40]. This defines a representation of the inverse image $\tilde{U}(M, N)$ of $U(M, N)$ in the metaplectic group $Mp(2(M+N), \mathbb{R})$ on $\overline{\mathcal{L}}$, which descends to a projective representation of $U(M, N)$, which we denote by $U_{\mathrm{hf}}(\tilde{U}(M, N))$. As pointed out in [41] and [4] (for $k = 1$), this representation is precisely the one obtained from geometric quantization (in a suitable cohomological variant) if half-forms are taken into account. This yields a first candidate for the quantization of the $U(M, N)$ action on $\mathbb{C}^{M+N}$.

The second possibility is to take the tensor product of the (restriction of) the metaplectic representation of $\tilde{U}(M, N)$ with the square-root of the determinant, which does define a unitary representation $U_{\mathrm{sq}}$ of $U(M, N)$ [41]; see [4] for a construction of this representation from geometric quantization. It is the representation which might be thought of as being defined by the physicists' second quantization on $\exp(\mathbb{C}^{M+N})$, as in the $U(M)$ case. However, since the action of $U(M, N)$ on $\mathbb{C}^{M+N}$ is not unitary, this second quantization is not, in fact, defined. In geometric quantization this lack of unitarity shows up through the non-existence of a totally complex invariant polarization on $S$ which is positive. Consequently, one needs to work with an indefinite such polarization [4], and this leads to complications that will eventually cause a shift in the representations one would naively expect to occur in the decomposition of the quantization of $S$.

For finite-dimensional $\mathcal{H} = \mathbb{C}^k$ we therefore have a suitable quantization of $S = \mathbb{C}^k \otimes \mathbb{C}^{M+N}$, namely the Hilbert space $\overline{\mathcal{L}}_k \equiv \otimes^k \overline{\mathcal{L}}$ (the Fock space realization of this space is not useful, so we drop the notation $\mathcal{F}$). Moreover, we have natural unitary representations $\otimes^k U_{\mathrm{sq/hf}}$ of $\tilde{U}(M, N)$ on $\overline{\mathcal{L}}_k$, which are quantizations of the symplectic action of $U(M, N)$ on $S$. Following our notation for $U(M)$, we refer to these representations as $U_{R,\mathrm{sq/hf}}^{-1}$.

In addition, the quantization of the $U(k)$ action on $S$ may be found (much more easily) from geometric quantization with or without half-forms. The latter case, in which we call the representation $U_{L,\mathrm{sq}}(U(k))$, is explicitly given in [17]. Its half-form variant $U_{L,\mathrm{hf}}(U(k))$ differs from it by the determinant representation raised to the power $(M - N)/2$.

It follows from the theory of Howe dual pairs [15] that $\overline{\mathcal{L}}_k$ decomposes discretely under these representations. Starting with $U_{L,\mathrm{sq}}(U(k)) \otimes U_{R,\mathrm{sq}}^{-1}(U(M, N))$, the explicit decomposition of $\overline{\mathcal{L}}_k$ is given in [17] as (remember that we have to take the conjugate of the $U(M, N)$ modules, but not of the $U(k)$ modules used in [17], since our $U(k)$ action is the

usual one; also, we use the conventions of [3] and [12] for labelling the highest weight, rather than those of [17] – this corresponds to an interchange of $\mathsf{m}$ and $\mathsf{n}$)

$$\overline{\mathcal{L}}_k \overset{\text{sq}}{\cong} \bigoplus_{(\mathsf{m},\mathsf{n})} \mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \overline{\mathcal{H}}^{U(M,N)}_{(\mathsf{m}+k,\mathsf{n})}, \tag{16}$$

where the sum is over all pairs $(\mathsf{m},\mathsf{n})$ as defined before, with zeros allowed, but neither $\mathsf{m}$ nor $\mathsf{n}$ allowed to be empty. $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ as a representation space of $U(k)$ was defined in subsection 2.3, and $\mathcal{H}^{U(M,N)}_{(\mathsf{m}+k,\mathsf{n})}$ carries the unitary representation of $U(M,N)$ with highest weight (not subject to further 'renormalization')

$$(m_1 + k, \ldots, m_i + k, \ldots, m_M + k, -n_N, \ldots, -n_j, \ldots, -n_1).$$

The decomposition under $U_{L,\text{hf}}(U(k)) \otimes U^{-1}_{R,\text{hf}}(U(M,N))$, on the other hand, reads [12]

$$\overline{\mathcal{L}}_k \overset{\text{hf}}{\cong} \bigoplus_{(\mathsf{m},\mathsf{n})} \mathcal{H}^{\tilde{U}(k)}_{(\mathsf{m}+\frac{1}{2}(M-N),\mathsf{n}-\frac{1}{2}(M-N))} \otimes \overline{\mathcal{H}}^{\tilde{U}(M,N)}_{(\mathsf{m}+\frac{1}{2}k,\mathsf{n}+\frac{1}{2}k)}, \tag{17}$$

where the highest weight $(\mathsf{m}+\frac{1}{2}k, \mathsf{n}+\frac{1}{2}k)$ is explicitly given by

$$(m_1 + k/2, \ldots, m_i + k/2, \ldots, m_M + k/2, -n_N - k/2, \ldots, n_j - k/2, \ldots, -n_1 - k/2),$$

whereas $\mathcal{H}_{(\mathsf{m}+\frac{1}{2}(M-N),\mathsf{n}-\frac{1}{2}(M-N))}$ is the tensor product of $\mathcal{H}_{(\mathsf{m},\mathsf{n})}$, and $\mathbb{C}$, carrying the determinant representation of $U(k)$ to the power $(M-N)/2$, cf. [12]).

Working with (16) for the sake of concreteness, we now wish to apply Rieffel induction from a suitable representation of $H = U(M,N)$ to $\overline{\mathcal{L}}_k$ in order to extract the copy of $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ for the value of $(\mathsf{m},\mathsf{n})$ selected by the representation we induce from. Firstly, we need a dense subspace $L \subset \overline{\mathcal{L}}_k$ such that the function $x \to (U^{-1}_{R,\text{sq}}(x)\psi, \varphi)$ is in $L^1(H)$ for all $\psi, \varphi \in L$. This is easily found: using the decomposition (16), we take $L$ to consist of vectors having a finite number of components in the decomposition, each component of which is in the tensor product of $\mathcal{H}^{U(k)}_{\cdots}$ and the dense subspace of $K$-finite vectors in the other factor. Since each function of the type $x \to (U(x)\psi, \varphi)$, where $U$ is in the discrete series, and $\psi$ and $\varphi$ are $K$-finite vectors, is in Harish-Chandra's Schwartz space [21] (which is a subspace of $L^1(H)$), this choice indeed satisfies the demand. (Based on the explicit realization of $\overline{\mathcal{L}}_k$ as a function space [17], a more 'geometric' choice of $L$ may also be found.)

As we are going to induce from holomorphic discrete series representations of $U(M,N)$, let us examine the tensor product $\overline{\mathcal{H}}^{U(M,N)}_{(\mathsf{m}_1,\mathsf{n}_1)} \otimes \mathcal{H}^{U(M,N)}_{(\mathsf{m}_2,\mathsf{n}_2)}$. Recall that $(\mathsf{m},\mathsf{n})$ (which here refers to the actual highest weight, rather than the dominant integral weight that is subject to renormalization, as sketched above) defines a unitary irreducible representation $U_{(\mathsf{m},\mathsf{n})}$ of the maximal compact subgroup $K = U(M) \times U(N)$ with highest weight $(m_1, \ldots, m_M, -n_N, \ldots, -n_1)$. By Theorem 2 in [38], the above tensor product is unitarily equivalent as a representation space of $U(M,N)$ to the representation induced (in the usual, Mackey, sense) from $\overline{U}_{(\mathsf{m}_1,\mathsf{n}_1)} \otimes U_{(\mathsf{m}_2,\mathsf{n}_2)}(K)$. Using the reduction-induction theorem, we can therefore decompose this induced representation as a direct sum over the representations induced from the components in the decomposition of $\overline{U}_{(\mathsf{m}_1,\mathsf{n}_1)} \otimes U_{(\mathsf{m}_2,\mathsf{n}_2)}(K)$.

Let us examine a generic representation $U^\kappa(H)$ (realized on the Hilbert space $\mathcal{H}^\kappa$ of functions $\psi : G \to \mathcal{H}_\kappa$ satisfying the equivariance condition $\psi(xk) = U_\kappa(k^{-1})\psi(x)$)

induced from an irreducible representation $U_\kappa(K)$. The Rieffel induction procedure produces the semi-definite form $(\cdot,\cdot)_0$ on $L \otimes \mathcal{H}_\chi$ (where, in this case, $\mathcal{H}_\chi = \mathcal{H}^{U(M,N)}_{(\mathsf{m},\mathsf{n})}$ for certain $(\mathsf{m},\mathsf{n})$). Using (16) and the previous paragraph, we find that $L \otimes \mathcal{H}_\chi$ is a certain dense subspace of a direct sum with components of the type $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^\kappa$, in which $H$ acts trivially on the first factor. By our construction of $L$, each element of $L \otimes \mathcal{H}_\chi$ only has components in a finite number of these Hilbert spaces, so that we can investigate each component separately. (Had the number of components of elements of $L$ been infinite, the study of $(\cdot,\cdot)_0$ would have been more involved, as this is an unbounded and non-closable quadratic form, so that $(\sum_i \psi_i, \varphi)_0 \neq \sum_i (\psi_i, \varphi)_0$ for infinite sums.)

Factorizing $\int_H dx = \int_N dn \int_K dk$ [21], it follows from the equivariance condition and the orthogonality relations for compact groups that in a given component $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^\kappa$ the expression $(\psi,\varphi)_0 = \int_H dx\, (\mathbb{I} \otimes U^\kappa(x)\psi, \varphi)$ vanishes unless $U_\kappa$ is the identity representation $U_{\mathrm{id}}$ of $K$. Given a highest weight representation $U_\chi(H)$ we Rieffel-induce from, there exists a unique pair $(\mathsf{m},\mathsf{n})$ for which $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^{\mathrm{id}}$ occurs in the decomposition of $\overline{\mathcal{L}}_k \otimes \mathcal{H}_\chi$ as a sum over induced representations of $H$ in the above sense.

Let $L^{\mathrm{id}}$ be the projection of $L \otimes \mathcal{H}_\chi$ onto this $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^{\mathrm{id}}$. We define $\tilde{V}: L^{\mathrm{id}} \to \mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ by linear extension of $\tilde{V}\psi_1 \otimes \psi_2 = \psi_1 \int_H dx\, \psi_2(x)$ (where $\psi_1 \in \mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ and $\psi_2 \in \mathcal{H}^{\mathrm{id}} \subset L^2(G)$). The integral exists by our assumptions on $L$; moreover, the explicit form of the inner product in $\mathcal{H}^{\mathrm{id}}$ (namely $(f,g) = \int_H dx\, f(x)\overline{g(x)}$, as $K$ is compact) leads to the equality $(\tilde{V}\psi, \tilde{V}\varphi) = (\psi,\varphi)_0$ (where the inner product on the left-hand side is the one in $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$). We now extend $\tilde{V}$ to a map $V$ from $L \otimes \mathcal{H}_\chi$ to $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ by putting it equal to zero on all spaces involving a factor $\mathcal{H}^\kappa$, where $\kappa \neq \mathrm{id}$ (and equal to $V$ on $L^{\mathrm{id}}$, of course). Clearly, by this and the preceding paragraph,

$$(V\psi, V\varphi) = (\psi,\varphi)_0. \tag{18}$$

We are now in a standard situation in the theory of Riefel induction, in which we can identify the null space of $(\cdot,\cdot)_0$ with the kernel of $V$, and the induced space $\mathcal{H}^\chi$ (which, we recall, is the completion of the quotient of $L \otimes \mathcal{H}_\chi$ by this null space in the inner product obtained from this form) with the closure of the image of $V$. It is clear from our definition of $L$ that the image of $V$ actually coincides with $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$. Also, the definition of the induced representation $U^\chi$ of $G = U(k)$ on $\mathcal{H}^\chi$ immediately implies that $U^\chi \simeq U_{(\mathsf{m},\mathsf{n})}$. Finally, note that (18) shows explicitly that $(\cdot,\cdot)_0$ is positive semi-definite, a fact which was already certified by Prop. 2 in [22].

Putting these arguments together, we have proved:

**Theorem 7** *Let $U(k)$ and $U(M,N)$ act on $S = \mathbb{C}^k \otimes \mathbb{C}^{M+N}$ (equipped with the symplectic form (3)) from the left and the right, respectively, in the natural way, and let $\overline{\mathcal{L}}_k$ be the quantization of $S$, with commuting representations of $U(k)$ and $U(M,N)$ on $\overline{\mathcal{L}}_k$ (which quantize the above symplectic actions) as given (up to conjugation of the representation of $U(M,N)$) by Kashiwara-Vergne [17].*

*Then Rieffel induction on $\overline{\mathcal{L}}_k$ from the holomorphic discrete series representation of $U(M,N)$ with highest weight $(\mathsf{m}+k,\mathsf{n})$ (that is, the highest weight with components $(m_1 + k, \ldots, m_M + k, -n_N, \ldots, -n_1)$) leads to an induced space $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$, which as a Rieffel-induced*

$U(k)$ *module carries the representation* $U_{(m,n)}(U(k))$ *(which is the Young product of the representation with Young diagram* m *and the conjugate of the representation with Young diagram* n*).*

*Moreover, the induced space is empty if one induces from a highest weight representation of* $U(M, N)$ *of the form* $(m, n)$ *in which at least one* $m_i$ *is smaller than* $k$*, or is not integral.*

## 3    Discussion

The last part of Theorem 7 is particularly unpleasant for the quantization theory of constrained system, for it shows that Theorem 5 cannot really be 'quantized' unless m or n are empty. For we would naturally induce from the holomorphic discrete series representation of $U(M, N)$ having the 'renormalized' highest weight corresponding to a coadjoint orbit characterized by $(m, n)$, as explained at the beginning of this subsection. But then for $k$ large enough the induced space will be empty, rather than consisting of $\mathcal{H}^{U(k)}_{(m,n)}$, as desired. As we have seen, the induction procedure is only successful if we induce from a representation with highest weight $(m + k, n)$, rather than from the ($k$-independent) renormalized weight we ought to use by first principles. This is bizarre, given that the original weight $(m, n)$ (or the orbit it corresponds to) knows nothing about $k$ or $U(k)$. In addition, even without this problem the induced space will often be empty, because the 'correct' renormalized highest weight one induces from may simply not occur in the Kashiwara-Vergne decomposition (16) because of the half-integral nature of its entries (which is a pure 'quantum' phenomenon). (In a rather different setting, the discrepancy for large $k$ between the 'decomposition' of $S$ into pairs of matched coadjoint orbits for $U(k)$ and $U(M, N)$, and the decomposition of $\overline{\mathcal{L}}_k$ under these groups, must have been noticed by Adams [2], who points out that there is a good correspondence for $k \leq \min(M, N)$ only.)

It is peculiar to the non-compact ($N \neq 0$) case that this difficulty even arises if the half-form correction to quantization is not applied. For (16) is the non-compact analogue of (13), and in the latter quantization clearly does commute with reduction. If we do incorporate half-forms, we obtain (17) for $U(M, N)$ and (15) for $U(M)$. In both cases the Rieffel induction process generically (that is, if $M \neq N$) fails to produce the correct representation of $U(k)$, even if one induces from a representation whose highest weight is renormalized (compared to the weight expected from the orbit correspondence) by the term $k/2$.

Finally, the passage from $\mathbb{C}^k$ to infinite-dimensional Hilbert spaces is tortuous whenever half-forms are used (the corrections being infinite for $k = \infty$), and in the non-compact case even without these. This is partly because of the $k$-dependence of the highest weights of $U(M, N)$, and partly because $\mathcal{L}$ does not contain the identity representation of $U(M, N)$ (recall that in the compact case we used the carrier space $\mathbb{C}\Omega$ of this representation as the fixed 'tail' vector to construct the von Neumann infinite tensor product from).

Clearly, this situation deserves further study. We do not think it is an artifact of our proposal of using Rieffel induction in the quantization of constrained systems. In fact, this technique comprises the only method known to us which is precise enough to bring the embarrassment to light.

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